an alternative semantics

<table>
<thead>
<tr>
<th>Definition</th>
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<tr>
<td>Let ( \mathcal{M} = (W, R, V) ) be a model. We define ([\varphi]_\mathcal{M} \subseteq W), the <strong>interpretation</strong> of a formula ( \varphi ) in the model ( \mathcal{M} ), inductively by</td>
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<tr>
<td>([p]_\mathcal{M} = V(p) ) ( (p \in \text{VAR}) )</td>
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<td>([\bot]_\mathcal{M} = \emptyset )</td>
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<td>([\top]_\mathcal{M} = W )</td>
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<td>([-\varphi]<em>\mathcal{M} = W \setminus [\varphi]</em>\mathcal{M} )</td>
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<tr>
<td>([\varphi \lor \psi]<em>\mathcal{M} = [\varphi]</em>\mathcal{M} \cup [\psi]_\mathcal{M} )</td>
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<td>([\varphi \land \psi]<em>\mathcal{M} = [\varphi]</em>\mathcal{M} \cap [\psi]_\mathcal{M} )</td>
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<tr>
<td>([\varphi \rightarrow \psi]<em>\mathcal{M} = \neg[\varphi]</em>\mathcal{M} \cup [\psi]_\mathcal{M} )</td>
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<td>([\Diamond \varphi]<em>\mathcal{M} = { w \in W \mid \exists v \ (Rwv \land v \in [\varphi]</em>\mathcal{M} ) } )</td>
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<td>([\Box \varphi]<em>\mathcal{M} = { w \in W \mid \forall v \ (Rwv \implies v \in [\varphi]</em>\mathcal{M} ) } )</td>
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(for \( X \subseteq W \) we write \( \neg X \) to denote the **complement** of \( X \), i.e., \( \neg X = W \setminus X \))
Lemma

The interpretation of $\varphi$ in $\mathcal{M} = (W, R, V)$ is the set of states of $\mathcal{M}$ where $\varphi$ is true:

$$\mathcal{M}, w \models \varphi \iff w \in [\varphi]_{\mathcal{M}}$$

Hence, we also have:

$$\mathcal{M} \models \varphi \iff [\varphi]_{\mathcal{M}} = W \quad (1)$$
evaluating substitution instances

Definition

Let $\mathcal{M} = (W, R, V)$ be a model, and $\sigma$ a substitution.
Define $\mathcal{M}^\sigma = (W, R, V^\sigma)$ where $V^\sigma$ is defined by

$$(V^\sigma)(p) = [\sigma(p)]_M \quad (p \in \text{VAR})$$

Lemma

$$[\varphi^\sigma]_M = [\varphi]_{M^\sigma} \quad (2)$$
validity is closed under substitution

Theorem

If a formula \( \varphi \) is valid in a frame \( \mathcal{F} \), then so are all its substitution instances:

\[
\mathcal{F} \models \varphi \quad \iff \quad \mathcal{F} \models \varphi^\sigma
\]

Proof: immediate from (1) and (2).
substitution as a tool to derive new validities

- substitution generates new valid formulas from old ones
- example: if $\mathcal{F} \models \Box \Diamond p \rightarrow \Diamond \Box p$ then $\mathcal{F} \models \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$ for all formulas $\varphi$
- if $\psi$ is a substitution instance of a propositional tautology then $\psi$ is a modal tautology
- let $\delta : \text{VAR} \rightarrow \text{FORM}$ be the substitution defined by
  \[ \delta(p) = \neg p \quad (p \in \text{VAR}) \]
- then we have:
  \[ \mathcal{F} \models \varphi \iff \mathcal{F} \models \varphi^\delta \]
- and so:
  \[ \mathcal{F} \models p \rightarrow \Diamond p \iff \mathcal{F} \models \Box p \rightarrow p \]
  \[ \mathcal{F} \models \Diamond \Diamond p \rightarrow \Diamond p \iff \mathcal{F} \models \Box p \rightarrow \Box \Box p \]
  \[ \mathcal{F} \models p \rightarrow \Box \Diamond p \iff \mathcal{F} \models \Diamond \Box p \rightarrow p \]
  \[ \mathcal{F} \models \Diamond p \rightarrow \Box \Diamond p \iff \mathcal{F} \models \Diamond \Box p \rightarrow \Box p \]
  etc.
\( \square \Box p \rightarrow p \) valid in all symmetric frames

let \( \text{Symm} = \{(W, R) \mid \forall xy (Rxy \implies Ryx)\} \)

\[ \text{Symm} \models \square \Box p \rightarrow p \]

1. let \( \mathcal{F} = (W, R) \in \text{Symm} \)
2. consider an arbitrary model \( \mathcal{M} \) based on \( \mathcal{F} \), and a state \( w \in W \) such that \( \mathcal{M}, w \models \square \Box p \). we have to show \( \mathcal{M}, w \models p \)
3. from 2 follows the existence of a state \( v \in W \) such that
   a. \( Rwv \) and
   b. \( \mathcal{M}, v \models \Box p \)
4. \( \mathcal{M}, u \models p \) for all \( u \) with \( Rvu \) (3b)
5. \( Rvw \) (3a & 1)
6. \( \mathcal{M}, w \models p \) (5 & 4)
7. \( \mathcal{M}, w \models \square \Box p \rightarrow p \) (2 & 6)
8. \( \text{Symm} \models \square \Box p \rightarrow p \)
truth versus validity

- validity is a **stable** form of truth
- validity is independent of valuation, truth is not, e.g.:
  - $\mathcal{F} \models p \rightarrow \diamond p$ implies $\mathcal{F} \models q \rightarrow \diamond q$
  - but $\mathcal{M} \models p \rightarrow \diamond p$ does not imply $\mathcal{M} \models q \rightarrow \diamond q$

- local truth (truth in a point of a model) is preserved by
  - modus ponens: if $\mathcal{M}, w \models \varphi \rightarrow \psi$ and $\mathcal{M}, w \models \varphi$ the $\mathcal{M}, w \models \psi$
- global truth (truth in all points of a model) is preserved by
  - modus ponens: if $\mathcal{M} \models \varphi \rightarrow \psi$ en $\mathcal{M} \models \varphi$ the $\mathcal{M} \models \psi$
  - necessitation: if $\mathcal{M} \models \varphi$ then $\mathcal{M} \models \Box \varphi$

- frame validity (truth in all models on the frame) is preserved by:
  - modus ponens: if $\mathcal{F} \models \varphi \rightarrow \psi$ and $\mathcal{F} \models \varphi$ then $\mathcal{F} \models \psi$
  - necessitation: if $\mathcal{F} \models \varphi$ then $\mathcal{F} \models \Box \varphi$
  - substitution: if $\mathcal{F} \models \varphi$ then $\mathcal{F} \models \varphi^\sigma$
The set of universally valid formulas, or modal tautologies:

- contains all propositional tautologies
- contains $\Box(p \to q) \to (\Box p \to \Box q)$
- is closed under modus ponens:
  if $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
- is closed under necessitation:
  if $\vdash \varphi$ then $\vdash \Box \varphi$
- is closed under substitution:
  if $\vdash \varphi$ then $\vdash \varphi^\sigma$
- and contains nothing else.
Definition

A modal formula $\varphi$ defines, or characterizes, a class $C$ of frames (a frame property) when

$$\mathcal{F} \in C \iff \mathcal{F} \models \varphi$$

In other words, $\varphi$ characterizes $C$ if

(⇒) $\varphi$ is valid in $C$: $C \models \varphi$

and

(⇐) $\varphi$ is invalid outside of $C$:

if $\mathcal{F} \notin C$ then $\mathcal{F} \not\models \varphi$
example: ♢□p → p characterizes symmetry

Example

\[ \mathcal{F} \in \text{Symm} \text{ if and only if } \mathcal{F} \models \Box \Box p \rightarrow p \]

(⇒) Symm ⊨ ♢□p → p (slide 13)

(⇐) by contraposition:

- let \( \mathcal{F} = (W, R) \not\in \text{Symm} \). we will show \( \mathcal{F} \not\models \Box \Box p \rightarrow p \)
- as \( R \) is not symmetric, there are states \( a \) and \( b \) (not necessarily distinct) such that \( Rab \) and \( \neg Rba \)
- let \( V \) on \( \mathcal{F} \) be such that \( V(p) = \{ x \in W \mid Rbx \} \) and \( \mathcal{M} = (\mathcal{F}, V) \)
- then \( \mathcal{M}, b \models \Box p \)
- by \( Rab \) also \( \mathcal{M}, a \models \Box \Box p \)
- also \( \mathcal{M}, a \not\models p \), as \( a \not\in V(p) \) since \( \neg Rba \)
- so \( \mathcal{M}, a \not\models \Box \Box p \rightarrow p \) and so \( \mathcal{F} \not\models \Box \Box p \rightarrow p \)
- conclusion: if \( \Box \Box p \rightarrow p \) is valid in a frame, then the frame is symmetric
example: $\diamond p \rightarrow \square p$ characterizes partial functionality

Example

$\mathcal{F} \in PF$ if and only if $\mathcal{F} \models \diamond p \rightarrow \square p$

where

$PF = \{ (W, R) \mid \forall xyz \ (Rxy \land Rxz \implies y = z) \}$
example

$$(W, R) \models \Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$$

if and only if

$$\{ (W, R) \mid \forall xyz \in W \ (Rxy \land Rxz \rightarrow Ryz \lor Ryz) \}$$