logical verification 2010-2011
exercises second-order propositional logic
and polymorphic lambda-calculus
with answers

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1. What is the type of the polymorphic identity?

   Answer: The polymorphic identity has type $\Pi a{:}\star\,.\,a \to a$.

2. Show how the polymorphic identity is used to get the identity on the type nat of natural numbers.

   Answer: The polymorphic identity:

   $$\lambda a : \star.\, \lambda x : a.\, x$$

   Instantiation of the polymorphic identity to the case of natural numbers:

   $$(\lambda a : \star.\, \lambda x : a.\, x)\, \mathsf{nat} : \mathsf{nat} \to \mathsf{nat}$$

   Note that $(\lambda a : \star.\, \lambda x : a.\, x)\, \mathsf{nat} \to_\beta \lambda x : \mathsf{nat}.\, x$.

3. Give the polymorphic version of the following function:
   $\lambda f{:}\mathsf{nat} \to \mathsf{bool} \to \mathsf{nat}.\, \lambda x{:}\mathsf{nat}.\, \lambda y{:}\mathsf{bool}.\, f\, x\, y$.

   (In the polymorphic variant neither nat nor bool occurs.)

   Answer: The polymorphic version of $\lambda f{:}\mathsf{nat} \to \mathsf{bool} \to \mathsf{nat}.\, \lambda x{:}\mathsf{nat}.\, \lambda y{:}\mathsf{bool}.\, f\, x\, y$:

   $$\lambda a : \star.\, \lambda b : \star.\, \lambda f : a \to b \to a.\, \lambda x : a.\, \lambda y : b.\, f\, x\, y$$

4. Explain why the following proof is not correct:

   $$\dfrac{\exists a.\, a \to b \qquad \dfrac{[a \to b^x]}{(a \to b) \to (a \to b)}\, I[x]\to}{a \to b}\, E\exists$$

   Answer: The proof does not follow the pattern of the $\exists$-elimination rule which is

   $$\dfrac{\exists x.\, A \qquad \forall x.\, (A \to B)}{B}\, E\exists$$

5. Show that $\forall a.\, ((\forall b.\, b) \to a)$ is a tautology.

   Answer: A proof that $\forall a.\, ((\forall b.\, b) \to a)$ is a tautology:

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^u]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[u] \to}{\forall a.\, ((\forall b.\, b) \to a)}\, I\forall$$

6. Give the $\lambda 2$-term corresponding to the formula $\forall a.\, ((\forall b.\, b) \to a)$.

   Answer: The $\lambda 2$-term corresponding to $\forall a.\, ((\forall b.\, b) \to a)$:

   $$\Pi a : \star.\, ((\Pi b : \star.\, b) \to a)$$

7. Give a $\lambda 2$-term that is an inhabitant of the answer to the previous exercise.

   Answer: An inhabitant of $\Pi a : \star.\, ((\Pi b : \star.\, b) \to a)$:

   $$\lambda a : \star.\, \lambda u : (\Pi b : \star.\, b).\, (u\, a)$$

8. Show that $(\forall c.\, ((a \to b \to c) \to c)) \to a$ is a tautology of second-order minimal propositional logic.

   Answer:

   not available yet

9. What is the impredicative definition of $\bot$ in second-order propositional logic?

   Answer: An impredicative definition of $\bot$:

   $$\forall a.\, a$$

10. (Following the previous exercise.) What is the corresponding term in $\lambda 2$?

    Answer: The corresponding term in $\lambda 2$:

    $$\Pi a : \star.\, a$$

11. Define the type $\mathsf{new\_or}$

    $$(\mathsf{new\_or}\, A\, B) = \Pi c{:}\, * \,.\, (A \to c) \to (B \to c) \to c$$

    Assume $\Gamma \vdash a : A$. Give an inhabitant of $(\mathsf{new\_or}\, A\, B)$.

    (NB: it is not asked to give the type derivation.)

    Answer:

    We assume $\Gamma \vdash a : A$. Then in the environment Γ the following term is an inhabitant of $(\mathsf{new\_or}\, A\, B)$:

    $$\lambda c : \star.\, \lambda u : A \to c.\, \lambda v : B \to c.\, (u\, a)$$

12. Assume $\mathsf{new\_or}$ as in the previous exercise, and in addition $\Gamma \vdash f : A \to D$, and $\Gamma \vdash g : B \to D$, and $\Gamma \vdash M : (\mathsf{new\_or}\, A\, B)$. Give an inhabitant of D.

    (NB: it is not asked to give the type derivation.)

    Answer:

    In the environment Γ, the following term is an inhabitant of D:

    $$M\, D\, f\, g$$

13. We define the booleans B and *true* (T) and *false* (F) as follows:

    $\mathsf{B} = \Pi a{:}*\,.\, a \to a \to a$
    $\mathsf{T} = \lambda a{:}*\,.\, \lambda x{:}a.\, \lambda y{:}a.\, x$
    $\mathsf{F} = \lambda a{:}*\,.\, \lambda x{:}a.\, \lambda y{:}a.\, y$

    Give a definition of negation in $\lambda 2$.

    Answer:

    A definition of not of type $\mathsf{B} \to \mathsf{B}$ (which is $\mathsf{B} \to \Pi a : \star.\, a \to a \to a$):

    $$\lambda b : \mathsf{B}.\, \lambda a : \star.\, \lambda x : a.\, \lambda y : a.\, b\, a\, y\, x$$

14. We assume $a : \star$. Give inhabitants in $\lambda 2$ of the following types:

    (a) $(\Pi b : \star.\, b) \to a$,
    (b) $a \to \Pi b : \star.\, (b \to a)$,
    (c) $a \to \Pi b : \star.\, ((a \to b) \to b)$.

    Answer:

    (a) An inhabitant of $(\Pi b : \star.\, b) \to a$:

    $$\lambda u : (\Pi b : \star. b).\, (u\, a)$$

    (b) An inhabitant of $a \to \Pi b : \star.\, (b \to a)$:

    $$\lambda x : a.\, \lambda b : \star.\, \lambda y : b.\, x$$

    (c) An inhabitant of $a \to \Pi b : \star.\, ((a \to b) \to b)$:

    $$\lambda x : a.\, \lambda b : \star.\, \lambda y : a \to b.\, (y\, x)$$