Exercises

1. Formulate the box version of “Modal Decomposition” (MLOM, p. 41, and slide number 8 of lecture 6), i.e., give necessary and sufficient conditions for validity of a modal sequent of the form

\[ \Box p, \Box \varphi_1, \ldots, \Box \varphi_k \implies \Box \psi_1, \ldots, \Box \psi_m, \Box \bar{q} \]

2. Consider both the sequent- and the tableau approach for the following formulas:

(a) \( \Diamond p \land \Diamond q \rightarrow \Diamond (p \land q) \),
(b) \( (\Box p \rightarrow \Box q) \rightarrow \Box (p \rightarrow q) \)

(see also MLOM p44).

3. Prove or disprove the validity of the following formulas in the temporal frame \( \mathcal{N} = (\mathbb{N}, \prec) \) of the natural numbers \( \mathbb{N} = \{0, 1, \ldots \} \) with the usual ordering \( \prec \):

(a) \( \Diamond \Box p \rightarrow \Box \Diamond p \)
(b) \( \Box \Diamond p \rightarrow \Diamond \Box p \)
(c) \( \langle F \rangle [F] \bot \)
(d) \( \langle P \rangle \top \rightarrow \langle P \rangle [P] \top \)
(e) \( \langle P \rangle \langle F \rangle q \rightarrow \langle \langle P \rangle q \lor q \lor \langle F \rangle q \rangle \)

4. Show that the formula

\[ \lambda = \Diamond p \land \Diamond q \rightarrow \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q) \]

defines right-linearity, that is, for all (not necessarily temporal) frames \( \mathcal{F} = (W, R) \):

\( \mathcal{F} \models \lambda \) if and only if \( R \) is right-linear

A relation \( R \) is right-linear if \( R_{xy} \land R_{xz} \) implies \( R_{yz} \lor y = z \lor R_{zy} \) for all \( x, y, z \).
5. Show that $\Diamond p \rightarrow \Diamond \Diamond p$ defines density.

6. Let $\tau$ and $\gamma$ abbreviate the following formulas:

$$
\tau = (\langle F \rangle[F]q \land \langle F \rangle\neg q) \rightarrow \langle F \rangle(\neg q \land [F]q)
$$

$$
\gamma = (\langle F \rangle[F]q \land \langle F \rangle\neg q) \rightarrow \langle F \rangle([F]q \land [P](F)\neg q)
$$

Consider the temporal frames $Z = (\mathbb{Z}, <)$, $Q = (\mathbb{Q}, <)$, and $R = (\mathbb{R}, <)$ of the integers, rational and real numbers, respectively, with their usual orderings.

(a) Show that $\tau$ is valid in $Z$.

(b) Show that $\tau$ is not valid in $Q$.

(c) Show that $\gamma$ is not valid in $Q$.

(d) Is $\gamma$ valid in $R$?