1. Consider the model \( \mathcal{M} = (W, R, V) \) given by the following picture:

(a) Write out the definitions of \( W \), \( R \) and \( V \).

(b) Show the following:
   (i) \( \mathcal{M}, a_1 \models \Box(p \lor q) \),
   (ii) \( \mathcal{M}, a_2 \models \Diamond q \rightarrow \Box \Diamond q \),
   (iii) \( \mathcal{M}, a_3 \models \Diamond p \rightarrow \Box(q \rightarrow \Box(p \rightarrow \Box p)) \).

(c) Show the following:
   (i) \( \mathcal{M} \not\models p \rightarrow \Diamond p \),
   (ii) \( \mathcal{M} \models \Box \Box \Box \neg q \),
   (iii) \( \mathcal{M} \not\models q \rightarrow (\Diamond q \rightarrow \Box(q \rightarrow \Diamond q)) \).

(d) Change the valuation \( V \) on the frame such that in the new model \( \mathcal{M}' = (W, R, V') \) it holds that: \( \mathcal{M}' \models \Box p \rightarrow p \).
2. The binary tree is the frame $\mathcal{B} = (W, R)$ where the domain $W$ is the set of all finite strings over the alphabet $\{0,1\}$:

$$W = \{0,1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \}.$$

Here $\varepsilon$ denotes the empty string, and the transition relation $R \subseteq W \times W$ is given by:

$$Rst \text{ if and only if } t = s0 \text{ or } t = s1$$

(a) Make a drawing of the first four levels of $\mathcal{B}$.

(b) Consider a valuation $V$ on $\mathcal{B}$ that makes $p$ true on all strings of even length. Show that $\mathcal{B}, V \models \Box \Diamond p \rightarrow \Box \Diamond p$.

(c) Let $V'$ be a valuation on $\mathcal{B}$ which makes the variable $p$ true on all strings whose first letter is 0, and $q$ on strings with first letter 1, so:

$$V'(p) = \{0w \mid w \in \{0,1\}^*\}$$
$$V'(q) = \{1w \mid w \in \{0,1\}^*\}$$

Use this valuation to show that the formula $\lambda$ defined by:

$$\lambda = \Diamond p \land \Diamond q \rightarrow \Diamond(p \land \Diamond q) \lor \Diamond(p \land q) \lor \Diamond(p \land q)$$

is not valid in the binary tree.

(d) Show that the formula $\Diamond \Diamond p \rightarrow \Diamond p$ is not valid in $\mathcal{B}$.

3. Prove or disprove universal validity of the following formulas:

(a) $\Box (p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$

(b) $\Box (p \land q) \rightarrow (\Box p \land \Box q)$

(c) $\Box p \rightarrow \Diamond p$

(d) $\Box (\Box p \rightarrow p) \rightarrow \Box p$