first approach

formulas from basic modal logic (BML),

models based on temporal frames

which are transitive and irreflexive

example: \((\{1, 2\} \cup 3, 4, <)\)

definition until:

\[M, t \models \phi U \psi \text{ iff } \exists v : t < v \land M, v \models \psi \land \forall u : t < u < v \rightarrow M, u \models \phi\]
second approach

basic temporal logic (BTL) formulas inductively defined by:

\[ p \mid \bot \mid \neg \phi \mid \phi \land \psi \mid \langle F \rangle \phi \mid \langle P \rangle \phi \]

truth and validity for basic temporal logic formulas in temporal frames

\[ \mathcal{M}, t \models p \text{ iff } t \in V(p) \]

\[ \mathcal{M}, t \models \bot \text{ never} \]

\[ \mathcal{M}, t \models \neg \phi \text{ iff } \mathcal{M}, t \not\models \phi \]

\[ \mathcal{M}, t \models \phi \land \psi \text{ iff } \mathcal{M}, t \models \phi \text{ and } \mathcal{M}, t \models \psi \]

\[ \mathcal{M}, t \models \langle F \rangle \phi \text{ iff for some } u \text{ with } t < u \text{ we have } \mathcal{M}, u \models \phi \]

\[ \mathcal{M}, t \models \langle P \rangle \phi \text{ iff for some } s \text{ with } s < t \text{ we have } \mathcal{M}, s \models \phi \]

alternatively: define \([F]\) using \(\langle F \rangle\) and similarly for \([P]\)
is basic temporal logic an extension?

yes: ‘past’ cannot be defined in basic modal logic

why?
we can define discreteness for linear temporal frames by

\[ q \rightarrow (F)[P](q \lor (F)q) \]

using next we can define discreteness by

\[ (F)\top \rightarrow \otimes\top \]

we can define right-linearity by

\[ (P)(F)q \rightarrow (P)q \lor q \lor (F)q \]
what to do with bisimulation?

the notion of bisimulation for basic modal logic is too weak for basic temporal logic

intuitively also the past should be taken into account
adapt bisimulation to basic temporal logic

$\emptyset \neq Z \subseteq T \times T'$ is a temporal bisimulation between $\mathcal{M} = (T, R, V)$ and $\mathcal{M'} = (T', R', V')$ if for all $(t, t') \in Z$ we have:

$t \in V(p) \text{ iff } t' \in V'(p) \text{ for all } p \in \text{Var}$

if $Rtu$ then there is $u'$ such that $R't'u'$ and $(u, u') \in Z$

if $R't'u'$ then there is $u$ such that $Rtu$ and $(u, u') \in Z$

if $Rst$ then there is $s'$ such that $R's't'$ and $(s, s') \in Z$

if $R's't'$ then there is $s$ such that $Rst$ and $(s, s') \in Z$
bisimulation and modal equivalence

\( w \) and \( w' \) are bisimilar using a temporal bisimulation

iff

in \( w \) and in \( w' \) the same basic temporal formulas are true

for finitely branching models

this is a special case of a more general result
until not definable in basic temporal logic

suppose until definable in basic temporal logic

let $\phi$ be a basic temporal formula defining until

in $a_1$ we have $pUq$, hence in $a_1$ we have $\phi$

$a_1$ is bisimilar with temporal bisimulation with $a'$

so $a' \models \phi$

contradiction with $a' \not\models pUq$

so until adds expressive power
property $P$ is modally definable if there exists a formula $\phi$ such that:

$\mathcal{F}$ has property $P$ iff $\mathcal{F} \models \phi$ \hspace{1cm} (for every frame $\mathcal{F}$)

how do we prove this for given $P$ and $\phi$ usually?

assume a frame $\mathcal{F}$ with property $P$, prove that $\mathcal{F} \models \phi$

assume a frame $\mathcal{F}$ without property $P$, prove that $\mathcal{F} \not\models \phi$

that is, find a valuation $V$ and a world $w$ in $\mathcal{F}$ such that $\mathcal{F}, V, w \not\models \phi$
remark: not modally definable frame properties

how do we prove that property \( P \) is not modally definable?

suppose (towards a contradiction) that \( P \) is modally definable by \( \phi \)

find ‘bisimilar’ frames \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) such that
\( \mathcal{F}_1 \) has property \( P \) but \( \mathcal{F}_2 \) does not have property \( P \)

we aim to prove towards a contradiction that \( \mathcal{F}_2 \models \phi \)

take a valuation \( V_2 \) for \( \mathcal{F}_2 \) arbitrary, and take an arbitrary state \( x_2 \) in \( \mathcal{F}_2 \)

transfer \( V_2 \) to a valuation \( V_1 \) for \( \mathcal{F}_1 \), and

take state \( x_1 \) in \( \mathcal{F}_1 \) such that
\( \mathcal{F}_1, V_1, w_1 \) and \( \mathcal{F}_2, V_2, w_2 \) are bisimilar

by assumption on \( \phi \) we have \( \mathcal{F}_1, V_1, w_1 \models \phi \)
by ‘bisimilar equivalent to modal equivalence’

we have \( \mathcal{F}_2, V_2, w_2 \models \phi \)
remark: operator definable

\[ \Delta \] a modal operator (modality) with \( n \) arguments

\( \Delta \) is definable if there is a formula \( \zeta(p_1, \ldots, p_n) \) such that

\[ \Delta(\phi_1, \ldots, \phi_n) \text{ modally equivalent to } \zeta(\phi_1, \ldots, \phi_n) \]

(for all formulas \( \phi_1, \ldots, \phi_n \))

how to show that an operator is not modally definable?

suppose it is definable by some formula

find two bisimilar worlds such that

in one the formula should hold and in the other it shouldn’t hold
overview

- temporal logic
- multi-modal logic
overview

- temporal logic
- multi-modal logic
multi-modal logic

we assume a set of labels $\mathcal{I}$

we consider for every label $i$ a box-formula $\langle i \rangle \phi$

so the formulas of multi-modal logic are, given $\mathcal{I}$, inductively defined by

$p | \bot | \neg \phi | \phi \land \psi | \langle i \rangle \phi | [i] \phi$

for $i \in \mathcal{I}$
instances of modal logic

we already know various instances of multi-modal logic:

$I = \emptyset$ gives propositional logic

$I = \{0\}$ gives basic modal logic

$I = \{F, P\}$ gives temporal logic

and we will see $I = \text{Prog}(A)$ gives propositional dynamic logic
especially interesting if there is some connection between the $R_i$

such as past is the inverse of future: $R_F = R_P^{-1}$
let $\mathcal{I}$ be a set of indices or labels

an $\mathcal{I}$-frame is a pair $(W, \{R_i \mid i \in \mathcal{I}\})$ with

$W \neq \emptyset$ a set of worlds or states

$R_i \subseteq W \times W$ for every $i \in \mathcal{I}$

an $\mathcal{I}$-model is a triple $(W, \{R_i \mid i \in \mathcal{I}\}, V)$ with

$(W, \{R_i \mid i \in \mathcal{I}\})$ an $\mathcal{I}$-frame and $V : Var \to \mathcal{P}(W)$ a valuation
multi-modal logic: truth and validity

let $\mathcal{M} = (W, \{R_i \mid i \in \mathcal{I}\}, V)$ be an $\mathcal{I}$-model

$\mathcal{M}, w \models \phi$ is defined by induction on the definition of formulas

important clauses:

$\mathcal{M}, w \models \langle \alpha \rangle \phi$ iff $\mathcal{M}, v \models \phi$ for some $v$ with $R_\alpha qv$

$\mathcal{M}, w \models [\alpha] \phi$ iff $\mathcal{M}, v \models \phi$ for all $v$ with $R_\alpha qv$

truth in a model, valid in a frame, universal validity are defined as before
example

use index set \( \{a, b, c\} \)

give a model with a world where the formula

\[ \langle a \rangle (\langle b \rangle [a] p \land [c] \neg \langle a \rangle p) \] is true