**Notation:** For a frame \((W, R)\) and a point \(x \in W\), we will use \(R[x]\) to denote the set of \(R\)-successors of \(x\) as follows: \(R[x] = \{y \mid Rxy\}\).

**Answers:**

1. (a)  
   \[ W = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} \]
   
   \[ R = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_5), (a_3, a_5), (a_4, a_6), (a_5, a_7), (a_6, a_7)\} \]
   
   \[ V(p) = \{a_1, a_2, a_5, a_7\} \]
   
   \[ V(q) = \{a_2, a_3, a_4, a_6\} \]

   (b) (ii) \(a_2 \not\models \diamond q \rightarrow \diamond \diamond q\) holds because \(a_2 \not\models \diamond q\), and this is because \(a_5 \not\in V(q)\) and \(a_5\) is the only \(R\)-successor of \(a_2\).

   (iii) To see that \(a_3 \models \diamond p \rightarrow \square (q \rightarrow \square (p \rightarrow \square p))\) we have to check that \(a_3 \models \square (q \rightarrow \square (p \rightarrow \square p))\) (because \(a_3 \models \diamond p\) holds). So we check \(x \models q \rightarrow \square (p \rightarrow \square p)\) for every \(x \in R[a_3] = \{a_5, a_6\}\):

   - \(a_5 \not\models q\), whence \(a_5 \not\models q \rightarrow \square (p \rightarrow \square p)\);
   - \(a_6 \models q\), so we need to check \(a_6 \not\models \square (p \rightarrow \square p)\), which means, as \(R[a_6] = \{a_7\}\), to check that \(a_7 \not\models p \rightarrow \square p\). This indeed holds since \(R[a_7] = \emptyset\), i.e., \(a_7\) is a blind world, and so, by the truth definition of \(\square\), we get \(a_7 \not\models \square p\), and therefore \(a_7 \not\models p \rightarrow \square p\).

   (c) (ii) \(\square \square \square q\) is satisfied in all points of \(\mathcal{M}\). The only point that has \(R^3\)-successors at all, is \(a_1\), namely \(a_7\) and we have \(a_7 \not\models q\). All other points vacuously satisfy \(\square \square \square \varphi\) for any formula \(\varphi\); for all \(a \neq a_1\), we have \(R^3[a] = \emptyset\).\(^1\)

   (iii) We have to find a point \(a_i\) such that \(\mathcal{M}, a_i \not\models q \rightarrow (\diamond q \rightarrow \square (q \rightarrow \diamond q))\), i.e., such that \(a_i \not\models q\) and \(a_i \models \diamond q\) but \(a_i \not\models (q \rightarrow \diamond q)\). The only candidates meeting the first two, are \(a_3\) and \(a_4\). In fact both will do, as the share \(a_6\) as their \(R\)-successor, and indeed in \(a_6\) the implication \(q \rightarrow \diamond q\) is false (we have \(a_6 \models q\) and \(a_6 \not\models \diamond q\)). Hence we have \(a_4 \not\models (q \rightarrow \diamond q)\).

\(^1\)This possibly sounds a bit vague, but recall that in the lecture we have seen that the composed modality \(\Box^n\) is interpreted by the composed relation \(R^n\), see the remark at the end of this note.
(d) Just let \( p \) be true in all worlds. Then \( \Box p \rightarrow p \) holds everywhere. In fact, this valuation \( (V'(p) = W) \) is the only option to make \( \Box p \rightarrow p \) globally true. The reason is that \( \Box p \) is true in \( a_7 \) independent of the valuation we choose. So, in order for the implication \( \Box p \rightarrow p \) to become true in \( a_7 \), we have to make \( p \) true in \( a_7 \). Reasoning backwards, we see that then also \( a_5 \) and \( a_6 \) need to have \( p \), and, in turn, also \( a_2, a_3, \) and \( a_4 \). At last, because of that, also \( a_1 \) must be in the valuation of \( p \).

2. (a) The first four levels of the complete binary tree \( \mathcal{B} \):

![Diagram of the complete binary tree]

(b) The given valuation \( V \) is that \( p \) holds at all and only the strings of even length. In order to show that the formula \( \Box \Diamond p \rightarrow \Box \Box p \) is true throughout the model \( (\mathcal{B}, V) \), we consider an arbitrary point \( s \in \{0,1\}^* \) in this model, and assume \( s \models \Box \Diamond p \). Our goal is to show \( s \not\models \Diamond \Diamond p \). From \( s \models \Box \Diamond p \) we obtain \( s0 \not\models \Diamond p \) (and also \( s1 \not\models \Diamond p \)). In turn, \( s0 \not\models \Diamond p \) means that \( s00 \not\models p \) or \( s01 \not\models p \). In both cases we see that \( s \) is of even length. This entails that \( s0 \not\models \Box p \) because both children of \( s0 \) have even length. We conclude \( s \not\models \Diamond \Diamond p \).

(c) Now we consider the valuation \( V' \) on \( \mathcal{B} \), which makes \( p \) true at all strings that start with a \( 0 \), and make \( q \) true at all points that start with a \( 1 \). So \( p \) holds everywhere in the left immediate subtree of the root \( \varepsilon \), whereas \( q \) holds in the entire right immediate subtree of \( \varepsilon \). Moreover, \( \varepsilon \not\in V'(p) \) and \( \varepsilon \not\in V'(q) \).

We have to show that the formula

\[
\Diamond p \land \Diamond q \rightarrow \Diamond (p \land q) \lor \Diamond (p \land q) \lor (\Diamond p \land q)
\]

is not true in all points of the model \( (\mathcal{B}, V') \). The only candidate to falsify this formula is \( \varepsilon \), as it is the only one that sees a point with \( p \) (namely \( 0 \)) and a point with \( q \) (namely \( 1 \)), so that \( \varepsilon \not\models \Diamond p \land \Diamond q \). All three disjuncts of the right-hand side of the implication are false in \( \varepsilon \):

2
- $0 \not\in \diamond q$ (by $00 \not\in q$ and $01 \not\in q$), so $0 \not\in p \land \diamond q$. As $1 \not\in p$, also $1 \not\in p \land \diamond q$. Hence $\varepsilon \not\in \diamond (p \land \diamond q)$.
- $0 \not\in q$, so $0 \not\in p \land q$, and $1 \not\in p \land q$. Hence $\varepsilon \not\in \diamond (p \land q)$.
- $0 \not\in q$, so $0 \not\in \diamond p \land q$. $1 \not\in \diamond p$ (by $10 \not\in p$ and $11 \not\in p$), so $1 \not\in \diamond p \land q$. Hence $\varepsilon \not\in \diamond (p \land q)$.

(d) To show that $\diamond \diamond p \rightarrow \diamond p$ is not valid in $\mathcal{B}$ we have to come up with a valuation $V''$ and a point $x$, such that $\mathcal{B}, V'', x \models \diamond \diamond p$ and $\mathcal{B}, V'', x \not\models \diamond p$. For example, we can take $V''(p) = \{00\}$. Then $\mathcal{B}, V'', \varepsilon \models \diamond \diamond p$ (since $\mathcal{B}, V''$, $0 \models \diamond p$), but $\mathcal{B}, V'', \varepsilon \not\models \diamond p$ (since $0 \not\in p$ and $1 \not\in p$). Another example is the model from (b).

3. (a) We have $\models \Box (p \rightarrow q) \rightarrow (\diamond p \rightarrow \diamond q)$.
(b) The formula $\Box (p \land q) \rightarrow (\diamond p \land \diamond q)$ is not universally valid. This means we have to give a concrete model and a point where the formula does not hold. Consider the one-point model $\langle \bullet, \emptyset, V \rangle$ without any arrows, and with $V$ irrelevant. In blind worlds boxed formulas always hold whereas diamonds always fail. So $\bullet \models \Box (p \land q)$ and $\bullet \not\models \diamond p$ and $\bullet \not\models \diamond q$. And so $\bullet \not\models \Box (p \land q) \rightarrow (\diamond p \land \diamond q)$.
(c) The formula $\Box (p \land q) \rightarrow (\Box p \land \Box q)$ is valid in all frames. Let $\mathcal{M} = (W, R, V)$ be an arbitrary model, $x$ an arbitrary point of $\mathcal{M}$, and assume $x \models \Box (p \land q)$. In order to show $x \models \Box p$ we consider an arbitrary $y$ with $Rxy$ (so the goal is to show $y \models p$). From the assumption $x \models \Box (p \land q)$ we know that $y \models p \land q$, and so $y \models p$. The argument for $x \models \Box q$ is analogous (pick an arbitrary $R$-successor $z$ of $x \ldots$). So we conclude $x \models \Box (p \land q) \rightarrow (\Box p \land \Box q)$. As $\mathcal{M}$ and $x$ were arbitrary we thus have shown universal validity of the formula.
(d) The formula $\Box p \rightarrow \diamond p$ is valid precisely in the serial frames, that is frames where every point has at least one successor. The one-point model mentioned under (b) forms a counterexample against universal validity: $\bullet \models \Box p$ but $\bullet \not\models \diamond p$.

Remark. In question 1 (c) (ii) we refer to the following fact:

$$\mathcal{M}, x \models \Box^n \varphi \quad \text{if and only if} \quad \mathcal{M}, y \models \varphi \quad \text{for all} \quad y \quad \text{such that} \quad x R^n y,$$

(1)

for all models $\mathcal{M} = (W, R, V)$, worlds $x \in W$, formulas $\varphi$ and natural numbers $n$. Here, for every $n \in \mathbb{N}$, the modality $\Box^n$, and the relation $R^n$
are defined by
\[
\begin{align*}
\Box^0 \varphi &= \varphi \\
\Box^{n+1} \varphi &= \Box \Box^n \varphi
\end{align*}
\]

\[R^0 = \text{Id} = \{(x, x) \mid x \in W\} \]

\[R^{n+1} = R \circ R^n,\]

where for binary relations \(S, T \subseteq W \times W\), their composition \(S \circ T\) is defined by
\[
S \circ T := \{(x, z) \mid \exists y. (x, y) \in S \land (y, z) \in T\}.
\]

(Note that \(S \circ \text{Id} = \text{Id} \circ S = S\) and so \(R^1 = R\).)

We prove (1) by induction on \(n\). If \(n = 0\), \(\Box^n \varphi = \varphi\) and \(R^n = \text{Id}\). So “\(M, y \models \varphi\) for all \(y\) such that \(xR^ny\)” boils down to “\(M, x \models \varphi\)”, and the equivalence holds trivially. For the induction step, fix an arbitrary \(n \in \mathbb{N}\) and assume the statement (1) holds for \(n\) (this is called the induction hypothesis, IH). We have to prove it holds for \(n + 1\) too. Here goes:

\[
M, x \models \Box^{n+1} \varphi \iff M, x \models \Box \Box^n \varphi
\]

\[
\iff \forall x' (xRx' \implies M, x' \models \Box^n \varphi)
\]

\[
\iff \forall x' (xRx' \implies \forall y (x'R^ny \implies M, y \models \varphi)) \quad \text{(by IH)}
\]

\[
\iff \forall x' \forall y (xRx' \implies (x'R^ny \implies M, y \models \varphi))
\]

\[
\iff \forall y \forall x' (xRx' \implies (x'R^ny \implies M, y \models \varphi))
\]

\[
\iff \forall y \forall x' ((xRx' \land x'R^ny) \implies M, y \models \varphi)
\]

\[
\iff \forall y (\exists x' (xRx' \land x'R^ny) \implies M, y \models \varphi)
\]

\[
\iff \forall y (xR^{n+1}y \implies M, y \models \varphi).
\]