Exercise 1. \((3+4+4\ \text{points})\)

Consider the \(\lambda\)-term \(M = (\lambda uv. u u)((\lambda x. x) z)((\lambda y. y) z)\)

(a) Depict \(M\) as a tree.

(b) Reduce \(M\) to \(\beta\)-normal form; give all reduction steps explicitly.

(c) Compute the substitution \((\lambda x. x y)[y := x]\).

Exercise 2. \((4+4\ \text{points})\)

(a) Give and explain an example of a \(\lambda\)-term that has a weak head normal form (WHNF) but does not have a normal form (NF).

(b) Show that the rightmost-outermost strategy is not normalizing.

Typing rules of the simply typed \(\lambda\)-calculus:

\[
\frac{}{\Gamma, x : A \vdash x : A} \\
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B} \\
\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash (FM) : B}
\]

Exercise 3. \((4+4\ \text{points})\)

(a) Give a typing derivation for the term \(\lambda x. \lambda y. (\lambda u. x) y\).

(b) Give and explain an example of a term that is terminating but not typable.
Exercise 4. \((4+4+4 \text{ points})\)

(a) We use the definition of the Church numerals: \(c_n = \lambda s. \lambda z.s^n(z)\).

Give a \(\lambda\)-term \(S\) for successor and show that \(S \, c_1 \xrightarrow{\beta} c_2\).

(b) We represents pairs using \(\pi = \lambda l r . z \, l \, r\).

Give a \(\lambda\)-term for the first projection-function \(\pi_1\), and show that we have \(\pi_1 (\pi \, P \, Q) \xrightarrow{\beta} P\).

(c) Show that

\[ T = (\lambda x. \lambda y. y \,(x \, x \, y)) \,(\lambda x. \lambda y. y \,(x \, x \, y)) \]

is a fixed-point combinator.

Exercise 5. \((4+4+4 \text{ points})\)

(a) Consider the definition of \(\text{foldr}\):

\[
\begin{align*}
\text{foldr} \, f \, z \, [] &= z \\
\text{foldr} \, f \, z \, (x:xs) &= f \, x \, (\text{foldr} \, f \, z \, xs)
\end{align*}
\]

Use \(\text{foldr}\) to define a function \(\text{concat}\) that takes as input a list of lists, and gives back as output the concatenation of the input-lists. An example of the use of \(\text{concat}\):

\[
\text{Prelude> concat \[\[1,2,3\] , \[4,5,6\]\]} \\
[1,2,3,4,5,6]
\]

(b) Evaluate step by step \(\text{concat} \, \[\[1,2,3\] , \[4,5\]\} .

(c) Give a definition of the function

\[
\text{range} :: \text{Integer} \to \text{Integer} \to [\text{Integer}]
\]

that takes as input two integers \(n\) and \(m\) and gives back as output the list of integers from \(n\) to \(m\); in case \(n > m\) then the output is the empty list.

An example of the use of \(\text{range}\):

\[
\text{Prelude> range 3 11} \\
[3,4,5,6,7,8,9,10,11]
\]
Derivation Rules Equational Logic.

Let \((S, \Sigma), E\) be an algebraic specification; let \(X\) be the set of variables.

**axioms** The equations in \(E\) are called axioms and are derivable from \(E\).

**reflexivity** Every equation \(t = t\) is derivable from \(E\).

**symmetry** if \(t = u\) is derivable from \(E\), then \(u = t\) is derivable from \(E\).

**transitivity** if \(t_1 = t_2\) and \(t_2 = t_3\) are derivable from \(E\), then \(t_1 = t_3\) is derivable from \(E\).

**congruence** Let \(t_1, \ldots, t_n\) be terms of sort \(s_1, \ldots, s_n\), and \(f : s_1 \times \cdots \times s_n \to s\). If \(t_i = u_i\) is derivable from \(E\) for all \(i \in \{1, \ldots, n\}\), then \(f(t_1, \ldots, t_n) = f(u_1, \ldots, u_n)\) is derivable from \(E\).

**substitution** Let \(\theta : X \to \text{Ter}_\Sigma(X)\) be a substitution. If \(t = u\) is derivable from \(E\), then \(\bar{\theta}(t) = \bar{\theta}(u)\) is derivable from \(E\).

**Exercise 6.** (5+5+5+4 points)

Consider a signature with sort \(S\) and function symbols \(a : \to S\), \(b : \to S\), and \(f : S \to S\). The set of equations consists of one equation, as follows:

\[
f(f(x)) = f(f(b))
\]

(a) Give a formal derivation (with all steps explicit) of \(f(f(a)) = f(f(f(b)))\).

(b) Give an initial model for the equational specification, and explain informally why it is an initial model.

(c) Give and explain a model with a carrier set consisting of three elements. The model may contain confusion but should not contain junk.

(d) Add one or more equation(s) to the original specification such that your model in (c) becomes an initial model.
Exercise 7. \((6+5+5+4\text{ points})\)

Consider the algebraic specification \(\text{Abc} = ((S, \Sigma), E)\) with

\[
S: \text{sorts} \quad I, V
\]

\[
\Sigma: \text{constants} \quad a: I \rightarrow I \\
b: I \rightarrow I \\
c: I \rightarrow I \\
e: \rightarrow V
\]

\[
e: \text{function in: } I \# V \rightarrow V
\]

\[
E: \text{equations} \quad [1] \quad \text{in}(x, \text{in}(y, z)) = \text{in}(y, \text{in}(x, z)) \\
[2] \quad \text{in}(x, \text{in}(x, y)) = \text{in}(x, y)
\]

For \(\text{in}(x, y)\) we may also use the infix-notation \(x \cdot y\).

The following three \(\Sigma\)-algebras \(\mathfrak{A}, \mathfrak{L}, \text{ and } \mathfrak{M}\) are defined:

\[
\mathfrak{A}: \quad I = V = \{0, 1\} \quad a_A \equiv 0 \quad e_A \equiv 1 \quad \text{in}_A(x, y) \equiv x \cdot y
\]

\[
\mathfrak{L}: \quad I = V = \{-1, 0, 1\} \quad a_L \equiv 0 \quad e_L \equiv 1 \quad \text{in}_L(x, y) \equiv x \cdot y
\]

\[
\mathfrak{M}: \quad I = \{1, 2, 3\} \quad a_M \equiv 1 \quad e_M \equiv \emptyset \quad \text{in}_M(x, y) \equiv \{x\} \cup y
\]

\(V = \mathcal{P}\{1, 2, 3\}\)

(Note that \(\cdot\) is multiplication, and \(\mathcal{P}\) stands for power set.)

(a) Which of these algebras is a model / are models for \(\text{Abc}\)?

Explain your answers informally.

(b) Investigate whether one of more of these models is an initial model for \(\text{Abc}\). Indicate junk and confusion if appropriate.

(c) Describe the term model of \(\text{Abc}\) by giving a representative from every equivalence class.

(d) If possible, give a homomorphism from \(\mathfrak{A}\) to \(\mathfrak{L}\). Explain your answer.

\(\text{The exam grade is (the total number of points plus 10) divided by 10.}\)