overview

more decidability

towards temporal logic

sequents: remarks

$A$ and $B$ in $A \Rightarrow B$ are multisets of formulas

we work up to associativity, commutativity, and idempotence of both $\Rightarrow$.

empty conjunction is true

indeed: adding true left does not help

empty disjunction is false

indeed: adding false right does not help
transformation of sequents: remark

\[ p_1, \ldots, p_n, \top \phi_1, \ldots, \top \phi_m \Rightarrow q_1, \ldots, q_k, \top \psi_1, \ldots, \top \psi_k \]

either we can decide validity because of the propositional part
if not, we go to a next world, leaving behind the propositional part
we then see if \( \phi_i \Rightarrow \psi_1, \ldots, \psi_k \) is valid for some \( i \in \{1, \ldots, m\} \)

transformation of sequents: termination

we can apply the transformation rules in any order

does the transformation always terminate?
intuitively yes, because the number of connectives decreases in every step
can the order in which the transformations are performed affect the result?
intuitively no, because the transformations are independent
(such intuitive arguments need to be completed and formalized)

decidability using semantic tableaux

due to Evert Beth (1908–1964)

method similar to analyze sequents
analyze the nature of possible models via the structure of the formula
a tableau is a finite tree of sequents
if all solid branches close this yields validity of initial sequent
if at least one branch does not close this yields a counterexample

tableaux: propositional part

\[ A, \neg \phi \bullet B \text{ gets successor } A \bullet \phi, B \]
\[ A \bullet \neg \phi, B \text{ gets successor } A, \phi \bullet B \]
\[ A, \phi \land \psi \bullet B \text{ gets successor } A, \phi, \psi \bullet B \]
\[ A \bullet \phi \land \psi, B \text{ gets successors } A \bullet \phi, B \text{ and } A, \psi \bullet B \]
\[ A, \phi \lor \psi \bullet B \text{ gets successors } A, \phi \bullet B \text{ and } A, \psi \bullet B \]
\[ A \bullet \phi \lor \psi, B \text{ gets successor } A \bullet \phi, \psi, B \]
\[ p_1, \ldots, p_n \bullet q_1, \ldots, q_m \text{ either closes or yields a countermodel} \]
if one branch does not close this yields a countermodel
extension to modal logic

\[ p_1, \ldots, p_n, \Diamond \phi_1, \ldots, \Diamond \phi_m \land q_1, \ldots, q_k, \Diamond \psi_1, \ldots, \psi_k \]

either closes,
or gives a valuation \( p_1, \ldots, p_n \) all true for this world, and gets \( m \) successors 
\( \phi_i \land \psi_1, \ldots, \psi_k \) for every \( i \in \{1, \ldots, m\} \)

if no branch closes this yields a countermodel

example

\[ \Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \]

overview
different approaches to temporal logic

more decidability
Pnueli

linear time logic
Branching time logic
Ben-Ari, Manna, Pnueli, Clarke and Emerson

more temporal logic
Kupferman, Venema

towards temporal logic
temporal model: definition

first approach to temporal logic

A frame $\mathcal{F} = (T, <)$ is a temporal frame if

1. $<$ is irreflexive: not $t < t$ for all $t$, and
2. $<$ is transitive: if $t < u$ and $u < v$ then $t < v$

so $(\mathbb{N}, <)$ is a temporal frame

A temporal model is a temporal frame with a valuation

new operators

We will consider new operators with a time-intuition

Next: $\otimes$

$\mathcal{M}, t \models \otimes \phi$ iff $\exists v : t < v \land (\neg \exists u : t < u < v) \land \mathcal{M}, v \models \phi$

Next is not definable in basic modal logic

Until: $U$

$\mathcal{M}, t \models U \psi$ iff $\exists v : t < v \land \mathcal{M}, v \models \psi \land \forall u : t < u < v \rightarrow \mathcal{M}, u \models \phi$

Until is not definable in basic modal logic

temporal model: example

Second approach to temporal logic

Formulas inductively defined by:

$p | \neg \phi | \phi \land \psi | \langle F \rangle \phi | \langle P \rangle \phi$

Truth and validity for temporal frames with modal clauses

$\mathcal{M}, t \models \langle F \rangle \phi$ iff $\exists u : t < u \land \mathcal{M}, u \models \phi$

$\mathcal{M}, t \models \langle P \rangle \phi$ iff $\exists u : u < t \land \mathcal{M}, u \models \phi$

basic temporal logic

$\mathcal{M} = (\mathbb{N}, <, V)$ with $V(q) = \{ n \mid n \geq 1000 \}$ and $V(r) = \{ 2n \mid n \in \mathbb{N} \}$

Then we have:

$0 \models \diamond \Box q$

$n \not\models \diamond \Box r$ for an arbitrary $n \in \mathbb{N}$

$\mathcal{M} \models \Box \diamond r$

$\mathcal{M} \models \Box \diamond \neg r$