overview

- truth and validity
- bisimulations
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valid in a frame is preserved under substitution

if \( \mathcal{F} \models \phi \) then \( \mathcal{F} \models \phi^\sigma \) for any substitution \( \sigma \)

how do we prove this?

use ‘alternative semantics’

how can we use this?

for example: consider the substitution \( \delta \) with \( \delta(p) = \neg p \)

\( \mathcal{F} \models p \rightarrow \Diamond p \) if and only if \( \mathcal{F} \models \Box p \rightarrow p \)

\( \mathcal{F} \models \Diamond p \rightarrow \Box \Diamond p \) if and only if \( \mathcal{F} \models \Diamond \Box p \rightarrow \Box p \)

etcetera
formulas characterizing frame properties

the formula $\phi$ characterizes the frame property $P$ means

$\mathcal{F}$ has property $P$ if and only if $\mathcal{F} \models \phi$

examples:

$\diamondsuit p \rightarrow \Box p$ characterizes $\forall xyz \ (Rxy \land Rxz \Rightarrow y = z)$

$\Box(\Box p \rightarrow q) \lor \Box(\Box q \rightarrow p)$ characterizes $\forall xy \ (Rxy \land Rxz \rightarrow Ryz \lor Rzy)$
preservation of truth and validity

local truth is preserved by modus ponens:
if $\mathcal{M}, w \models \phi \rightarrow \psi$ and $\mathcal{M}, w \models \phi$ then $\mathcal{M}, w \models \psi$

global truth is preserved by modus ponens and by necessitation:
if $\mathcal{M} \models \phi$ then $\mathcal{M} \models \Box \phi$

frame validity is preserved by modus ponens, necessitation, and substitution:
if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$
the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
\[ \vdash \Box (p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

modus ponens:
if \( \vdash \phi \rightarrow \psi \) and \( \vdash \phi \) then \( \vdash \psi \)

necessitation:
if \( \vdash \phi \) then \( \vdash \Box \phi \)

substitution:
if \( \vdash \phi \) then \( \vdash \phi^\sigma \)
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distinguishable and indistinguishable states

we can distinguish state 1 from state 3
we cannot distinguish state 2 from state 4
Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models.

A non-empty relation $Z \subseteq W \times W'$ is a **bisimulation**, notation $Z : \mathcal{M} \leftrightarrow \mathcal{M}'$, if for all pairs $(w, w') \in Z$ we have the following:

- $w \in V(p)$ if and only if $w' \in V'(p)$
- if $Rwv$ then for some $v' \in W'$ we have $R'w'v'$ and $vZv'$
- if $R'w'v'$ then for some $v \in W$ we have $Rwv$ and $vZv'$
bisimulation: base

\[ \mathcal{M} = (W, R, V) \quad \quad \quad \mathcal{M}' = (W', R', V') \]

if \( wZw' \) then for all \( p \in \text{Var} \) we have \( p \in V(p) \) if and only if \( p \in V'(p) \)
bisimulation: base

\[ M = (W, R, V) \]
\[ M' = (W', R', V') \]

if \( wZw' \) then for all \( p \in \text{Var} \) we have \( p \in V(p) \) if and only if \( p \in V'(p) \)
bisimulation: zig

\[ M = (W, R, V) \]

\[ M' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ \mathcal{M} = (W, R, V) \]

\[ \mathcal{M}' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ M = (W, R, V) \quad \exists v' \in W' \quad M' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ \mathcal{M} = (W, R, V) \quad \exists v' \in W' \quad \mathcal{M}' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)

\[ \text{if } wZw' \text{ and } Rwv \text{ then there exists } v' \in W' \text{ such that } R'w'v' \text{ and } vZv' \]
bisimulation: zig

$$
\mathcal{M} = (W, R, V) \quad \exists v' \in W' \quad \mathcal{M}' = (W', R', V')
$$

if $wZw'$ and $Rwv$ then there exists $v' \in W'$ such that $R'w'v'$ and $vZv'$
bisimulation: zag

\[ M = (W, R, V) \quad \quad \quad M' = (W', R', V') \]

if \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rwv \) and \( vZv' \)
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\[ M = (W, R, V) \]
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\[ M = (W, R, V) \quad \exists v] \in W \quad M' = (W', R', V') \]

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\[ \mathcal{M} = (W, R, V) \quad \exists v \in W \quad \mathcal{M}' = (W', R', V') \]

if \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rwv \) and \( vZv' \)
bisimilarity: definition

two models \( calM = (W, R, V) \) and \( \mathcal{M}' = (W', R', V') \) are bisimilar, notation \( \mathcal{M} \leftrightarrow \mathcal{M}' \), if there exists a bisimulation \( Z \subseteq W \times W' \)

two pointed models \( \mathcal{M}, w \) and \( calM', w' \) are bisimilar notation \( \mathcal{M}, w \leftrightarrow \mathcal{M}', w' \) or \( w \leftrightarrow w' \), if there exists a bisimulation \( Z : \mathcal{M} \leftrightarrow \mathcal{M}' \) such that \( (w, w') \in Z \)
bisimilarity: example

give a bisimulation relating 2 and 4
non-bisimilarity: example
non-bisimilarity: example

```
    a
   /\  
  /   \  
 p----b----q
 |   / \   |
 | /   \  |
 c----d----

 1
 /\  
/   \  
2----3
 |
 |
 p 4----q
 |
 |
 5
```
non-bisimilarity: example
non-bisimilarity: example
non-bisimilarity: example
bisimularity: example

\( \mathcal{N} = (\mathbb{N}, S) \)

\( S = \{(n, n+1) \mid n \in \mathbb{N}\} \)

\( V(p) = \{2n \mid n \in \mathbb{N}\} \)

\( \mathcal{F} = (\{e, o\}, R) \)

\( R = \{(e, o), (o, e)\} \)

\( U(p) = \{e\} \)

0 and e are bisimilar
bisimilarity: property

bisimilarity is an equivalence relation between models
modal equivalence: definition

two states (not necessarily in the same model) are modally equivalent if they satisfy exactly the same formulas
Theorem: bisimilarity implies modal equivalence

If two states are bisimilar then they are modally equivalent.
Theorem: modal equivalence implies bisimilarity for finitely branching models:

If two states are modally equivalent then they are bisimilar.

Why is the condition needed?