overview

- semantics
- game semantics
- alternative semantics

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formulas of modal logic

\[ p \mid \bot \mid \phi \rightarrow \psi \mid \Box \phi \mid \Diamond \phi \]

we could define \( \Box \) using \( \Diamond \) or vice versa:

\[
\Box \phi := \neg \Diamond \neg \phi \\
\Diamond \phi := \neg \Box \neg \phi
\]

we will consider different (equivalent) sets of connectives

- necessity: true in all accessible states or in all possible worlds
- possibility: true in at least one accessible state
truth and validity

\[(W, R), w \models \phi\]
\[(W, R, V) \models \phi\]
\[(W, R) \models \phi\]
\[\models \phi\]

what is omitted is implicitly universally quantified

more examples

\[\models \Box(\phi \land \psi) \leftrightarrow \Box \phi \land \Box \psi\]
\[\models \Diamond(\phi \lor \psi) \leftrightarrow \Diamond \phi \lor \Diamond \psi\]
\[\models \Box \phi \lor \Box \psi \rightarrow \Box(\phi \lor \psi)\]
\[\not\models \Box(\phi \lor \psi) \rightarrow \Box \phi \lor \Box \psi\]
\[\models \Diamond(\phi \land \psi) \rightarrow \Diamond \phi \land \Diamond \psi\]
\[\not\models \Diamond \phi \land \Diamond \psi \rightarrow \Diamond(\phi \land \psi)\]
\[\models \Diamond(\phi \land \psi) \rightarrow \Diamond \phi \land \Diamond \psi\]
\[\models \Diamond \phi \land \Diamond \psi \rightarrow \Diamond(\phi \land \psi)\]

examples of (in)valid axioms and rules

\[\models \phi\] if \(\phi\) a tautology of propositional logic
\[\models \Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)\]

modus ponens: if \(\models \phi \rightarrow \psi\) and \(\models \phi\) then \(\models \psi\)

necessitation: if \(\models \phi\) then \(\models \Box \phi\)
\[\not\models \phi \rightarrow \Box \phi\]
\[\not\models \Box \phi \rightarrow \phi\]

satisfiability

\(\phi\) is satisfiable in a model \(\mathcal{M}\)
if there is a world \(w\) in \(\mathcal{M}\) such that \(\mathcal{M}, w \models \phi\)

\(\phi\) is satisfiable
if there exist a model \(\mathcal{M}\) and a world \(w\) in \(\mathcal{M}\) such that \(\mathcal{M}, w \models \phi\)

\(\phi\) and \(\psi\) are semantically equivalent
if for all \(\mathcal{M}\) and \(w\) we have \(\mathcal{M}, w \models \phi\) if and only if \(\mathcal{M}, w \models \psi\)
semantics and syntax

we will also see various proof systems for modal logic
and soundness and completeness theorems

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modal formulas distinguishing states

consider a frame with four states \{1, 2, 3, 4\} and accessibility relation
\[1 \rightarrow 2 \rightarrow 3 \text{ and } 4 \rightarrow 1, 4 \rightarrow 3\]

find for every point \(i\) a distinguishing formula \(\phi_i\)
that is, a formula such that \(i \models \phi_j\) if and only if \(i = j\) for all
\(i, j \in \{1, 2, 3, 4\}\)

game semantics: setting

another approach to local truth

a model \(M = ((W, R), V)\), and a world \(w \in W\), and a formula \(\phi\)
there are two players:
Verifier (V) claims that \(\phi\) is true in \(w\)
Falsifier (F) claims that \(\phi\) is false in \(w\)
a position is a pair \((w, \phi)\) with \(w \in W\) a world and \(\phi\) a formula
a move from a position \((w, \phi)\) is determined by the main operator of \(\phi\)
game semantics: position determines move

\((t, \phi_1 \lor \phi_2)\): V chooses a disjunct \(\phi_i\); play continues with \((t, \phi_i)\)

\((t, \phi_1 \land \phi_2)\): F chooses a conjunct \(\phi_i\); play continues with \((t, \phi_i)\)

\((t, \diamond \phi)\): V chooses a successor \(u\) of \(t\); play continues with \((u, \phi)\)

\((t, \Box \phi)\): F chooses a successor \(u\) of \(t\); play continues with \((u, \phi)\)

\((t, \neg \phi)\): players switch roles; play continues with \((t, \phi)\)

\((t, p)\): if \(p\) is true in \(t\) then \(V\) wins; otherwise \(F\) wins

who should but cannot choose a successor loses

game tree

a complete game tree for \(\phi\) and \((M, w)\)

starts with \((w, \phi)\) and contains all possible moves

game semantics: strategy

a strategy for player \(P\) is a method to select moves

player \(P\) has a winning strategy

if there is a strategy ensuring that \(P\) wins the game

Theorem explaining the connection with the truth definition:

\(\phi\) is true in \(M\) in \(s\) \iff V has a winning strategy for \(M, s, \phi\)

overview
alternative semantics

intuition:

given a model $\mathcal{M} = (W, R, V)$,

the interpretation $\llbracket \phi \rrbracket_\mathcal{M}$ of a formula $\phi$

is the set of worlds in which $\phi$ is true