overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations

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prop1: schemes preservation under substitution

Sometimes we consider a formula as a scheme:

\[ p \rightarrow q \rightarrow p \]

\[ (r \land s) \rightarrow \bot \rightarrow (r \land s) \]

Let \( \sigma : \text{Var} \rightarrow \text{Form} \) be a substitution and \( \nu : \text{Var} \rightarrow \{0, 1\} \) a valuation

- If \( \models \phi \nu = 1 \) then not necessarily \( \models \phi^{\sigma} \nu = 1 \)
- If \( \models \phi \nu = 1 \) (or: \( \models \phi \)) then \( \models \phi^{\sigma} \)
definition: substitution

mapping $\sigma : \text{Var} \to \text{Form}$ that induces a mapping $(\cdot)\sigma : \text{Form} \to \text{Form}$:

$$
\begin{align*}
  p\sigma &= \sigma(p) \\
  \bot\sigma &= \bot \\
  \top\sigma &= \top \\
  (X\phi)\sigma &= X(\phi\sigma) \quad \text{for } X \in \{\neg, \Box, \Diamond\} \\
  (\phi * \psi)\sigma &= (\phi\sigma) * (\psi\sigma) \quad \text{for } * \in \{\lor, \land, \to\}
\end{align*}
$$

valid in a frame is preserved under substitution

we will see:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi\sigma$ for any substitution $\sigma$

how do we prove this?

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (W, R)$ and let $\sigma$ be a substitution

let $V$ be a valuation and let $w \in W$

we have to show $\mathcal{F}, V, w \models \phi\sigma$

we would like to use $\mathcal{F}, V\sigma, w \models \phi$

with $V\sigma$ mapping $p$ to the set of worlds in which $\sigma(p)$ is true

use 'alternative semantics'

modal prop1: what validity is preserved under substitution?

let $\sigma : \text{Var} \to \text{Form}$ be a substitution

if $((W, R), V), w \models \phi$ then not necessarily $((W, R), V), w \models \phi\sigma$

if $((W, R), V) \models \phi$ then not necessarily $((W, R), V) \models \phi\sigma$

if $(W, R) \models \phi$ then $(W, R) \models \phi\sigma$?

then also: if $\models \phi$ then $\models \phi\sigma$

alternative semantics

Let $\mathcal{M} = (W, R, V)$ be a model.
We define $\llbracket \phi \rrbracket_\mathcal{M} \subseteq W$, the interpretation of a formula $\phi$ in the model $\mathcal{M}$, inductively by

$$
\begin{align*}
  \llbracket p \rrbracket_\mathcal{M} &= V(p) \\
  \llbracket \bot \rrbracket_\mathcal{M} &= \emptyset \\
  \llbracket \top \rrbracket_\mathcal{M} &= W \\
  \llbracket \neg \phi \rrbracket_\mathcal{M} &= W \setminus \llbracket \phi \rrbracket_\mathcal{M} \\
  \llbracket \phi \lor \psi \rrbracket_\mathcal{M} &= \llbracket \phi \rrbracket_\mathcal{M} \cup \llbracket \psi \rrbracket_\mathcal{M} \\
  \llbracket \phi \land \psi \rrbracket_\mathcal{M} &= \llbracket \phi \rrbracket_\mathcal{M} \cap \llbracket \psi \rrbracket_\mathcal{M} \\
  \llbracket \phi \to \psi \rrbracket_\mathcal{M} &= \neg \llbracket \phi \rrbracket_\mathcal{M} \cup \llbracket \psi \rrbracket_\mathcal{M} \\
  \llbracket \Diamond \phi \rrbracket_\mathcal{M} &= \{w \in W | \exists v R v w \land v \in \llbracket \phi \rrbracket_\mathcal{M}\} \\
  \llbracket \Box \phi \rrbracket_\mathcal{M} &= \{w \in W | \forall v R v w \Rightarrow v \in \llbracket \phi \rrbracket_\mathcal{M}\}
\end{align*}
$$

(for $X \subseteq W$ we write $\neg X$ to denote the complement of $X$, i.e., $\neg X = W \setminus X$)
lemma: intuition about interpretation is correct

the interpretation $\llbracket \phi \rrbracket_M$ of formula $\phi$ in model $M = (W, R, V)$

is the set of worlds in which $\phi$ is true:

$M, w \models \phi$ if and only if $w \in \llbracket \phi \rrbracket_M$

consequence of the lemma:

$M \models \phi$ if and only if $\llbracket \phi \rrbracket_M = W$

evaluating substitution instances

let $M = (W, R, V)$ be a model and let $\sigma$ be a substitution

definition: a substitution $\sigma$ applied to a valuation:

$V^\sigma(p) = \llbracket \sigma(p) \rrbracket_M$

lemma:

$\llbracket \phi^\sigma \rrbracket_{(W, R, V)} = \llbracket \phi \rrbracket_{(W, R, V^\sigma)}$

theorem: validity is closed under substitution

if $F \models \phi$ then $F \models \phi^\sigma$ for any substitution $\sigma$

assume $F \models \phi$ with $F = (W, R)$ and let $\sigma$ be a substitution

we have to show $F \models \phi^\sigma$

let $V$ be a valuation and let $w \in W$

we have $F, V^\sigma, w \models \phi$

hence $w \in \llbracket \phi \rrbracket_{F, V^\sigma}$

hence $w \in \llbracket \phi^\sigma \rrbracket_{F, V}$

hence $F, V, w \models \phi^\sigma$

hence $F \models \phi^\sigma$

consequences of this theorem

take a propositional tautology, apply a substitution, then we have a modal tautology

how can we use this?

for example: consider the substitution $\delta$ with $\delta(p) = \neg p$

$F \models p \rightarrow \Diamond p$ if and only if $F \models \Box p \rightarrow p$

$F \models \Diamond p \rightarrow \Box \Diamond p$ if and only if $F \models \Diamond \Box p \rightarrow \Box p$
preservation of truth and validity

local truth is preserved by modus ponens:
if $M, w \models \phi \rightarrow \psi$ and $M, w \models \phi$ then $M, w \models \psi$

global truth is preserved by modus ponens and by necessitation:
if $M \models \phi$ then $M \models \Box \phi$

frame validity is preserved by modus ponens, necessitation, and substitution:
if $F \models \phi$ then $F \models \phi^\sigma$

example frame characterization

if $F = (W, R)$ is symmetric then $F \models \Diamond \Box p \rightarrow p$

let $F = (W, R)$ be a symmetric frame
let $M = (F, V)$ be a model based on $F$, and let $w \in W$
assume $M, w \models \Diamond \Box p$; we have to show $M, w \models p$
because $w \models \Diamond \Box p$ there is a $v \in W$ with $Rvw$ and $v \models \Box p$
because $F$ is symmetric we have $Rvw$
because $v \models \Box p$ we have $w \models p$
so $M, w \models \Diamond \Box p \rightarrow p$
so $F \models \Diamond \Box p \rightarrow p$

the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
$\vdash \Box (p \rightarrow q) \rightarrow \Box p \rightarrow \Box q$

modus ponens:
if $\vdash \phi \rightarrow \psi$ and $\vdash \phi$ then $\vdash \psi$

necessitation:
if $\vdash \phi$ then $\vdash \Box \phi$

substitution:
if $\vdash \phi$ then $\vdash \phi^\sigma$

example frame characterization

if $F \models \Diamond \Box p \rightarrow p$ then $F$ is symmetric

assume $F = (W, R)$ is not symmetric; we will show $F \not\models \Diamond \Box p \rightarrow p$
because $F$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$
(note: a non-symmetric frame has at least two states)
define $V$ such that $V(p) = \{ x \in W \mid Rbx \}$ and let $M = (F, V)$
then we have $M, b \models \Box p$
because $Rab$ we have $M, a \models \Diamond \Box p$
because not $Rba$ we have $a \notin V(p)$ so we have $M, a \not\models p$
so $M, a \not\models \Diamond \Box p \rightarrow p$
so $F \not\models \Diamond \Box p \rightarrow p$
frame characterization

we have shown:

the formula $\lozenge \Box p \rightarrow p$ characterizes the frame property symmetry

in general:

the formula $\phi$ characterizes the frame property $P$ means $\mathcal{F}$ has property $P$ if and only if $\mathcal{F} \models \phi$

more examples of frame characterizations

$\lozenge p \rightarrow \Box p$ characterizes $\forall xyz \ (Rxy \land Rxz \Rightarrow y = z)$

$\Box p \rightarrow \lozenge p$ characterizes $\forall x \exists y Rxy$

$\Box (\Box p \rightarrow q) \lor \Box (\Box q \rightarrow p)$ characterizes $\forall xy \ (Rxy \land Rxz \rightarrow Ryz \lor Ryz)$

overview

distinguishable and indistinguishable states

- preservation of truth and validity
- characterizations of frame properties
- bisimulations

we can distinguish state 1 from state 3
we cannot distinguish state 2 from state 4
Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models.

A non-empty relation $Z \subseteq W \times W'$ is a bisimulation, notation $Z : \mathcal{M} \leftrightarrow \mathcal{M}'$, if for all pairs $(w, w') \in Z$ we have the following:

- $w \in V(p)$ if and only if $w' \in V'(p)$
- if $Rwv$ then for some $v' \in W'$ we have $R'w'v'$ and $vZv'$
- if $R'w'v'$ then for some $v \in W$ we have $Rwv$ and $vZv'$

If $wZw'$ then for all $p \in \text{Var}$ we have $w \in V(p)$ if and only if $w' \in V'(p)$.

**bisimulation: zig**

- If $wZw'$ and $Rwv$ then there exists $v' \in W'$ such that $R'w'v'$ and $vZv'$

**bisimulation: zag**

- If $wZw'$ and $R'w'v'$ then there exists $v \in W$ such that $Rwv$ and $vZv'$
two models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ are bisimilar, notation $\mathcal{M} \leftrightarrow \mathcal{M}'$, if there exists a bisimulation $Z \subseteq W \times W'$

two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are bisimilar notation $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$ or $w \leftrightarrow w'$, if there exists a bisimulation $Z : \mathcal{M} \leftrightarrow \mathcal{M}'$ such that $(w, w') \in Z$

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non-bisimilarity: example

give a bisimulation relating 2 and 4