Probabilistic Reasoning

Independence Representation

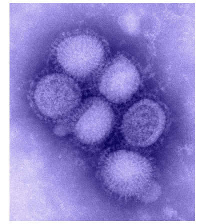
Peter Lucas

peterl@cs.ru.nl

Institute for Computing and Information Sciences

Radboud University Nijmegen

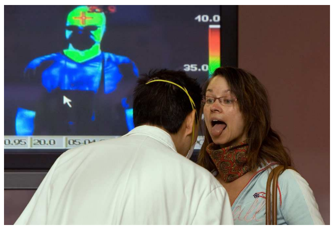
Have you got mexican flu?

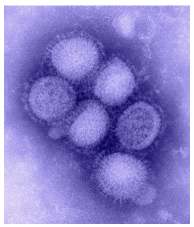


$$
\begin{aligned}
P(m,c,s) &= 0.009215 \\
P(m,\bar{c},s) &= 0.000485 \\
P(m,c,\bar{s}) &= 0.000285 \\
P(m,\bar{c},\bar{s}) &= 1.5 \cdot 10^{-5} \\
P(\bar{m},c,s) &= 9.9 \cdot 10^{-6} \\
P(\bar{m},\bar{c},s) &= 0.0098901 \\
P(\bar{m},c,\bar{s}) &= 0.0009801 \\
P(\bar{m},\bar{c},\bar{s}) &= 0.97912
\end{aligned}
$$

- M: mexican flu; C: chills; S: sore throat
- Probability of mexican flu and sore throat?
- Probability of mexican flu given sore throat?

Have you got mexican flu?





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$$

- M: mexican flu; C: chills; S: sore throat
- Probability of mexican flu and sore throat? 0.0097
- Probability of mexican flu given sore throat? 0.495

Probabilistic reasoning

Joint probability distribution $P(X) = P(X_1, X_2, \ldots, X_n)$

- marginalisation:

$$P(Y) = \sum_Z P(Y,Z), \quad \text{with } X = Y \cup Z$$

- conditional probabilities:

$$P(Y \mid Z) = \frac{P(Y,Z)}{P(Z)}$$

- Bayes' theorem:

$$P(Y \mid Z) = \frac{P(Z \mid Y)P(Y)}{P(Z)}$$

Probabilistic reasoning (cont)

Examples:

$P(m,s) = P(m,c,s) + P(m,\bar{c},s) = 0.009215 + 0.000485 = 0.0097$

$P(m \mid s) = P(m,s)/P(s) = 0.0097/0.0196 = 0.495$

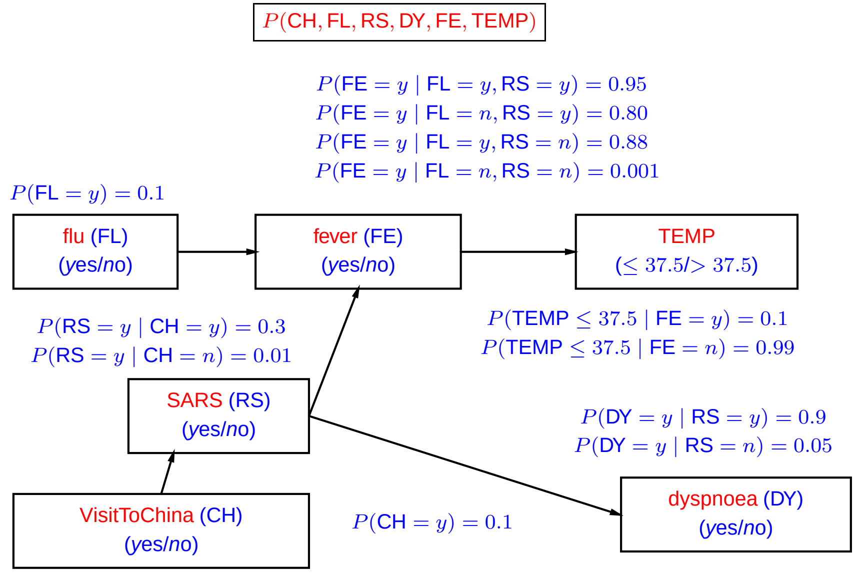
Note that:

- Mainly interested in conditional probability distributions:

$$P(Z \mid \mathcal{E}) = P^{\mathcal{E}}(Z)$$

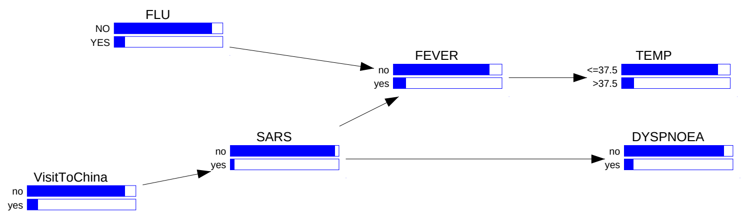
  for (possibly empty) evidence $\mathcal{E}$ (instantiated variables)
- Tendency to focus on conditional probability distributions of single variables
- Many efficient reasoning algorithms exist

Bayesian networks

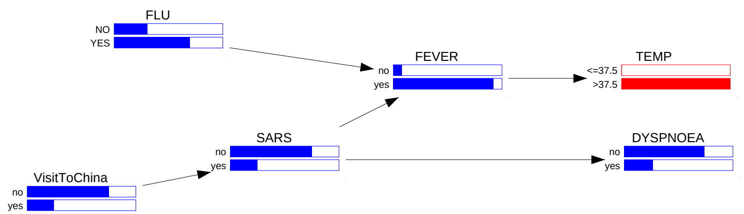


Reasoning: evidence propagation

- Nothing known:

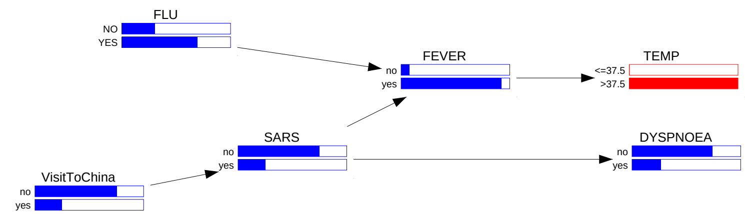


- Temperature >37.5 °C:

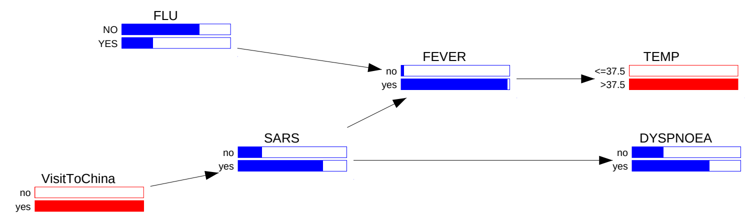


Reasoning: evidence propagation

- Temperature >37.5 °C:



- I just returned from China:



Definition Bayesian network

A Bayesian network $\mathcal{B}$ is a pair $\mathcal{B} = (G, P)$, where:

- (Qualitative part) $G = (V(G), A(G))$ is an acyclic directed graph, with
  - $V(G) = \{v_1, v_2, \ldots, v_n\}$, a set of vertices (nodes)
  - $A(G) \subseteq V(G) \times V(G)$ a set of arcs
- (Quantitative part) $P(X_{V(G)})$ is a joint probability distribution, such that

$$P(X_{V(G)}) = \prod_{v \in V(G)} P(X_v \mid X_{\pi(v)})$$

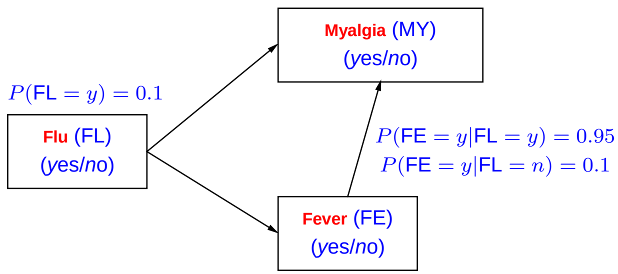
where $\pi(v)$ denotes the set of parents of vertex v in G

Markov independence

A Bayesian network

$P(\text{FL}, \text{MY}, \text{FE})$

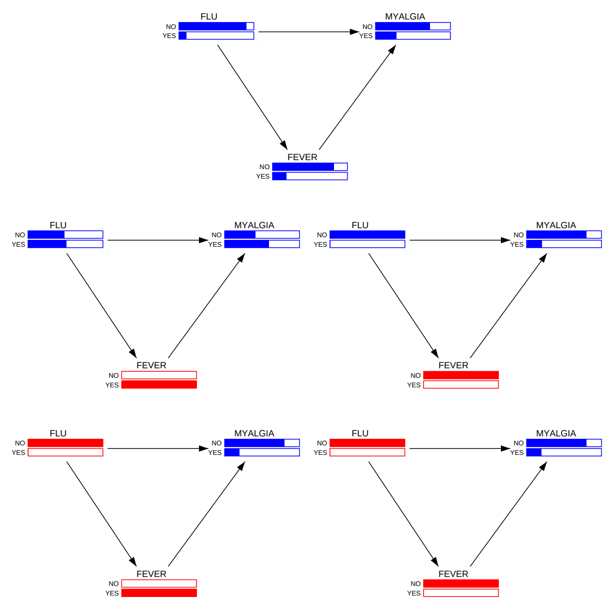
$P(\text{MY} = y | \text{FL} = y, \text{FE} = y) = 0.96$
$P(\text{MY} = y | \text{FL} = y, \text{FE} = n) = 0.96$
$P(\text{MY} = y | \text{FL} = n, \text{FE} = y) = 0.20$
$P(\text{MY} = y | \text{FL} = n, \text{FE} = n) = 0.20$



Thus: $P(\text{FL}, \text{MY}, \text{FE}) = P(\text{MY}|\text{FL}, \text{FE})P(\text{FE}|\text{FL})P(\text{FL})$

Example: $P(\neg fl, my, fe) = 0.20 \cdot 0.1 \cdot 0.9 = 0.018$

Independence and reasoning

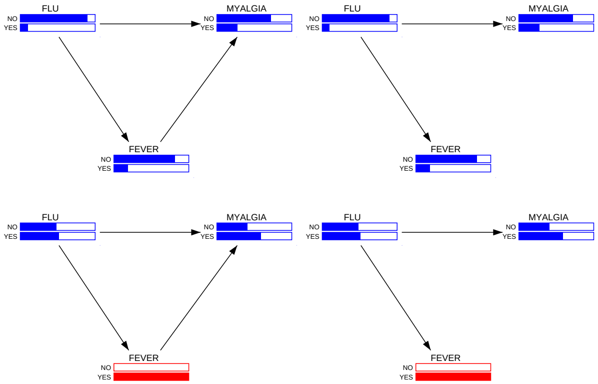


Independence and reasoning

Conclusion: the arc from FEVER to MYALGIA can be removed, and hence only

$$P(\text{MY} \mid \text{FL}) \; (= P(\text{MY} \mid \text{FL}, \text{FE}))$$

need be specified



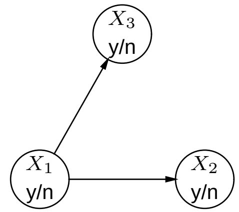
Independence relation

Let $X, Y, Z \subseteq V$ be *sets of (random) variables*, and let P be a probability distribution of V then X is called conditionally independent of Y given Z, denoted as

$$X \perp\!\!\!\perp_P Y \mid Z, \quad \text{iff} \quad P(X \mid Y, Z) = P(X \mid Z)$$

Note: This relation is completely defined in terms of the probability distribution P, but there is *a relationship to graphs*, for example:

$$\{X_2\} \perp\!\!\!\perp_P \{X_3\} \mid \{X_1\}$$



How to define an independence relation?

- List all the instances of $\perp\!\!\!\perp$
- List some of the instances of $\perp\!\!\!\perp$ and add axioms from which other instances can be derived
- Define a joint probability distribution P and look into the numbers to see which instances of the independence relation $\perp\!\!\!\perp$ hold (this yields $\perp\!\!\!\perp_P$)
- Use a graph to encode $\perp\!\!\!\perp$, which yields $\perp\!\!\!\perp_G$ (so, what type of graph — directed, undirected, chain?)

Explicit enumeration

Consider $V = \{1, 2, 3, 4\}$ and $\perp\!\!\!\perp$:

$\{1\} \perp\!\!\!\perp \{4\} \mid \varnothing$	$\{4\} \perp\!\!\!\perp \{2\} \mid \{1\}$	$\{2\} \perp\!\!\!\perp \{4\} \mid \varnothing$
$\{4\} \perp\!\!\!\perp \{3\} \mid \{1\}$	$\{3\} \perp\!\!\!\perp \{4\} \mid \varnothing$	$\{4\} \perp\!\!\!\perp \{2,3\} \mid \{1\}$
$\{4\} \perp\!\!\!\perp \{1\} \mid \varnothing$	$\{1\} \perp\!\!\!\perp \{4\} \mid \{2\}$	$\{4\} \perp\!\!\!\perp \{2\} \mid \varnothing$
$\{3\} \perp\!\!\!\perp \{4\} \mid \{2\}$	$\{4\} \perp\!\!\!\perp \{3\} \mid \varnothing$	$\{1,3\} \perp\!\!\!\perp \{4\} \mid \{2\}$
$\{1,2\} \perp\!\!\!\perp \{4\} \mid \varnothing$	$\{4\} \perp\!\!\!\perp \{1\} \mid \{2\}$	$\{1,3\} \perp\!\!\!\perp \{4\} \mid \varnothing$
$\{4\} \perp\!\!\!\perp \{3\} \mid \{2\}$	$\{2,3\} \perp\!\!\!\perp \{4\} \mid \varnothing$	$\{4\} \perp\!\!\!\perp \{1,3\} \mid \{2\}$
$\{4\} \perp\!\!\!\perp \{1,2\} \mid \varnothing$	$\{1\} \perp\!\!\!\perp \{4\} \mid \{3\}$	$\{4\} \perp\!\!\!\perp \{1,3\} \mid \varnothing$
$\{2\} \perp\!\!\!\perp \{4\} \mid \{3\}$	$\{4\} \perp\!\!\!\perp \{2,3\} \mid \varnothing$	$\{1,2\} \perp\!\!\!\perp \{4\} \mid \{3\}$
$\{1,2,3\} \perp\!\!\!\perp \{4\} \mid \varnothing$	$\{1\} \perp\!\!\!\perp \{2\} \mid \{4\}$	$\{4\} \perp\!\!\!\perp \{1,2,3\} \mid \varnothing$
$\{2\} \perp\!\!\!\perp \{1\} \mid \{4\}$	$\{1\} \perp\!\!\!\perp \{2\} \mid \varnothing$	$\{3\} \perp\!\!\!\perp \{4\} \mid \{1,2\}$
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$\{2\} \perp\!\!\!\perp \{1,4\} \mid \varnothing$	$\{1\} \perp\!\!\!\perp \{4\} \mid \{2,3\}$	$\{1\} \perp\!\!\!\perp \{2,4\} \mid \varnothing$
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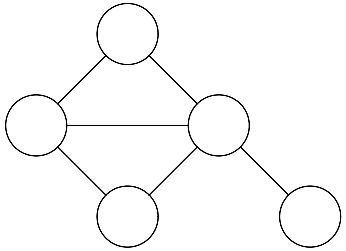
$\{3\} \perp\!\!\!\perp \{4\} \mid \{1\}$	$\{4\} \perp\!\!\!\perp \{1\} \mid \{3\}$	$\{2,3\} \perp\!\!\!\perp \{4\} \mid \{1\}$
$\{4\} \perp\!\!\!\perp \{2\} \mid \{3\}$		

As an undirected graph

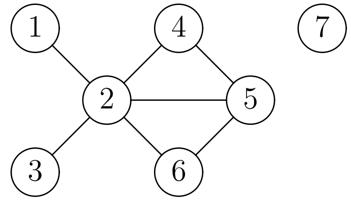


Basic idea:

- Each variable V is represented as a vertex in an undirected graph $G = (V(G), E(G))$, with set of vertices $V(G)$ and set of edges $E(G)$
- the independence relation $\perp\!\!\!\perp_G$ is encoded as the absence of edges; a missing edge between vertices u and v indicates that random variables X_u and X_v are (conditionally) independent $=$ (u-)separation

Example

Consider the following undirected graph G:



- $\{1\} \perp\!\!\!\perp_G \{3,6\} \mid \{2\}$
- $\{4\} \perp\!\!\!\perp_G \{6\} \mid \{2,5\}$
- $\{4\} \perp\!\!\!\perp_G \{6\} \mid \{1,2,3,5\}$
- $\{1\} \not\!\perp\!\!\!\perp_G \{5\} \mid \{4\}$, as the path $1 - 2 - 5$ does not contain 4
- $\{1,5,6\} \perp\!\!\!\perp_G \{7\} \mid \varnothing$

D-map and I-map for $\perp\!\!\!\perp_P$

Let P be probability distribution of X. Let $G = (V(G), E(G))$ be an undirected graph, then for each $U, W, Z \subseteq V(G)$:

- G is called an undirected dependence map, D-map for short, if

$$X_U \perp\!\!\!\perp_P X_W \mid X_Z \Rightarrow U \perp\!\!\!\perp_G W \mid Z$$

- G is called an undirected independence map, I-map for short, if

$$U \perp\!\!\!\perp_G W \mid Z \Rightarrow X_U \perp\!\!\!\perp X_W \mid X_Z$$

- G is called an undirected perfect map, or P-map for short, if G is both a D-map and an I-map, or, equivalently

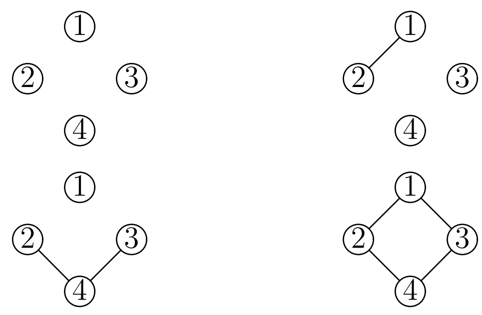
$$X_U \perp\!\!\!\perp_P X_W \mid X_Z \Longleftrightarrow U \perp\!\!\!\perp_G W \mid Z$$

Examples D-maps

Let $V = \{1,2,3,4\}$ be a set and X_V the corresponding set of random variables, and consider the independence relation $\perp\!\!\!\perp_P$, defined by

$$\{X_1\} \perp\!\!\!\perp_P \{X_4\} \mid \{X_2, X_3\}$$
$$\{X_2\} \perp\!\!\!\perp_P \{X_3\} \mid \{X_1, X_4\}$$

The following undirected graphs are examples of D-maps:

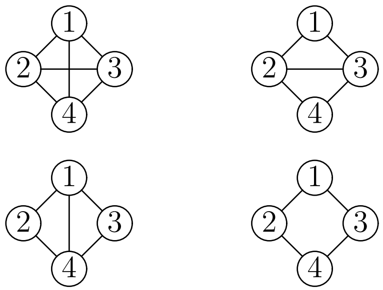


Examples of I-maps

Let $V = \{1, 2, 3, 4\}$ be a set with random variables X_V, and consider the independence relation $\perp\!\!\!\perp_P$:

$$\{X_1\} \perp\!\!\!\perp_P \{X_4\} \mid \{X_2, X_3\}$$
$$\{X_2\} \perp\!\!\!\perp_P \{X_3\} \mid \{X_1, X_4\}$$

The following undirected graphs are examples of I-maps:



(So, what is the P-map?)

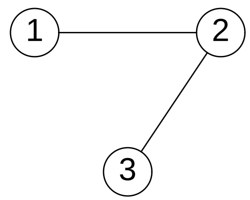
Markov network

A pair $\mathcal{M} = (G, P)$, where

- $G = (V(G), E(G))$ is an *undirected* graph with set of vertices $V(G)$ and set of edges $E(G)$,
- P is a joint probability distribution of $X_{V(G)}$, and
- G is an *I-map* of P

is said to be a Markov network or Markov random field

Example $\mathcal{M} = (G, \phi) = (G, P)$:

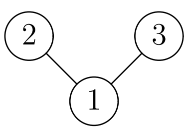


Potential:
$\phi(X_1, X_2, X_3) = \psi(X_1, X_2)\tau(X_2, X_3)$,

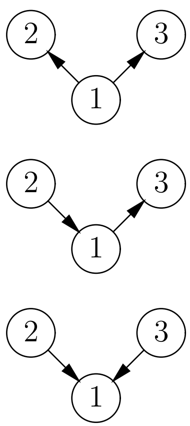
or joint probability distribution:
$P(X_1, X_2, X_3) = \frac{P(X_1, X_2)P(X_2, X_3)}{P(X_2)}$

Expressiveness: directed vs undirected

Directed graphs are more subtle when it comes to expressing independence information than undirected graphs



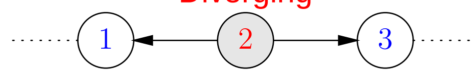
vs



d-Separation: *3* situations

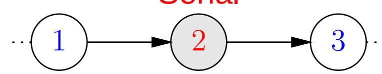
A chain k (= path in undirected underlying graph) in an acyclic directed graph $G = (V(G), A(G))$ can be blocked:

Diverging



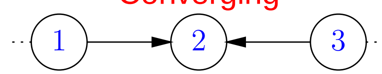
2 blocks (d-separates) 1 and 3: $\{1\} \perp\!\!\!\perp \{3\} \mid \{2\}$

Serial



2 blocks (d-separates) 1 and 3: $\{1\} \perp\!\!\!\perp \{3\} \mid \{2\}$

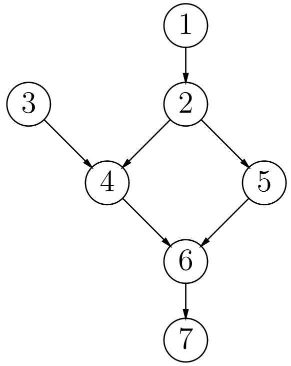
Converging



2 d-connects 1 and 3: $\{1\} \not\perp\!\!\!\perp \{3\} \mid \{2\}$

(same holds for successors of 2); note $\{1\} \perp\!\!\!\perp \{3\} \mid \varnothing$

Example blockage



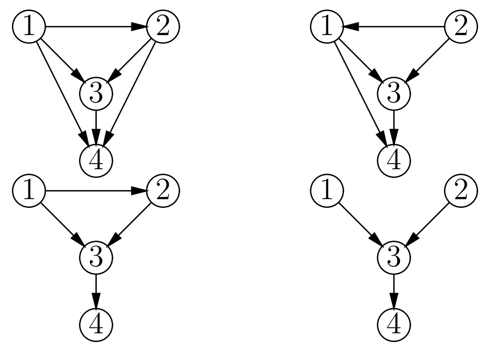
- The chain $4, 2, 5$ from 4 to 5 is blocked by $\{2\}$
- The chain $1, 2, 5, 6$ from 1 to 6 is blocked by $\{5\}$, and also by $\{2\}$ and $\{2, 5\}$
- The chain $3, 4, 6, 5$ from 3 to 5 is blocked by $\{4\}$ and $\{4, 6\}$, but *not* by $\{6\}$

Examples directed I-maps

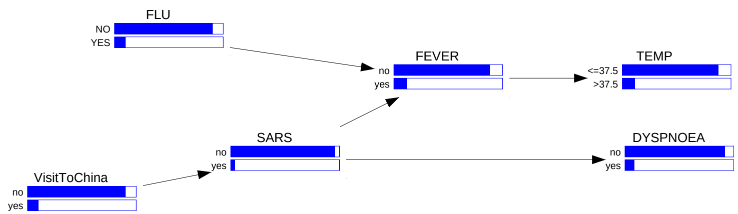
Consider the following independence relation $\perp\!\!\!\perp_P$:

$$\begin{array}{rcl} \{X_1\} & \perp\!\!\!\perp_P & \{X_2\} \mid \varnothing \\ \{X_1, X_2\} & \perp\!\!\!\perp_P & \{X_4\} \mid \{X_3\} \end{array}$$

and the following directed I-maps of P:



Find the independences



Examples:

- FLU $\perp\!\!\!\perp$ VisitToChina $\mid \varnothing$
- FLU $\perp\!\!\!\perp$ SARS $\mid \varnothing$
-
-

Relationship directed and undirected graphs

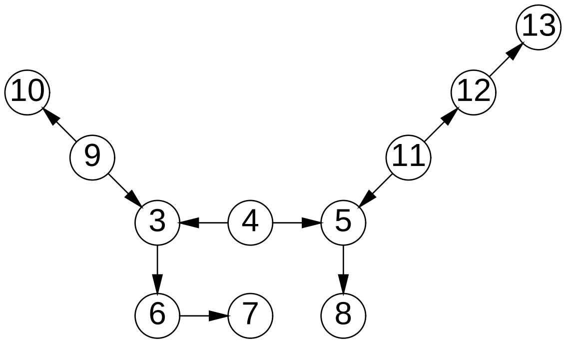
- Directed graphs contain independences that become dependences after conditioning (instantiating variables)
- Undirected graphs do not have this property
- However, undirected subgraphs can be generated, by making potentially dependent parents of a child dependent

Example:

Moralisation

Let G be an acyclic directed graph; its associated undirected moral graph G^m can be constructed by moralisation:

1. add lines to all non-connected vertices, which have a common child, or descendant of a common child, and
2. replace each arc with a line in the resulting graph



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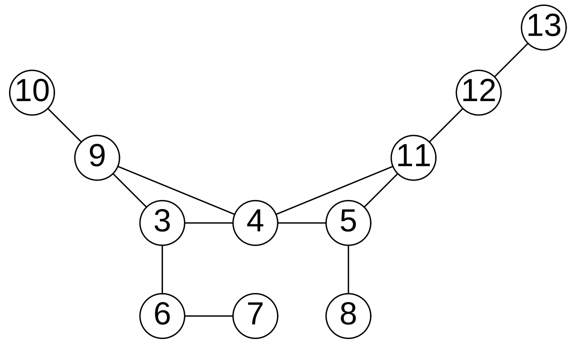
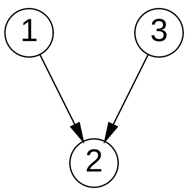


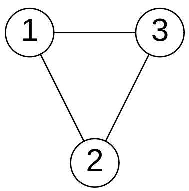
Comments

- Resulting undirected (moral) graph is an I-map of the associated probability distribution
- However, it contains too many dependences!

Example: $\{1\} \perp\!\!\!\perp_G^d \{3\} \mid \varnothing$, whereas $\{1\} \not\perp\!\!\!\perp_{G^m} \{3\} \mid \varnothing$



Original

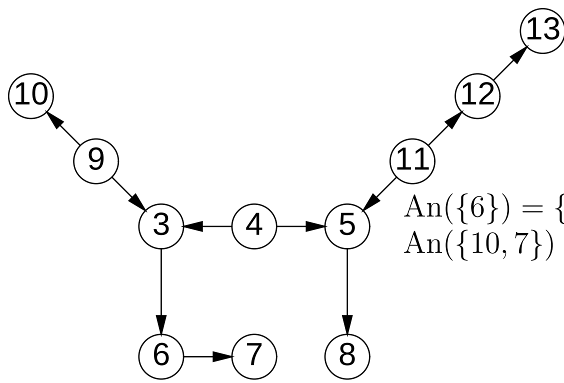


Moral Graph

- Conclusion: make moralisation 'dynamic' (i.e. a function of the set on which we condition)
- For this the notion of 'ancestral set' is required

Ancestral set

Let $G = (V(G), A(G))$ be an acyclic directed graph, then if for $W \subseteq V(G)$ it holds that $\pi(v) \subseteq W$ for all $v \in W$, then W is called an ancestral set of W. $\mathrm{An}(W)$ denotes the smallest ancestral set containing W



$\mathrm{An}(\{6\}) = \{3, 4, 6, 9\}$
$\mathrm{An}(\{10, 7\}) = \{7, 6, 3, 4, 9, 10\}$

‘Dynamic’ moralisation

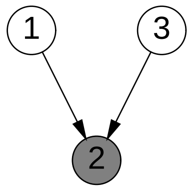
Let P be a joint probability distribution of a Bayesian network $\mathcal{B} = (G, P)$, then

$$X_U \perp\!\!\!\perp_P X_V \mid X_W$$

holds iff U and V are (u-)separated by W in the moral induced subgraph G^m of G with vertices $\mathrm{An}(U \cup V \cup W)$

Example:

$$X_1 \not\perp\!\!\!\perp_P X_3 \mid X_2; \quad \mathrm{An}(\{1,2,3\}) = \{1,2,3\}$$



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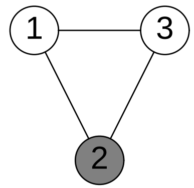

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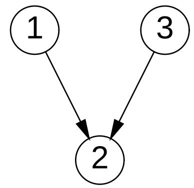
holds iff U and V are (u-)separated by W in the moral induced subgraph G^m of G with vertices $\mathrm{An}(U \cup V \cup W)$

Example:

$$X_1 \perp\!\!\!\perp_P X_3 \mid \varnothing; \quad \mathrm{An}(\{1,3\}) = \{1,3\}$$

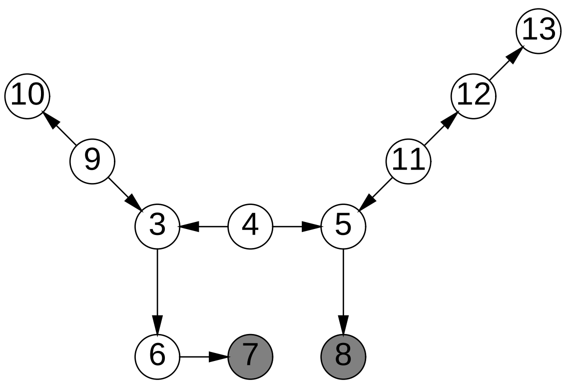


Original

Moral Graph

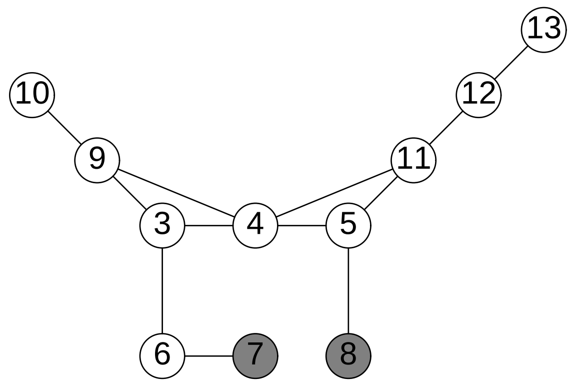
Example (1)

$$\{10\} \not\perp\!\!\!\perp_G^d \{13\} \mid \{7,8\}$$



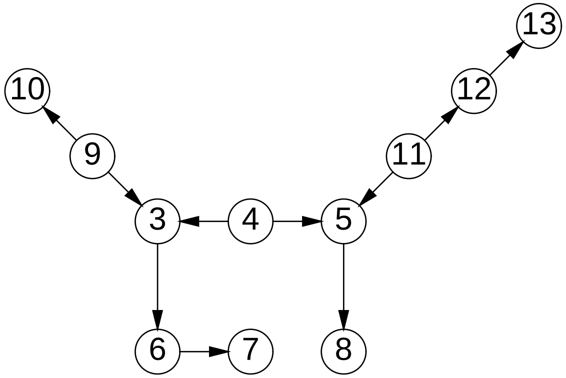
Example (1)

$$\{10\} \not\perp\!\!\!\perp_{G^m_{\mathrm{An}(\{10,7,8,13\})}} \{13\} \mid \{7,8\}$$



Example (2)

$\{10\} \perp\!\!\!\perp^{d}_{G} \{13\} \mid \varnothing$



Example (2)

$\{10\} \not\perp\!\!\!\perp_{G^{m}_{\mathrm{An}(\{10,13\})}} \{13\} \mid \varnothing$

Conclusions

- Conditional independence is defined as a logic that supports:
  - symbolic reasoning about dependence and independence information
  - makes it possible to abstract away from the numerical detail of probability distributions
  - the process of assessing probability distributions
- Looking at graphs makes it easier to find probability distributions that are equivalent (important in learning)
- Conditional independence is currently being extended towards causal independence (a logic of causality) = maximal ancestral graphs