A frame $\mathcal{F} = (W, R)$ is

- **symmetric** if $\forall xy (Rxy \rightarrow Ryx)$
- **serial** if $\forall x (\exists y (Rxy))$
- **partially functional** if $\forall xyz (Rxy \land Rxz \rightarrow y = z)$
- **functional** if serial and partially functional

1. Let $\mathcal{F} = (W, R)$ be a frame.

   (a) Show that the following implications hold:
   
   i. if $\mathcal{F}$ is serial, then $\mathcal{F} \vDash \Box p \rightarrow \Diamond p$
   
   ii. if $\mathcal{F}$ is serial, then $\mathcal{F} \vDash \Diamond \top$

   iii. if $\mathcal{F}$ is partially functional, then $\mathcal{F} \vDash \Diamond p \rightarrow \Box p$

   (b) Show that the modal formulas in (a) define the corresponding frame property. In other words, show that the reverse implications hold as well.

2. Let $\mathcal{N}$ be the frame $(\mathbb{N}, S)$ of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the successor relation $S$ defined by:

   $$Smn \text{ if and only if } n = m + 1,$$

   and recall the binary tree $\mathcal{B} = (\{0, 1\}^*, R)$ from last week with $R$ defined by:

   $$Rst \text{ if and only if } t = s0 \text{ or } t = s1$$

   (a) Define a valuation $U$ on $\mathcal{N}$ such that:

   $$\mathcal{B}, V, \varepsilon \leftrightarrow \mathcal{N}, U, 0$$

   where $V$ is a valuation on $\mathcal{B}$ such that:

   $$V(p) = \{ w \in \{0, 1\}^* \mid w \text{ is of even length} \}.$$

   Show that this is the only possibility for $U$. 


(b) Let $V'$ be a valuation on $B$ such that:

$$V'(p) = \{ 0w \mid w \in \{0,1\}^* \}$$
$$V'(q) = \{ 1w \mid w \in \{0,1\}^* \} .$$

Show that there exists no valuation $U'$ on $N$ such that:

$$B, V', \varepsilon \models N, U', 0$$

3. Consider the following two frames $F$ and $F'$

Here it is to be understood that Frame $F$ has infinitely many paths of finite length 1, 2, 3, ... Frame $F'$ is like $F$ but additionaly has one infinite path.

Argue that there cannot be a bisimulation $Z$ between $F$ and $F'$ such that $wZw'$.

4. (a) Give an example of a formula $\varphi$ and a model $M$ such that neither $M \models \varphi$ nor $M \models \neg \varphi$.

(b) Give an example of a formula $\varphi$, a frame $F$ and two models $M$ and $M'$ based on $F$ such that $M \models \varphi$ and $M' \models \neg \varphi$.

5. Consider the frame $F = (W,R)$ with set of worlds $W = \{1, 2, 3\}$ and $R = \{(1,1), (1,2), (1,3), (2,3)\}$. Give for every $i \in W$ a formula $\phi_i$ that is for any valuation true in $i$, but not in any $j \in W \setminus \{i\}$. So $F, V, i \models \phi_i$ for any $V$, but for $i \neq j$ we have for all $V$ that $F, V, j \not\models \phi_i$.  

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