



1. (a) Show that  $(B \rightarrow (A \rightarrow B) \rightarrow C) \rightarrow B \rightarrow C$  is a tautology of minimal prop1.

**Answer:** A proof that  $(B \rightarrow (A \rightarrow B) \rightarrow C) \rightarrow B \rightarrow C$  is a tautology:

$$\frac{\frac{\frac{[B \rightarrow (A \rightarrow B) \rightarrow C^x] \quad [B^y]}{(A \rightarrow B) \rightarrow C} E \rightarrow \quad \frac{[B^y]}{A \rightarrow B} I[z] \rightarrow}{\frac{C}{B \rightarrow C} I[y] \rightarrow} E \rightarrow}{\frac{C}{B \rightarrow C} I[y] \rightarrow} I[x] \rightarrow}{(B \rightarrow (A \rightarrow B) \rightarrow C) \rightarrow B \rightarrow C} I[x] \rightarrow$$

- (b) Give the type derivation in simply typed  $\lambda$ -calculus corresponding to the proof of 1.(a).

**Answer:** The corresponding type derivation in simply typed  $\lambda$ -calculus:

$$\frac{\frac{\frac{\Gamma_1 \vdash x : B \rightarrow (A \rightarrow B) \rightarrow C \quad \Gamma_1 \vdash y : B}{\Gamma_1 \vdash (xy) : (A \rightarrow B) \rightarrow C} \quad \frac{\Gamma_2 \vdash y : B}{\Gamma_1 \vdash \lambda z : A. y : A \rightarrow B}}{\Gamma_1 \vdash (xy) (\lambda z : A. y) : C}}{\Gamma_0 \vdash \lambda y : B. (xy) (\lambda z : A. y) : B \rightarrow C}}{\vdash \lambda x : (B \rightarrow (A \rightarrow B) \rightarrow C). \lambda y : B. (xy) (\lambda z : A. y) : (B \rightarrow (A \rightarrow B) \rightarrow C) \rightarrow B \rightarrow C}$$

We use:

$$\begin{aligned} \Gamma_0 &= \{x : B \rightarrow (A \rightarrow B) \rightarrow C\} \\ \Gamma_1 &= \{x : B \rightarrow (A \rightarrow B) \rightarrow C, y : B\} \\ \Gamma_2 &= \{x : B \rightarrow (A \rightarrow B) \rightarrow C, y : B, z : A\} \end{aligned}$$

2. (a) Show that  $(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$  is a tautology of minimal prop1.

**Answer:** A proof that  $(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$  is a tautology:

$$\frac{\frac{\frac{[A \rightarrow A \rightarrow B^x] \quad [A^y]}{A \rightarrow B} E \rightarrow \quad \frac{[A^y]}{A \rightarrow B} I[z] \rightarrow}{\frac{B}{A \rightarrow B} I[y] \rightarrow} E \rightarrow}{\frac{B}{A \rightarrow B} I[y] \rightarrow} I[x] \rightarrow}{(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B} I[x] \rightarrow$$

- (b) Give the type derivation in simply typed  $\lambda$ -calculus corresponding to the proof of 2.(a).

**Answer:** The corresponding typing derivation:

$$\frac{\frac{\frac{\Gamma_1 \vdash x : A \rightarrow A \rightarrow B \quad \Gamma_1 \vdash y : A}{\Gamma_1 \vdash (xy) : A \rightarrow B}}{\Gamma_1 \vdash ((xy)y) : B}}{\Gamma_0 \vdash \lambda y : A. ((xy)y) : A \rightarrow B}}{\vdash \lambda x : A \rightarrow A \rightarrow B. \lambda y : A. ((xy)y) : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B}$$

We use:

$$\begin{aligned} \Gamma_0 &= \{x : A \rightarrow A \rightarrow B\} \\ \Gamma_1 &= \{x : A \rightarrow A \rightarrow B, y : A\} \end{aligned}$$

3. (a) Show that the formula  $((A \rightarrow B \rightarrow A) \rightarrow B) \rightarrow B$  is a tautology of minimal prop1.

**Answer:** A proof that  $((A \rightarrow B \rightarrow A) \rightarrow B) \rightarrow B$  is a tautology:

$$\frac{\frac{[(A \rightarrow B \rightarrow A) \rightarrow B^x] \quad \frac{\frac{[A^y] \quad I[z] \rightarrow}{B \rightarrow A} \quad I[y] \rightarrow}{A \rightarrow B \rightarrow A} \quad E \rightarrow}{B} \quad I[x] \rightarrow}{((A \rightarrow B \rightarrow A) \rightarrow B) \rightarrow B}$$

- (b) Give the type derivation in simply typed  $\lambda$ -calculus corresponding to the proof of 3.(a).

**Answer:** The corresponding typing derivation:

$$\frac{\frac{\frac{\Gamma_2 \vdash y : A}{\Gamma_1 \vdash \lambda z : B. y : B \rightarrow A}}{\Gamma_0 \vdash x : (A \rightarrow B \rightarrow A) \rightarrow B} \quad \Gamma_0 \vdash \lambda y : A. \lambda z : B. y : A \rightarrow B \rightarrow A}{\Gamma_0 \vdash (x(\lambda y : A. \lambda z : B. y)) : B}}{\vdash \lambda x : ((A \rightarrow B \rightarrow A) \rightarrow B). (x(\lambda y : A. \lambda z : B. y)) : ((A \rightarrow B \rightarrow A) \rightarrow B) \rightarrow B}$$

We use:

$$\begin{aligned} \Gamma_0 &= \{x : (A \rightarrow B \rightarrow A) \rightarrow B\} \\ \Gamma_1 &= \{x : (A \rightarrow B \rightarrow A) \rightarrow B, y : A\} \\ \Gamma_2 &= \{x : (A \rightarrow B \rightarrow A) \rightarrow B, y : A, z : B\} \end{aligned}$$

4. (a) Show that  $((A \rightarrow B) \rightarrow C \rightarrow D) \rightarrow C \rightarrow B \rightarrow D$  is a tautology of minimal prop1.

**Answer:**

A proof that  $((A \rightarrow B) \rightarrow C \rightarrow D) \rightarrow C \rightarrow B \rightarrow D$  is a tautology:

$$\frac{[C^y] \quad \frac{[(A \rightarrow B) \rightarrow C \rightarrow D^x] \quad \frac{\frac{B}{A \rightarrow B} \quad I[u] \rightarrow}{C \rightarrow D} \quad E \rightarrow}{D} \quad E \rightarrow}{\frac{\frac{D}{B \rightarrow D} \quad I[z] \rightarrow}{C \rightarrow B \rightarrow D} \quad I[y] \rightarrow}{((A \rightarrow B) \rightarrow C \rightarrow D) \rightarrow C \rightarrow B \rightarrow D} \quad I[x] \rightarrow}$$

- (b) Give the type derivation in simply typed  $\lambda$ -calculus corresponding to the proof of 4.(a).

**Answer:**

The corresponding typing derivation:

$$\frac{\Gamma_2 \vdash y : C \quad \frac{\frac{\Gamma_2 \vdash x : (A \rightarrow B) \rightarrow C \rightarrow D \quad \frac{\Gamma_3 \vdash z : B}{\Gamma_2 \vdash \lambda u : A. z : A \rightarrow B}}{\Gamma_2 \vdash x(\lambda u : A. z) : C \rightarrow D}}{\Gamma_2 \vdash x(\lambda u : A. z)y : D}}{\Gamma_1 \vdash \lambda z : B. x(\lambda u : A. z)y : B \rightarrow D}}{\Gamma_0 \vdash \lambda y : C. \lambda z : B. x(\lambda u : A. z)y : C \rightarrow B \rightarrow D}}{\vdash \lambda x : (A \rightarrow B) \rightarrow C \rightarrow D. \lambda y : C. \lambda z : B. x(\lambda u : A. z)y : ((A \rightarrow B) \rightarrow C \rightarrow D) \rightarrow C \rightarrow B \rightarrow D}}$$

We use:

$$\begin{aligned} \Gamma_0 &= \{x : (A \rightarrow B) \rightarrow C \rightarrow D\} \\ \Gamma_1 &= \{x : (A \rightarrow B) \rightarrow C \rightarrow D, y : C\} \\ \Gamma_2 &= \{x : (A \rightarrow B) \rightarrow C \rightarrow D, y : C, z : B\} \\ \Gamma_3 &= \{x : (A \rightarrow B) \rightarrow C \rightarrow D, y : C, z : B, u : A\} \end{aligned}$$

5. (a) What is the definition of a detour in a natural deduction proof?

**Answer:** A detour is an introduction rule for a connective immediately followed by an elimination rule for the *same* connective.

- (b) Give a proof of  $A \rightarrow A \rightarrow A$  in first-order minimal propositional logic that contains a detour.

**Answer:** A proof of  $A \rightarrow A \rightarrow A$  in first-order minimal propositional logic that contains a detour:

$$\frac{\frac{\frac{[A^z]}{A \rightarrow A} I[z] \rightarrow \quad E \rightarrow^{A^x}}{A \rightarrow A} I[y] \rightarrow}{A \rightarrow A \rightarrow A} I[x] \rightarrow$$

- (c) Give the  $\lambda$ -term that corresponds to the proof of 5.(b).

Which part corresponds to the detour?

Give the normal form of the  $\lambda$ -term.

**Answer:**

The  $\lambda$ -term that corresponds to the proof of 5b:

$$M = \lambda x : A. \lambda y : A. ((\lambda z : A. z) x)$$

The part corresponding to the detour is the redex  $(\lambda z : A. z) x$

The normal form of the  $\lambda$ -term  $M$ :

$$\lambda x : A. \lambda y : A. x$$

6. Show that the following formulas are tautologies of prop1:

(a)  $((A \rightarrow \perp) \rightarrow A) \rightarrow A \rightarrow \neg\neg A \rightarrow A$ ,

**Answer:** A proof that  $((A \rightarrow \perp) \rightarrow A) \rightarrow A \rightarrow \neg\neg A \rightarrow A$  is a tautology:

$$\frac{\frac{\frac{[(A \rightarrow \perp) \rightarrow \perp^y] \quad [A \rightarrow \perp^z]}{E \rightarrow} \quad \frac{\perp}{A} E\perp}{(A \rightarrow \perp) \rightarrow A} I[z] \rightarrow}{\frac{A}{((A \rightarrow \perp) \rightarrow \perp) \rightarrow A} I[y] \rightarrow} E \rightarrow \frac{((A \rightarrow \perp) \rightarrow \perp) \rightarrow A}{((A \rightarrow \perp) \rightarrow A) \rightarrow A} I[x] \rightarrow$$

(An instance of Peirce's Law implies an instance of double negation.)

(b)  $\neg\neg((A \rightarrow B) \rightarrow A) \rightarrow A$ ,

**Answer:** A proof that  $\neg\neg((A \rightarrow B) \rightarrow A) \rightarrow A$  is a tautology;  
we abbreviate  $((A \rightarrow B) \rightarrow A) \rightarrow A \rightarrow \perp$  as  $C$ :

$$\frac{\frac{\frac{[C^x] \quad \frac{[A^z]}{((A \rightarrow B) \rightarrow A) \rightarrow A} I[u] \rightarrow}{E \rightarrow} \quad \frac{\perp}{B} E\perp}{A \rightarrow B} I[z] \rightarrow}{\frac{A}{((A \rightarrow B) \rightarrow A) \rightarrow A} I[y] \rightarrow} E \rightarrow \frac{((A \rightarrow B) \rightarrow A) \rightarrow A}{\neg\neg((A \rightarrow B) \rightarrow A) \rightarrow A} I[x] \rightarrow$$

(c)  $(A \vee \neg A) \rightarrow ((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B)) \rightarrow A$ ,

**Answer:** A proof that  $(A \vee \neg A) \rightarrow ((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B)) \rightarrow A$  is a tautology;

we abbreviate  $((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B))$  as  $D$

$$\frac{\frac{\frac{[C^y]}{\neg A \rightarrow \neg B} E\wedge \quad \frac{[\neg A^u]}{\neg B} E \rightarrow \quad \frac{[C^y]}{\neg A \rightarrow B} E\wedge \quad \frac{[\neg A^u]}{B} E \rightarrow}{\neg A \rightarrow A} E \rightarrow}{\frac{[A^z]}{A \rightarrow A} I[z] \rightarrow} E \vee \frac{\frac{\perp}{A} E\perp}{\neg A \rightarrow A} I[u] \rightarrow}{\frac{A}{(A \vee \neg A) \rightarrow C \rightarrow A} I[x] \rightarrow} E \vee$$

(d)  $\neg\neg((A \vee \neg A) \rightarrow ((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B)) \rightarrow A)$ .

**Answer:**

A proof that  $\neg\neg((A \vee \neg A) \rightarrow ((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B)) \rightarrow A)$  is a tautology; we abbreviate  $((A \vee \neg A) \rightarrow ((\neg A \rightarrow B) \wedge (\neg A \rightarrow \neg B)))$  by  $D$ :

$$\frac{\frac{[D \rightarrow \perp^v] \quad \mathcal{P}}{\perp} \quad E \rightarrow}{(D \rightarrow \perp) \rightarrow \perp} I[v] \rightarrow$$

with  $\mathcal{P}$  the proof of exercise 6c.

7. Replace in the following terms the ?'s by simple types, such that we obtain typable  $\lambda$ -terms.

- (a)  $\lambda x:?. \lambda y:?. x$
- (b)  $\lambda x:?. \lambda y:?.(x y)$
- (c)  $\lambda x:?. \lambda y:?. x y y$
- (d)  $\lambda x:?. \lambda y:?. x (x y)$
- (e)  $\lambda x:?. \lambda y:?. \lambda z:?. x (y z)$
- (f)  $\lambda x:?. \lambda y:?. \lambda z:?. y (\lambda u:?. x)$
- (g)  $\lambda x:?. \lambda y:?. \lambda z:?. (\lambda u:?. y) x z$
- (h)  $\lambda x:?. \lambda y:?. x ((\lambda z:?. y)y)$
- (i)  $\lambda x:?. \lambda y:?. \lambda z:?. z ((\lambda u:?. y) x)$

**Answer:**

- (a)  $\lambda x:A. \lambda y:B. x$
- (b)  $\lambda x : A \rightarrow B. \lambda y : A.(x y)$
- (c)  $\lambda x:A \rightarrow A \rightarrow B. \lambda y:A. x y y$
- (d)  $\lambda x:A \rightarrow A. \lambda y:A. x (x y)$
- (e)  $\lambda x:B \rightarrow C. \lambda y:A \rightarrow B. \lambda z:A. x (y z)$
- (f)  $\lambda x:B. \lambda y:(A \rightarrow B) \rightarrow C. \lambda z:D. y (\lambda u:A. x)$
- (g)  $\lambda x : A. \lambda y : B \rightarrow C. \lambda z : B. (\lambda u : A. y) x z$
- (h)  $\lambda x : A \rightarrow B. \lambda y : A. x ((\lambda z : A. y)y)$
- (i)  $\lambda x:A. \lambda y:B. \lambda z:B \rightarrow C. z ((\lambda u:A. y) x)$

8. Give four different closed normal forms of type  $(A \rightarrow A) \rightarrow A \rightarrow A$ .

**Answer:**

$\lambda x:A \rightarrow A. x$   
 $\lambda x:A \rightarrow A. \lambda y:A. y$   
 $\lambda x:A \rightarrow A. \lambda y:A. x y$   
 $\lambda x:A \rightarrow A. \lambda y:A. x (x y)$   
 $\lambda x:A \rightarrow A. \lambda y:A. x (x (x y))$