overview

- temporal frames
- basic temporal logic
- towards multi-modal logic
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temporal model: definition

first approach to temporal logic: a special case of basic modal logic

a frame $\mathcal{F} = (T, <)$ is a temporal frame if

$<$ is irreflexive: not $t < t$ for all $t$, and
$<$ is transitive: if $t < u$ and $u < v$ then $t < v$

so $(\mathbb{N}, <)$ is a temporal frame

a temporal model is a temporal frame with a valuation
properties of temporal frames

we consider some properties of temporal frames

with the intuition of ‘time’ in mind
right-linearity

intuition: all future points are related

definition: \((x < y) \land (x < z) \rightarrow (y < z) \lor (y = z) \lor (z < y)\)

right-linearity is modally definable

by \((\Diamond p \land \Diamond q) \rightarrow \Diamond (p \land q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)\)

and also by \(\Box((p \land \Box p) \rightarrow q) \lor \Box((q \land \Box q) \rightarrow p)\)
right-branching

**intuition:** right-branching is not-right-linear

so some point has two unrelated points in the future

**definition:** there exist \( x, y, z \) such that \( x < y \) and \( x < z \) but

\[
\neg(y < z) \land y \neq z \land \neg(z < y)
\]

right-branching is not modally definable

why?
intuition: every point with a successor has an immediate successor

definition: \((x < y) \rightarrow \exists z: x < z \land \neg \exists u: (x < u) \land (u < z)\)

discreteness is modally definable in basic temporal logic (later)
intuition: between any two points is a third one

definition: $x < z \rightarrow \exists y (x < y \land y < z)$

density is modally definable: by $\Diamond p \rightarrow \Diamond \Diamond p$

neither dense nor discrete: $\{0, 1\} \cup [2, 3]$ with normal $<$ ordering

both dense and discrete: $\{0\}$ with empty relation
new operators

we will consider new(?) operators with a time-intuition

next: \( \boxdot \)

\[ \mathcal{M}, t \models \boxdot \phi \iff \exists v \, t < v \land (\neg \exists u : t < u < v) \land \mathcal{M}, v \models \phi \]

next is not definable in basic modal logic

until: \( U \)

\[ \mathcal{M}, t \models \phi U \psi \iff \exists v : t < v \land \mathcal{M}, v \models \psi \land \forall u : t < u < v \rightarrow \mathcal{M}, u \models \phi \]

until is not definable in basic modal logic
diamond and next and until

in a temporal frame

\(\Diamond p\) is equivalent to \(\top Up\)

\(\bigotimes p\) is equivalent to \(\perp Up\)
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basic temporal logic

second approach to temporal logic

formulas inductively defined by:

\[ p | \bot | \neg \phi | \phi \land \psi | \langle F \rangle \phi | \langle P \rangle \phi \]

truth and validity for basic temporal formulas in temporal frames

\[ M, t \models p \text{ iff } t \in V(p) \]

\[ M, t \models \bot \text{ never} \]

\[ M, t \models \neg \phi \text{ iff } M, t \not\models \phi \]

\[ M, t \models \phi \land \psi \text{ iff } M, t \models \phi \text{ and } M, t \models \psi \]

\[ M, t \models \langle F \rangle \phi \text{ iff for some } u \text{ with } t < u \text{ we have } M, u \models \phi \]

\[ M, t \models \langle P \rangle \phi \text{ iff for some } s \text{ with } s < t \text{ we have } M, s \models \phi \]
basic temporal logic: remark

we can define $[F]\phi$ as $\neg\langle F\rangle\neg\phi$ and $[P]\phi$ as $\neg\langle P\rangle\neg\phi$

alternatively we can define

$\mathcal{M}, t \models [F]\phi$ iff for all $u$ with $t < u$ we have $\mathcal{M}, u \models \phi$

$\mathcal{M}, t \models [P]\phi$ iff for all $s$ with $s < u$ we have $\mathcal{M}, s \models \phi$
is basic temporal logic an extension?

yes: past cannot be defined in basic modal logic

why?
defining properties in basic temporal logic

we can define discreteness by

\[ q \rightarrow \langle F \rangle [P] (q \lor \langle F \rangle q) \]

using next we can define discreteness by

\[ \langle F \rangle T \rightarrow \otimes T \]

we can define right-linearity by

\[ \langle P \rangle \langle F \rangle q \rightarrow \langle P \rangle q \lor q \lor \langle F \rangle q \]
adapt bisimulation to basic temporal logic

\[ \emptyset \neq Z \subseteq T \times T' \] is a temporal bisimulation between \( \mathcal{M} = (T, R, V) \) and \( \mathcal{M}' = (T', R', V') \) if for all \( (t, t') \in Z \) we have:

\[ t \in V(p) \text{ iff } t' \in V'(p) \] for all \( p \in \text{Var} \)

if \( Rtu \) then there is \( u' \) such that \( R't'u' \) and \( (u, u') \in Z \)

if \( R't'u' \) then there is \( u \) such that \( Rtu \) and \( (u, u') \in Z \)

if \( Rst \) then there is \( s' \) such that \( R's't' \) and \( (s, s') \in Z \)

if \( R's't' \) then there is \( s \) such that \( Rst \) and \( (s, s') \in Z \)
bisimulation and modal equivalence

$w$ and $w'$ are bisimilar using a temporal bisimulation

iff

in $w$ and in $w'$ the same basic temporal formulas are true

for finitely branching models

this is a special case of a more general result
until not definable in basic temporal logic
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multi-modal logic

we assume a set of labels $\mathcal{I}$

we consider for every label $i$ a relation $R_i$

we consider for every label $i$ a box-formula $\langle i \rangle \phi$

we already know various instances of multi-modal logic:

$\mathcal{I} = \emptyset$ gives propositional logic

$\mathcal{I} = \{0\}$ gives basic modal logic

$\mathcal{I} = \{F, P\}$ gives temporal logic

and we will see $\mathcal{I} = \text{Prog}(A)$ gives propositional dynamic logic