

Integrable Systems in n -dimensional Riemannian Geometry

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Abstract

In this paper we show that if one writes down the structure equations for the evolution of a curve embedded in an n -dimensional Riemannian manifold with constant curvature this leads to a symplectic, a Hamiltonian and an hereditary operator. This gives us a natural connection between finite dimensional geometry, infinite dimensional geometry and integrable systems. Moreover one finds a Lax pair in \mathfrak{o}_{n+1} with the vector modified Korteweg-De Vries equation (vmKDV)

$$\mathbf{u}_t = \mathbf{u}_{xxx} + \frac{3}{2} \|\mathbf{u}\|^2 \mathbf{u}_x$$

as integrability condition. We indicate that other integrable vector evolution equations can be found by using a different Ansatz on the form of the Lax pair. We obtain these results by using the *natural* or *parallel* frame and we show how this can be gauged by a generalized Hasimoto transformation to the (usual) *Frenêt* frame. If one chooses the curvature to be zero, as is usual in the context of integrable systems, then one loses information unless one works in the natural frame.

1 Introduction

The study of the relationship between finite-dimensional differential geometry and partial differential equations which later came to be known as integrable systems, started in the 19th century. Liouville found and solved the equation describing minimal surfaces in 3-dimensional Euclidean space [Lio53]. Bianchi solved the general Goursat problem for the sine-Gordon equation, which arises in the theory of pseudospherical surfaces [Bia53, Bia27].

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Much later Hasimoto [Has72] found the relation between the equations for curvature and torsion of vortex filament flow and the nonlinear Schrödinger equation, which led to many new developments, cf. [MW83, LP91, DS94, LP96, YS98, Cal00].

The similarity between the structure equations for connections in differential geometry and Lax pair equations for integrable systems has intrigued many researchers from both fields of mathematics. We refer the interested reader to the following books: [BE00, Gue97, H el01, H el02, RS02]. A good introductory review is [Pal97].

Recently, we showed in [MBSW02] that if a flow of curves in a 3-dimensional Riemannian manifold with constant curvature \varkappa follows an arc-length preserving geometric evolution, the evolution of its curvature and torsion is always a Hamiltonian flow with respect to the pencil $\mathcal{E} + \mathcal{D} + \varkappa\mathcal{C}$, where \mathcal{E} , \mathcal{D} and \mathcal{C} are compatible Hamiltonian structures. However, the close geometric relationship remains here: the triplet is obtained solely from the intrinsic geometry of curves on 3-dimensional Riemannian manifolds with constant curvature.

Once one has two compatible Hamiltonian operators, one can construct an hereditary operator, from which a hierarchy of integrable equations can be computed. They are all Hamiltonian with respect to two different Hamiltonian operators, that is, biHamiltonian as defined in [Mag78]. Thus Poisson geometry is very important in the study of integrable systems, cf. [Olv93, Dor93].

The goal of the present paper is to generalize this analysis to arbitrary dimension and see how much of the infinite dimensional geometric structure is still present in this case.

The Hasimoto transformation is a Miura transformation, which is induced by a gauge transformation from the Fren et frame (the obvious frame to choose from the traditional differential geometry point of view) to the parallel or natural frame, cf. [DS94]. On the basis of *loc.cit.*, Langer and Perline [LP00] advocated the use of the natural frame in the n -dimensional situation in the context of vortex-filament flow.

We show in this paper that the appreciation of the exact relationship between the underlying finite dimensional geometry and the infinite dimensional geometry has been complicated by the use of the Fren et frame. In fact, the operators that lead to biHamiltonian systems **naturally** come out of the computation of the structure equations. This fact is true in general, but only when one uses the parallel or natural frame does this conclusion come out automatically. This will be the first ingredient of our approach.

To relate our results to the classical situation in terms of the curvatures of the evolving curve in the Fren et frame we can of course say that this relation is obvious since there must exist a gauge transform between the two connections, defined by the Cartan matrices specifying our frame. Nevertheless, we give its explicit construction, and produce a generalized Hasimoto transformation in arbitrary dimension, with complete proof, as announced in [Wan03]. This insures that anything that can be formulated abstractly can also be checked by direct (though complicated) computations. Since the equations are derived from the computation of the curvature tensor which behaves very nicely under

transformations, we feel that these direct calculations are the very last thing to try. In this kind of problem the abstract point of view seems to be much more effective computationally. With hindsight one might say that Hasimoto was the first to exploit the natural frame to show the integrability of the equations for the curvature and torsion of a curve embedded in a three-dimensional Riemannian manifold.

Here we should mention the fact that the generalization of the Hasimoto also plays a role in [LP00], but it is in a different direction; the focus is on the complex structure and this is generalized to the $2n$ -dimensional situation.

A second ingredient in our approach is the fact that we assume the Riemannian manifold to have (nonzero) constant curvature. If one takes the curvature equal to zero and one uses the natural frame, then one can still recover the symplectic and cosymplectic operators which give rise to a biHamiltonian system. But if one uses another frame, then information gets lost and one is faced with the difficult task of recreating this information in order to get the necessary operators and Poisson geometry. In our opinion, this has been a major stumbling block in the analysis of the relation between high-dimensional geometry and integrability.

A third ingredient is the study of symmetric spaces. The relation between symmetric spaces and integrability was made explicit in the work of Fordy and others [AF87, FK83], There it is shown how to construct a Lax pair using the structure of the Lie algebra. This construction does not work in the semidirect product \mathfrak{euc}_n , but there we use the fact that we can map the elements in the n -dimensional euclidean algebra to the $n + 1$ -dimensional orthogonal algebra. This leads immediately to the construction of Lax pairs, combining the methods in [TT01] and [LP00].

Putting together these three elements, natural frame, constant curvature and extension to a symmetric semisimple Lie algebra, gives us complete control over all the integrability issues one might like to raise. We have tried to formulate things in such a way that it is clear what to do in other geometries. The theory, however, is not yet strong enough to guarantee success in other geometries, so this will remain an area of future research.

If we try to understand the success of the natural frame in this context, then it seems that the key is the natural identification of the Lie algebra $\mathfrak{euc}_n/\mathfrak{o}_n$ with the adjoint orbit of \mathfrak{o}_n^* . In the natural frame, this identification is simple to see and can be performed by putting a complex structure on \mathfrak{o}_{n+1} . To give a geometrical explanation of this, one may have to formulate everything in terms of Poisson reduction of the Kac-Moody bracket of $SO(n)$, but we make no attempt to do so here. The main point we will make in this paper is that the whole construction can be explicitly computed. Once this is done, one can then choose a more conceptual approach later on. We have indicated the general character of the approach where appropriate.

The paper is organized as follows. In section 2 we derive the structure equations, using Cartan's moving frame method, and find that the equation is of the type

$$\mathbf{u}_t = \Phi_1 \mathbf{h} + \varkappa \Phi_2 \mathbf{h}, \quad \Phi_2 = Id_{n-1}.$$

In section 4 we show that Φ_1 is an hereditary operator, and we draw the conclusion that if we had been computing in any other frame, resulting in the equation

$$\bar{\mathbf{u}}_t = \bar{\Phi}_1 \bar{\mathbf{h}} + \varkappa \bar{\Phi}_2 \bar{\mathbf{h}}, \quad (1)$$

then $\bar{\Phi}_1 \bar{\Phi}_2^{-1}$ is also hereditary.

We then proceed, by taking $\mathbf{h} = \mathbf{u}_1$, the first x -derivative of \mathbf{u} , to derive a vector mKDV equation. In [DS94] it was predicted (or derived by general considerations) that the equation should be a vmKDV equation, but there are two versions of vmKDV equation, cf. [SW01b], and the prediction does not say which version it should be. That question is now settled.

In section 3 we state explicitly the formula for the generalized Hasimoto transformation, which transforms the Frenét frame into the natural frame and in Appendix B we prove this.

We conclude in section 5 by constructing a Lax pair which has this equation as its integrability condition. This is done as follows. The geometric problem is characterized by two Lie algebras, \mathfrak{o}_n and \mathfrak{euc}_n . The usual Lie algebraic construction of Lax pairs starts with a symmetric Lie algebra $\mathfrak{h}^0 + \mathfrak{h}^1$, with

$$[\mathfrak{h}^0, \mathfrak{h}^0] \subset \mathfrak{h}^0, [\mathfrak{h}^0, \mathfrak{h}^1] \subset \mathfrak{h}^1, [\mathfrak{h}^1, \mathfrak{h}^1] \subset \mathfrak{h}^0. \quad (2)$$

If we think of \mathfrak{o}_n as \mathfrak{h}^0 , then we cannot take \mathfrak{euc}_n as $\mathfrak{h}^0 + \mathfrak{h}^1$, since \mathfrak{h}^1 is then abelian. Although it trivially obeys all the inclusions, this triviality kills all the actions we need to construct the Lax pair. But if we take $\mathfrak{o}_{n+1} = \mathfrak{h}^0 + \mathfrak{h}^1$, then things fit together nicely and we can simply copy the construction in [TT01]. Of course, we can identify \mathfrak{h}^1 with $\mathfrak{euc}_n/\mathfrak{o}_n$ as vectorspaces, and this should be the main consideration (apart from the requirements in (2)) in other geometries when one tries to find such a *symmetric extension* of the geometrically given \mathfrak{h}^0 .

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2 Moving Frame Method

Our approach, although restricted to the Riemannian case, is also suitable for arbitrary geometries. As a general reference to the formulation of geometry in terms of Lie algebras, Lie groups and connections we refer to [Sha97].

We consider a curve (denoted by \mathbf{e}_0 if we want to stress its relation to the other vectors in the moving frame, or γ , if we just want to concentrate on the curve itself), parametrized by arclength x and evolving geometrically in

time t , which is imbedded in a Riemannian manifold M . We choose a frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ as a basis for $T_{\mathbf{e}_0}M$, and a dual basis of one forms $\tau_1, \dots, \tau_n \in \bigwedge^1 M$, so that $\tau_i(\mathbf{e}_j) = \delta_i^j$. Choosing a frame is equivalent to choosing an element in $g \in G = Euc(n, \mathbb{R}) = O(n, \mathbb{R}) \times \mathbb{R}^n$. We write $g = \begin{pmatrix} 1 & \mathbf{e}_0 \\ 0 & \mathbf{e} \end{pmatrix}$. Let $\Gamma(TM)$ be the space of smooth sections, that is, maps $\sigma : M \rightarrow TM$ such that $\pi\sigma$ is the identity map on M , where π is the natural projection of the tangent space on its base M . We consider the \mathbf{e}_i as sections in TM , at least locally, that is, elements in $\Gamma(TM)$, varying \mathbf{e}_0 . We assume that there is a Lie algebra \mathfrak{g} and a subalgebra \mathfrak{h} such that $T_{\mathbf{e}_0}M \simeq \mathfrak{g}/\mathfrak{h}$ as vector spaces.

We now define a connection $d : \bigwedge^p M \otimes \Gamma(TM) \rightarrow \bigwedge^{p+1} M \otimes \Gamma(TM)$ as follows. Let $\omega \in \bigwedge^p M$. Then

$$d\omega \otimes \sigma = d\omega \otimes \sigma + (-1)^p \omega \otimes d\sigma.$$

Notice that we use d as a cochain map from $\bigwedge^p M$ to $\bigwedge^{p+1} M$, $p \in \mathbb{N}$ in the ordinary de Rham complex. We extend the connection to act on \mathbf{e}_0 as follows.

$$d\mathbf{e}_0 = \sum_i \tau_i \otimes \mathbf{e}_i, \quad \tau_i \in \bigwedge^1 M$$

with $\tau = (\tau_1, \dots, \tau_n) \in C^1(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ a differential 1-form. We now suppose that $d\mathbf{e}_i = \sum_j \omega_{ij} \otimes \mathbf{e}_j$, with $\omega_{ij} \in \bigwedge^1 M$, that is, $\omega \in C^1(\mathfrak{g}/\mathfrak{h}, \mathfrak{h})$. This connection naturally extends to a connection $d : \bigwedge^p M \otimes \Gamma(G) \rightarrow \bigwedge^{p+1} M \otimes \Gamma(G)$. We may write our basic structure equations as

$$dg = Ag = \begin{pmatrix} 0 & \tau \\ 0 & \omega \end{pmatrix} g \tag{3}$$

where A is the vector potential or Cartan matrix. We remark that $d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y])$. Differentiating both sides of (3), we have

$$\begin{aligned} d^2g &= \begin{pmatrix} 0 & d\tau \\ 0 & d\omega \end{pmatrix} g - \begin{pmatrix} 0 & \tau \\ 0 & \omega \end{pmatrix} \begin{pmatrix} 0 & d\mathbf{e}_0 \\ 0 & d\mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} 0 & d\tau \\ 0 & d\omega \end{pmatrix} g - \begin{pmatrix} 0 & \tau \\ 0 & \omega \end{pmatrix} \begin{pmatrix} 0 & \tau \\ 0 & \omega \end{pmatrix} g \\ &= \begin{pmatrix} 0 & d\tau - \tau \wedge \omega \\ 0 & d\omega - \omega \wedge \omega \end{pmatrix} g = Fg \end{aligned}$$

We draw the conclusion that

$$d^2 = F = \begin{pmatrix} 0 & T \\ 0 & \Omega \end{pmatrix} = \begin{pmatrix} 0 & d\tau - \tau \wedge \omega \\ 0 & d\omega - \omega \wedge \omega \end{pmatrix},$$

where $F \in C^2(\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$, of which the 2-form $\Omega \in C^2(\mathfrak{g}/\mathfrak{h}, \mathfrak{h})$ is called the *curvature*, the 2-form T is called *torsion* and $\omega \wedge \omega$ is often denoted by $\frac{1}{2}[\omega, \omega]$.

We are now in a position to compute a few things explicitly. Let $X = \mathbf{e}_{0*} \frac{\partial}{\partial x} = D_x$ and $Y = \mathbf{e}_{0*} \frac{\partial}{\partial t} = D_t$, that is, the induced vectorfields tangent to the imbedded curve of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$, respectively.

Then, using the fact that $[X, Y] = 0$, we obtain

$$\begin{aligned} F(X, Y) &= \\ &= \begin{pmatrix} 0 & \Upsilon(X, Y) \\ 0 & \Omega(X, Y) \end{pmatrix} = \begin{pmatrix} 0 & d\tau(X, Y) - \tau(X) \wedge \omega(Y) \\ 0 & d\omega(X, Y) - \omega(X) \wedge \omega(Y) \end{pmatrix} \\ &= \begin{pmatrix} 0 & X\tau(Y) - Y\tau(X) - \tau(X) \wedge \omega(Y) \\ 0 & X\omega(Y) - Y\omega(X) - \omega(X) \wedge \omega(Y) \end{pmatrix} \\ &= \begin{pmatrix} 0 & D_x\tau(Y) - D_t\tau(X) - \tau(X) \wedge \omega(Y) \\ 0 & D_x\omega(Y) - D_t\omega(X) - \omega(X) \wedge \omega(Y) \end{pmatrix} \end{aligned} \quad (4)$$

$$= [D_x + \begin{pmatrix} 0 & \tau(X) \\ 0 & \omega(X) \end{pmatrix}, D_t + \begin{pmatrix} 0 & \tau(Y) \\ 0 & \omega(Y) \end{pmatrix}] \quad (5)$$

3 Hasimoto transformation

In 1975, Bishop discovered the same transformation as Hasimoto when he studied the relations between two different frames to frame a curve in 3-dimensional Euclidean space, namely the Frenét frame and parallel (or natural) frame, cf. [Bis75]. More explicitly, let the orthonormal basis $\{T, N, B\}$ along the curve be the Frenét frame, that is,

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}_x = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

The matrix in this equation is called the Cartan matrix. Now we introduce the following new basis

$$\begin{pmatrix} T \\ N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad \theta = \int \tau dx. \quad (6)$$

Its frame equation is

$$\begin{pmatrix} T \\ N^1 \\ N^2 \end{pmatrix}_x = \begin{pmatrix} 0 & \kappa \cos \theta & \kappa \sin \theta \\ -\kappa \cos \theta & 0 & 0 \\ -\kappa \sin \theta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N^1 \\ N^2 \end{pmatrix}.$$

We call the basis $\{T, N^1, N^2\}$ the parallel frame. This geometric meaning of Hasimoto transformation was also pointed out in [DS94], where the authors studied the connection between the differential geometry and integrability.

In this section, we give explicit formula of such a transformation in arbitrary dimension n , which has the exact same geometric meaning as the Hasimoto transformation. Therefore, we call it generalized Hasimoto transformation. The

existence of such a transformation is clear from the geometric point of view, and was mentioned and implicitly used in several papers such as [Bis75] and [LP00].

First we give some notation. Denote the Cartan matrix of the Frenêt frame $\bar{\mathbf{e}}$ by $\bar{\omega}(D_x)$ ($\bar{\mathbf{e}}_x = \bar{\omega}(D_x)\bar{\mathbf{e}}$) and that of the parallel frame \mathbf{e} by $\omega(D_x)$ ($\mathbf{e}_x = \omega(D_x)\mathbf{e}$), i.e.,

$$\bar{\omega}(D_x) = \begin{pmatrix} 0 & \bar{u}^{(1)} & 0 & \cdots & 0 \\ -\bar{u}^{(1)} & 0 & \bar{u}^{(2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & -\bar{u}^{(n-3)} & 0 & \bar{u}^{(n-2)} & 0 \\ 0 & 0 & -\bar{u}^{(n-2)} & 0 & \bar{u}^{(n-1)} \\ 0 & 0 & 0 & -\bar{u}^{(n-1)} & 0 \end{pmatrix}$$

and, letting $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(n-1)})^\top$,

$$\omega(D_x) = \begin{pmatrix} 0 & \mathbf{u}^\top \\ -\mathbf{u} & 0 \end{pmatrix}.$$

The orthogonal matrix that keeps the first row and rotates the i -th and j -th row with the angle θ_{ij} is denoted by R_{ij} , where $2 \leq i < j \leq n$. For example, when $n = 3$, the orthogonal matrix

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & \sin \theta_{23} \\ 0 & -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix}$$

is the transformation between two frames, i.e., $\bar{\mathbf{e}} = R_{23}\mathbf{e}$, comparing to (6). Let us compute the conditions such that

$$\bar{\omega}(D_x)R_{23} - \frac{\partial R_{23}}{\partial x} = R_{23}\omega(D_x).$$

Then we have $\bar{u}^{(2)} = D_x \theta_{23}$, $u^{(1)} = \bar{u}^{(1)} \cos \theta_{23}$ and $u^{(2)} = \bar{u}^{(1)} \sin \theta_{23}$. This is the famous Hasimoto transformation, i.e., $\phi = u^{(1)} + iu^{(2)} = \bar{u}^{(1)} \exp(i \int \bar{u}^{(2)} dx)$.

Therefore, the generalized Hasimoto transformation, which by definition transforms the Frenêt frame into the natural frame, can be found by computing the orthogonal matrix, which gauges the Cartan matrix of the Frenêt frame into that of the parallel frame in n -dimensional space.

Theorem 1. *Let $n \geq 3$. The orthogonal matrix R gauges the standard Frenêt frame into the natural frame, that is,*

$$\bar{\omega}(D_x)R - D_x R = R\omega(D_x),$$

where $R = R_{n-1,n} \cdots R_{3,n} \cdots R_{34}R_{2,n} \cdots R_{24}R_{23}$. This leads to the components

of \mathbf{u} satisfying the Euler transformation

$$\begin{cases} u^{(1)} = \bar{u}^{(1)} \cos \theta_{23} \cos \theta_{24} \cdots \cos \theta_{2n}, \\ u^{(2)} = \bar{u}^{(1)} \sin \theta_{23} \cos \theta_{24} \cdots \cos \theta_{2n}, \\ u^{(3)} = \bar{u}^{(1)} \sin \theta_{24} \cdots \cos \theta_{2n}, \\ \dots\dots \\ \dots\dots \\ u^{(n-2)} = \bar{u}^{(1)} \sin \theta_{2,n-1} \cos \theta_{2n}, \\ u^{(n-1)} = \bar{u}^{(1)} \sin \theta_{2n}, \end{cases}$$

and the curvatures in the standard Frenét frame satisfying

$$\bar{u}^{(i)} = \prod_{j=i+2}^n \frac{\cos \theta_{i,j}}{\cos \theta_{i+1,j}} A_i, \quad i = 2, 3, \dots, n-1, \quad (7)$$

where the A_i are generated by the recursive relation

$$A_i = D_x \theta_{i,i+1} + \sin \theta_{i-1,i+1} A_{i-1}, \quad A_1 = 0, \quad 2 \leq i \leq n-1. \quad (8)$$

There are $\frac{1}{2}(n-2)(n-3)$ constraints on the rotating angles:

$$D_x \theta_{ij} = \prod_{l=i+2}^j \frac{\cos \theta_{il}}{\cos \theta_{i+1,l}} \sin \theta_{i+1,j} A_i - \prod_{l=i+1}^{j-1} \frac{\cos \theta_{i-1,l}}{\cos \theta_{i,l}} \sin \theta_{i-1,j} A_{i-1}, \quad (9)$$

where $4 \leq j \leq n$, $2 \leq i \leq j-2$.

Proof. It is obvious that $R = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$, where T is an orthogonal $(n-1) \times (n-1)$ -matrix and the first row of T equals is $\frac{\mathbf{u}^\top}{\bar{u}^{(1)}}$, that is,

$$(\cos \theta_{23} \cos \theta_{24} \cdots \cos \theta_{2n}, \sin \theta_{23} \cos \theta_{24} \cdots \cos \theta_{2n}, \dots, \sin \theta_{2n}).$$

This implies that $\bar{\mathbf{v}}^\top T = \mathbf{u}^\top$, where $\bar{\mathbf{v}}^\top = (\bar{u}^{(1)}, 0, \dots, 0)$. Since $TT^\top = I$, we have $T\mathbf{u} = \bar{\mathbf{v}}$. Rewrite $\bar{\omega}(D_x) = \begin{pmatrix} 0 & \bar{\mathbf{v}}^\top \\ -\bar{\mathbf{v}} & \underline{\omega} \end{pmatrix}$, and compute

$$\begin{aligned} & \bar{\omega}(D_x)R - D_x R - R\omega(D_x) \\ &= \begin{pmatrix} 0 & \bar{\mathbf{v}}^\top \\ -\bar{\mathbf{v}} & \underline{\omega} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & D_x T \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 0 & \mathbf{u}^\top \\ -\mathbf{u} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{u}^\top \\ -\bar{\mathbf{v}} & \underline{\omega} T \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & D_x T \end{pmatrix} - \begin{pmatrix} 0 & \mathbf{u}^\top \\ -\bar{\mathbf{v}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \underline{\omega} T - D_x T \end{pmatrix} \end{aligned}$$

Therefore, to prove the statement, we only need to solve $\underline{\omega} T = D_x T$, that is, the matrix T gauges $\underline{\omega}$ into zero, which is proved in Appendix B. \square

4 Hereditary operator in Riemannian Geometry

In this section, we present a hereditary operator that arises in a natural way from the geometric arelength-preserving evolution of curves in n -dimensional Riemannian manifold with constant curvature. The operator is the product of a cosymplectic (Hamiltonian) operator and a symplectic operator.

Theorem 2. *Let $\gamma(x, t)$ be a family of curves on \mathbf{M} satisfying a geometric evolution system of equations of the form*

$$\gamma_t = \sum_{l=1}^n h^{(l)} \mathbf{e}_l \quad (10)$$

where $\{\mathbf{e}_l, l = 1, \dots, n\}$ is the natural frame of γ , and where h_l are arbitrary smooth functions of the curvatures $u^{(i)}, i = 1, \dots, n-1$ and their derivatives with respect to x .

Assume that x is the arc-length parameter and that evolution (10) is arc-length preserving.

Then, the curvatures $\mathbf{u} = (u^{(1)}, \dots, u^{(n-1)})^\top$ satisfy the evolution

$$\mathbf{u}_t = \mathfrak{R}\mathbf{h} - \varkappa\mathbf{h}, \quad \mathbf{h} = (h^{(2)}, \dots, h^{(n)})^\top,$$

where the operator \mathfrak{R} is hereditary and defined as follows:

$$\mathfrak{R} = D_x^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \mathbf{u}_1 D_x^{-1} \langle \mathbf{u}, \cdot \rangle - \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} \mathbf{u}_1, \cdot \rangle,$$

and where the J_{ij} are anti-symmetric matrices with nonzero entry of (i, j) being 1 if $i < j$, that is, $(J_{ij})_{kl} = \delta_k^i \delta_j^l - \delta_l^i \delta_k^j$. Moreover, \mathfrak{R} can be written as $\mathfrak{H}\mathfrak{J}$, where \mathfrak{J} is a symplectic operator defined by $\mathfrak{J} = D_x + \mathbf{u} D_x^{-1} \mathbf{u}^\top$, and \mathfrak{H} is a cosymplectic (or Hamiltonian) operator, defined by

$$\mathfrak{H} = D_x + \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} (J_{ij} \mathbf{u})^\top.$$

Proof. In the Riemannian case, \mathfrak{h} is \mathfrak{o}_n and \mathfrak{g} is \mathfrak{euc}_n . By fixing the frame, we know the value of $\omega(D_x)$ and $\tau(D_x)$. We assumed our frame to be the natural (or parallel) frame, see [Bis75]. So we have $\mathbf{e}_1 = X$. Due to $d\mathbf{e}_0 = \tau \otimes \mathbf{e}$, we know

$$\tau(D_x) = (1, 0, \dots, 0).$$

The Cartan matrix of the natural frame is by definition

$$\omega(D_x) = \begin{pmatrix} 0 & u^{(1)} & u^{(2)} & \dots & u^{(n-1)} \\ -u^{(1)} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -u^{(n-1)} & 0 & 0 & \dots & 0 \end{pmatrix},$$

that is, $\omega_{ij}(D_x) = \delta_1^i u^{(j-1)} - \delta_1^j u^{(i-1)}$.

Now we use the moving frame method in section 2 to derive the evolution equation of the curvatures of a smooth curve $\gamma(x, t)$ in n -dimensional Riemannian manifold satisfying a geometric evolution of the form

$$\gamma_t = \sum_{i=1}^n h^{(i)} \mathbf{e}_i, \quad (11)$$

where $\{\mathbf{e}_i, i = 1 \cdots n\}$ is the natural frame and $h^{(i)}$ are arbitrary smooth functions of the curvatures $u^{(i)}, i = 1 \cdots n - 1$, and their derivatives with respect to the arclength parameter x .

The given curve (11) implies $h^{(i)} = \tau_i(D_t)$. So the equation (4) reads

$$\mathbb{T}_i(D_x, D_t) = D_x h^{(i)} - \omega_{1i}(D_t) + u^{(i-1)} h^{(1)} - \delta_1^i \sum_{j=2}^n h^{(j)} u^{(j-1)}$$

and we find

$$\omega_{1i}(D_t) = D_x h^{(i)} + u^{(i-1)} h^{(1)} - \delta_1^i \sum_{j=2}^n h^{(j)} u^{(j-1)} - \sum_{j=2}^n \mathbb{T}_i(D_x, \mathbf{e}_j) h^{(j)}. \quad (12)$$

We redefine ω by

$$\tilde{\omega}_{1i} = \omega_{1i} + \mathbb{T}_i(D_x, \cdot).$$

This does not influence our previous calculations since $\omega(D_x) = \tilde{\omega}(D_x)$, and reflects the fact that we can define the torsionless connection, i.e., Riemannian connection. It is customary to call the geometry Riemannian if the torsion is zero. We now write ω_{1i} for $\tilde{\omega}_{1i}$ to avoid the complication of the notation and rewrite (12) as

$$\omega_{1i}(D_t) = D_x h^{(i)} + u^{(i-1)} h^{(1)} - \delta_1^i \sum_{j=2}^n h^{(j)} u^{(j-1)}. \quad (13)$$

We need $\omega_{11} = 0$ for the consistence, i.e.,

$$0 = D_x h^{(1)} - \sum_{j=2}^n h^{(j)} u^{(j-1)}.$$

Geometrically it means that the evolution is arc-length preserving. By eliminating $h^{(1)}$ in (13), we obtain, taking the integration constants equal to zero,

$$\omega_{1i}(D_t) = D_x h^{(i)} + u^{(i-1)} D_x^{-1} \sum_{j=2}^n h^{(j)} u^{(j-1)} = \mathfrak{J} \mathbf{h}, \quad 1 < i \leq n,$$

with $\mathbf{h} = (h_2, \dots, h_n)$. This defines a symplectic operator $\mathfrak{J} = D_x + \mathbf{u}D_x^{-1}\mathbf{u}^\top$, cf. [Wan03], as will be proved in Proposition 1 in Appendix A. We now do the same for \mathfrak{o}_n -component in (4). Assuming $j > i$ (ω is antisymmetric), we have

$$\begin{aligned} \Omega_{ij}(D_x, D_t) &= D_x \omega_{ij}(D_t) - D_t(\delta_1^i u^{(j-1)} - \delta_1^j u^{(i-1)}) \\ &\quad - \sum_{l=1}^n (\delta_1^i u^{(l-1)} - \delta_1^l u^{(i-1)}) \omega_{lj}(D_t) + \sum_{l=1}^n \omega_{il}(D_t) (\delta_1^l u^{(j-1)} - \delta_1^j u^{(l-1)}). \end{aligned}$$

This leads to

$$D_t u^{(j-1)} = D_x \omega_{1j}(D_t) - \sum_{l=2}^n u^{(l-1)} \omega_{lj}(D_t) - \Omega_{1j}(D_x, D_t), \quad j = 2, \dots, n$$

when $i = 1$, and when $i > 1$,

$$D_x \omega_{ij}(D_t) = \Omega_{ij}(D_x, D_t) - u^{(i-1)} \omega_{1j}(D_t) + u^{(j-1)} \omega_{1i}(D_t).$$

We combine these to

$$\begin{aligned} D_t u^{(j-1)} &= -\Omega_{1j}(D_x, D_t) + D_x \omega_{1j}(D_t) \\ &\quad - \sum_{l=2}^n u^{(l-1)} D_x^{-1} (-u^{(l-1)} \omega_{1j}(D_t) + u^{(j-1)} \omega_{1l}(D_t)) \quad j = 2, \dots, n \end{aligned}$$

Assume that the curvature of Riemannian manifold is constant, i.e., the only nonzero entries in $\Omega(\mathbf{e}_i, \mathbf{e}_j)$ are

$$\Omega_{ij}(\mathbf{e}_i, \mathbf{e}_j) = -\Omega_{ji}(\mathbf{e}_i, \mathbf{e}_j) = \varkappa, \quad j > i.$$

Then $\Omega_{1j}(D_x, D_t) = h_j$. The above formula defines in the case of constant curvature a cosymplectic (or Hamiltonian) operator \mathfrak{H} (cf. [Wan03]) given by

$$\mathfrak{H} = D_x + \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} (J_{ij} \mathbf{u})^\top,$$

where the J_{ij} , given by $(J_{ij})_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$, are the generators of \mathfrak{o}_n and such that $D_t \mathbf{u} = \mathfrak{H} \omega_1(D_t)$. The fact that the operator is indeed Hamiltonian will be given in Proposition 2 in Appendix A. The operator is weakly non-local, as defined in [MN01].

It corresponds to what is called the *second Poisson operator* P_u , defined by

$$P_u = D_x + \pi_1 ad(u) + ad(u) D_x^{-1} \pi_0 ad(u),$$

in [TT01], but looks simpler by the choice of frame.

We write the evolution equation for the curvatures $u^{(j)}$ as

$$\mathbf{u}_t = \mathfrak{R} \mathbf{h} - \varkappa \mathbf{h},$$

where the operator $\mathfrak{R} = \mathfrak{H} \mathfrak{J}$ is explicitly given as

$$\mathfrak{R} = D_x^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \mathbf{u}_1 D_x^{-1} \langle \mathbf{u}, \cdot \rangle - \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} \mathbf{u}_1, \cdot \rangle. \quad (14)$$

We prove that \mathfrak{R} is hereditary in Proposition 3 in Appendix A. \square

Remark 1. *The recursion operator can be used to generate infinitely many compatible symplectic and cosymplectic weakly non-local operators starting with \mathfrak{J} and \mathfrak{H} . That the result is indeed weakly non-local follows from the techniques in [SW01a, MN01].*

Remark 2. *A special case arises for $n = 3$: one has*

$$\mathfrak{R} = -(J_{12}(D_x + \mathbf{u}D_x^{-1}\mathbf{u}))^2 = -R_{NLS}^2,$$

where R_{NLS} is the recursion operator of the Nonlinear Schrödinger (NLS) equation.

Let us now take $h^{(j)} = u_1^{(j-1)}$ in Theorem 2. Then we obtain

$$\mathbf{u}_t = \mathbf{u}_3 + \frac{3}{2}\langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u}_1 - \varkappa \mathbf{u}_1 \text{ (vmKDV)},$$

which is one of two versions of vector mKDV equations appearing in [SW01b].

Corollary 1. *The hereditary operator defined in (14) is a recursion operator of the vector mKDV equation $\mathbf{u}_t = \mathbf{u}_3 + \frac{3}{2}\langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u}_1$.*

Although we give the explicit formula of the generalized Hasimoto transformation in section 3, it is difficult to use it to compute the formula (1). It is easier to obtain it directly using the moving frame method in section 2.

Theorem 3. *Let us assume that in the Frenét frame, the curvatures of a geometric curve $\gamma_t = \sum_{l=1}^n h^{(l)} \bar{\mathbf{e}}_l$ under arc-length preserving satisfy*

$$\bar{\mathbf{u}}_t = \bar{\Phi}_1 \bar{\mathbf{h}} - \varkappa \bar{\Phi}_2 \bar{\mathbf{h}}, \quad \bar{\mathbf{h}} = (h^{(2)}, \dots, h^{(n)})^\top. \quad (15)$$

There exists an operator \mathcal{Q} such that $\bar{\Phi}_1 \mathcal{Q}$ and $\bar{\Phi}_2 \mathcal{Q}$ are compatible Hamiltonian operators and $\mathcal{Q} = T \mathfrak{H} T^\top \bar{\Phi}_2^$, where \mathfrak{H} is defined in Theorem 2 and T is defined in the proof of Theorem 1.*

Proof. First we notice that $\bar{\mathbf{h}} = T \mathbf{h}$ in Riemannian manifold. The Hasimoto transformation is also a Miura transformation between \mathbf{u}_t and $\bar{\mathbf{u}}_t$, that is,

$$\mathbf{u}_t = D_{\mathbf{u}} \bar{\mathbf{u}}_t = (\mathfrak{R} - \varkappa) \mathbf{h} = D_{\mathbf{u}} (\bar{\Phi}_1 - \varkappa \bar{\Phi}_2) \bar{\mathbf{h}}.$$

Therefore, $D_{\mathbf{u}}^{-1} = \bar{\Phi}_2 T$ and $\mathfrak{R} = D_{\mathbf{u}} \bar{\Phi}_1 T$. We know that \mathfrak{H} and $\mathfrak{R} \mathfrak{H}$ are compatible Hamiltonian operators. Thus

$$D_{\mathbf{u}}^{-1} \mathfrak{H} D_{\mathbf{u}}^{-1,*} = \bar{\Phi}_2 T \mathfrak{H} T^\top \bar{\Phi}_2^* \quad \text{and} \quad D_{\mathbf{u}}^{-1} \mathfrak{R} \mathfrak{H} D_{\mathbf{u}}^{-1,*} = \bar{\Phi}_1 T \mathfrak{H} T^\top \bar{\Phi}_2^*$$

are also compatible. By now, we prove the statement. \square

The operator \mathcal{Q} can be computed explicitly and it is independent on the angles in the Hasimoto transformation.

5 A Lax pair of vector mKDV

For the Lie algebra background to find Lax pairs, see [FK83, For90]. The computation of the Lax pair is known in the literature. In the case of \mathfrak{o}_n , two derivations, both using different $\mathbb{Z}/2$ -gradings, are known to us: [TT01, LP00]. Both derivations at some point use some not so obvious steps and we noted that the steps that were obvious in one were not obvious in the other. So it seemed like a natural idea to use both $\mathbb{Z}/2$ -gradings and do the derivation in a completely obvious way.

We now identify \mathbf{e}_1 with an element in \mathfrak{o}_{n+1} which we call L^{10} using the identification

$$\mathbf{euc}_n \ni \begin{pmatrix} 0 & v^\top \\ 0 & \omega \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} 0 & v^\top \\ -v & \omega \end{pmatrix} \in \mathfrak{o}_{n+1}$$

Let $L = \phi_* A(D_x)$, where A is defined in equation (3) and let λ be the norm of the tangent vector to the curve, that is, this vector equals $\lambda \mathbf{e}_1$. Then

$$\begin{aligned} L &= L^{01} + \lambda L^{10} = \\ &= \begin{pmatrix} 0 & 0, & 0, & \cdots, & 0 \\ 0 & 0, & u^{(1)}, & \cdots, & u^{(n-1)} \\ 0 & -u^{(1)}, & 0, & \cdots, & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -u^{(n-1)}, & 0, & \cdots, & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1, & 0, & \cdots, & 0 \\ -1 & 0, & 0, & \cdots, & 0 \\ 0 & 0, & 0, & \cdots, & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0, & 0, & \cdots, & 0 \end{pmatrix}. \end{aligned}$$

In order to get some control over our calculations, we introduce two $\mathbb{Z}/2$ -gradings. We do this by partitioning the $n+1$ -dimensional matrix into $(1, n) \times (1, n)$ -blocks, and into $(2, n-1) \times (2, n-1)$ -blocks. We then grade the diagonal blocks with 0 and the off-diagonal blocks with 1. This way we can write \mathfrak{o}_{n+1} as

$$\mathfrak{o}_{n+1} = \mathfrak{g}^{00} + \mathfrak{g}^{01} + \mathfrak{g}^{10} + \mathfrak{g}^{11}.$$

We see that $L^{01} \in \mathfrak{g}^{01}$ and $L^{10} \in \mathfrak{g}^{10}$, which explains the notation. Observe that the first grading is consistent with the parity of the power of λ , the second is not. So one does expect the computation to be homogeneous in the first grading and inhomogeneous with respect to the second. The second grading codes some useful facts. If $X \in \ker ad(L^{10})$, then $X \in \mathfrak{g}^0$ and if $X \in \text{im } ad(L^{10})$, then $X \in \mathfrak{g}^{-1}$. If $X \in \mathfrak{g}^{10}$, then $X \in \langle L^{10} \rangle_{\mathbb{R}}$. Furthermore, $ad(L^{10})$ gives us a complex structure on $\mathfrak{g}^{01} + \mathfrak{g}^{10}$. We now let $A(D_t) = M = M_3 + \lambda M_2 + \lambda^2 M_1 + \lambda^3 M_0$. The reader may want to experiment with lower order expansions of M .

From the curvature equation (5)

$$F(D_x, D_t) = [D_x + L, D_t + M]$$

it follows that the λ^4 term vanishes, that is: $ad(L^{10})M_0 = 0$. We simply choose $M_0 = -L^{10}$. The next order λ^3 gives us

$$ad(L^{10})M_1 = -D_x M_0 - [L^{01}, M_0].$$

or $M_1 + L^{01} \in \ker ad(L^{10})$. We take $M_1 = -L^{01}$.

On the λ^2 level we find

$$ad(L^{10})M_2 = -D_x M_1 - [L^{01}, M_1] = -D_x L^{01}.$$

We can solve this by letting $M_2 = -D_x ad(L^{10})L^{01} + \beta L^{10}$, with β some arbitrary function. We find on the λ level

$$ad(L^{10})M_3 = D_x^2 ad(L^{10})L^{01} - \beta_x L^{10} + [L^{01}, D_x ad(L^{10})L^{01}] - \beta[L^{01}, L^{10}]$$

or

$$ad(L^{10})M_3 = ad(L^{10})(D_x^2 L^{01} + \beta L^{01}) - \beta_x L^{10} - ad(L^{01})ad(D_x L^{01})L^{10}$$

Looking at the grading, we see that $ad(L^{01})ad(D_x L^{01})L^{10} = \kappa(L^{01}, D_x L^{01})L^{10}$, where κ is a bilinear form with values in \mathbb{R} . We find that on the span of L^{10} and L^{01} the operator $ad(L^{01})ad(D_x L^{01})$ behaves like

$$\langle \mathbf{u}, \mathbf{u} \rangle D_x - \langle \mathbf{u}, \mathbf{u}_1 \rangle.$$

In other words,

$$ad(L^{10})M_3 = ad(L^{10})(D_x^2 L^{01} + \beta L^{01}) - \beta_x L^{10} + \langle \mathbf{u}, \mathbf{u}_1 \rangle L^{10}$$

So we take $\beta = \nu + \frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle$, $\nu \in \mathbb{R}$. It follows that $M_3 = D_x^2 L^{01} + (\nu + \frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle)L^{01} + X^{00}$. The last equation (and this gives us the obstruction to solving the structure equations) reads

$$F(D_x, D_t) = D_x M_3 - D_t L^{01} + [L^{01}, M_3].$$

This implies

$$\begin{aligned} D_t L^{01} &= D_x^3 L^{01} + \langle \mathbf{u}, \mathbf{u}_1 \rangle L^{01} + (\nu + \frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle) D_x L^{01} + D_x X^{00} \\ &\quad + D_x [L^{01}, D_x L^{01}] + [L^{01}, X^{00}] - F(D_x, D_t) \end{aligned}$$

We take $X^{00} = -[L^{01}, D_x L^{01}]$ and obtain

$$\begin{aligned} D_t L^{01} &= D_x^3 L^{01} + \langle \mathbf{u}, \mathbf{u}_1 \rangle L^{01} + (\nu + \frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle) D_x L^{01} \\ &\quad + ad(L^{01})ad(D_x L^{01})L^{01} - F(D_x, D_t) \\ &= D_x^3 L^{01} + \langle \mathbf{u}, \mathbf{u}_1 \rangle L^{01} + (\nu + \frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle) D_x L^{01} \\ &\quad + (\langle \mathbf{u}, \mathbf{u} \rangle D_x - \langle \mathbf{u}, \mathbf{u}_1 \rangle) L^{01} - F(D_x, D_t) \\ &= D_x^3 L^{01} + (\nu + \frac{3}{2}\langle \mathbf{u}, \mathbf{u} \rangle) D_x L^{01} - F(D_x, D_t) \end{aligned}$$

Further computation shows that the Killing form of L^{01} is given by

$$K(L^{01}, L^{01}) = -2(n-2)\langle \mathbf{u}, \mathbf{u} \rangle,$$

and we can write, at least for $n > 2$,

$$D_t L^{01} = D_x^3 L^{01} + (\nu - \frac{3}{4(n-2)}K(L^{01}, L^{01}))D_x L^{01} - F(D_x, D_t).$$

We now make the choice $D_t = \sum_{i=1}^{n-1} u^{(i)} u_1^{(i)} \mathbf{e}_1 + \sum_{i=1}^{n-1} u_1^{(i)} \mathbf{e}_{i+1}$, which is consistent with the condition $\omega_{11} = 0$. We let $R_{ijkl} = \Omega_{ij}(\mathbf{e}_k, \mathbf{e}_l)$. Constant curvature means that $R_{ijij} = \varkappa$ for $i \neq j$ and the other R coefficients are zero. We obtain

$$D_t L^{01} = D_x^3 L^{01} + \left(\nu - \frac{3}{4(n-2)} K(L^{01}, L^{01})\right) D_x L^{01} - \varkappa D_x L^{01}.$$

Observe that the role of the integration constant and the curvature is the same. This may explain some of the success of the zero-curvature method.

We can *flatten* the equation by absorbing the curvature \varkappa in the connection by taking the integration constant ν equal to \varkappa :

$$D_t L^{01} = D_x^3 L^{01} - \frac{3}{4(n-2)} K(L^{01}, L^{01}) D_x L^{01}$$

or

$$\mathbf{u}_t = \mathbf{u}_3 + \frac{3}{2} \|\mathbf{u}\|^2 \mathbf{u}_1 \text{ (vmKDV)}.$$

From the point of view of integrability this flattening does not have any influence, since \mathbf{u}_1 is a trivial symmetry of the equation. We remark that while we have found the vmKDV equation through the Lax pair construction, it also provides a Lax pair for vmKDV, taking its values in \mathfrak{o}_{m+1} , if \mathbf{u} is an $(m-1)$ -vector, cf. [TT01].

A Geometric structures

In this Appendix we collect the proofs that the operators \mathfrak{J} , \mathfrak{H} and \mathfrak{R} are symplectic, cosymplectic and hereditary, respectively. The fact that \mathfrak{R} is hereditary is equivalent to the compatibility of the Hamiltonian structures \mathfrak{J}^{-1} and \mathfrak{H} .

These operators have nonlocal terms by the occurrence of D_x^{-1} . We refer to [SW01a] for results to on the locality of the generated symmetries and cosymmetries.

Proposition 1. *The operator $\mathfrak{J} = D_x + \mathbf{u} D_x^{-1} \mathbf{u}^\top$ is symplectic.*

Proof. As is proved in [Dor93], to show that \mathfrak{J} is symplectic it suffices that

$$\int (\langle \mathfrak{J}'[h_1] h_2, h_3 \rangle + \langle \mathfrak{J}'[h_3] h_1, h_2 \rangle + \langle \mathfrak{J}'[h_2] h_3, h_1 \rangle) dx = 0.$$

One has $\mathfrak{J}'[h_1] = h_1 D_x^{-1} \mathbf{u}^\top + \mathbf{u} D_x^{-1} h_1^\top$. Then

$$\begin{aligned} \langle \mathfrak{J}'[h_1] h_2, h_3 \rangle &= \\ &= \langle h_1 D_x^{-1} \mathbf{u}^\top h_2, h_3 \rangle + \langle \mathbf{u} D_x^{-1} h_1^\top h_2, h_3 \rangle \\ &= D_x^{-1} (\langle \mathbf{u}, h_2 \rangle) \langle h_1, h_3 \rangle + D_x^{-1} (\langle h_1, h_2 \rangle) \langle \mathbf{u}, h_3 \rangle \\ &\equiv D_x^{-1} (\langle \mathbf{u}, h_2 \rangle) \langle h_1, h_3 \rangle - D_x^{-1} (\langle \mathbf{u}, h_3 \rangle) \langle h_1, h_2 \rangle, \end{aligned}$$

and the result follows upon summation of the 3 cyclically permuted terms. \square

Proposition 2. *The operator \mathfrak{H} , given by*

$$\mathfrak{H} = D_x + \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} (J_{ij} \mathbf{u})^\top,$$

where the J_{ij} , given by $(J_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, are the generators of \mathfrak{o}_{n-1} , is a cosymplectic (or Hamiltonian) operator.

Proof. We prove this by checking the conditions of Theorem 7.8 and Corollary 7.21 in [Olv93].

The associated bi-vector of \mathfrak{H} is by definition

$$\begin{aligned} \Theta_{\mathfrak{H}} &= \frac{1}{2} \int (\theta \wedge \mathfrak{H}\theta) dx \\ &= \frac{1}{2} \int \left(\theta \wedge \left(\theta_1 + \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} (J_{ij} \mathbf{u})^\top \theta \right) \right) dx \\ &= \frac{1}{2} \int \theta \wedge \theta_1 dx + \frac{1}{2} \sum_{i < j} \int \langle J_{ij} \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} dx, \end{aligned}$$

where $\theta = (\vartheta^1, \dots, \vartheta^n)$ and $\vartheta_i^j = \frac{\partial^i \vartheta^j}{\partial x^i}$, etc. Here \wedge means that one needs to take the ordinary inner product between the vectors θ and $\mathfrak{H}\theta$. The elements of these vectors are then multiplied using the ordinary wedge product. We need to check the vanishing of the Schouten bracket $[\mathfrak{H}, \mathfrak{H}]$ which is equivalent to the Jacobi identity for the Lie bracket defined by \mathfrak{H} .

$$\begin{aligned} [\mathfrak{H}, \mathfrak{H}] &= \text{Pr } \mathbf{v}_{\mathfrak{H}\theta}(\Theta_{\mathfrak{H}}) = \\ &= \sum_{i < j} \int (\langle J_{ij} \mathfrak{H}\theta, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1}) dx \\ &= \sum_{i < j} \int J_{ij} \theta_1 \wedge \theta \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} dx \\ &+ \sum_{i < j} \sum_{k < l} \int \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} J_{ij} J_{kl} \mathbf{u} \wedge \theta \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} dx. \end{aligned}$$

The first term is zero because

$$\begin{aligned} J_{ij} \theta_1 \wedge \theta \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} &= \\ &= (\theta_1^j \wedge \theta^i - \theta_1^i \wedge \theta^j) \wedge D_x^{-1} (\mathbf{u}^j \theta^i - \mathbf{u}^i \theta^j) \\ &= (\theta_1^j \wedge \theta^i + \theta^j \wedge \theta_1^i) \wedge D_x^{-1} (\mathbf{u}^j \theta^i - \mathbf{u}^i \theta^j) \\ &\equiv -\theta^j \wedge \theta^i \wedge (\mathbf{u}^j \theta^i - \mathbf{u}^i \theta^j) \\ &= 0. \end{aligned}$$

The second term is zero because

$$\begin{aligned}
& \sum_{i < j} \sum_{k < l} \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} \wedge \langle J_{ij} J_{kl} \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \\
&= \frac{1}{2} \sum_{i < j} \sum_{k < l} \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} \wedge \langle [J_{ij}, J_{kl}] \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \\
&= \sum_{i < j, k < l, i < k} \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} \wedge \langle \delta_{lj} J_{ik} \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \\
&\quad - \sum_{i < j, k < l, i < l} \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} \wedge \langle \delta_{jk} J_{il} \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \\
&\quad + \sum_{i < j, k < l, j < l} \langle J_{kl} \mathbf{u}, \theta \rangle_{-1} \wedge \langle \delta_{ik} J_{jl} \mathbf{u}, \theta \rangle \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \\
&= \sum_{i < j < k} D_x (\langle J_{jk} \mathbf{u}, \theta \rangle_{-1} \wedge \langle J_{ij} \mathbf{u}, \theta \rangle_{-1} \wedge \langle J_{ik} \mathbf{u}, \theta \rangle_{-1}) \\
&\equiv 0.
\end{aligned}$$

Thus the result follows. \square

Proposition 3. *The operator*

$$\mathfrak{R} = D_x^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \mathbf{u}_1 D_x^{-1} \langle \mathbf{u}, \cdot \rangle - \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} \mathbf{u}_1, \cdot \rangle$$

is hereditary.

Proof. Notice that for the matrices J_{ij} we have

$$\sum_{i < j}^n J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} \mathbf{u}_1, P \rangle = (D_x^{-1} P \mathbf{u}_1^\top) \mathbf{u} - (D_x^{-1} \mathbf{u}_1 P^\top) \mathbf{u}. \quad (16)$$

We recall that an hereditary operator \mathfrak{R} is characterized by the property that $\mathfrak{R} D_{\mathfrak{R}}[P](Q) - D_{\mathfrak{R}}[\mathfrak{R}P](Q)$ is symmetric with respect to vectors P and Q , cf. [FF82].

We first compute $D_{\mathfrak{R}}[P](Q)$, meanwhile we drop the terms that are symmetric with respect to P and Q by using "≡".

$$\begin{aligned}
D_{\mathfrak{R}}[P](Q) &= 2 \langle \mathbf{u}, P \rangle Q + P_x D_x^{-1} \langle \mathbf{u}, Q \rangle + \mathbf{u}_1 D_x^{-1} \langle P, Q \rangle \\
&\quad - \sum_{i < j}^n J_{ij} P D_x^{-1} \langle J_{ij} \mathbf{u}_1, Q \rangle - \sum_{i < j}^n J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} P_x, Q \rangle \\
&\equiv \langle \mathbf{u}, P \rangle Q + P_x D_x^{-1} \langle \mathbf{u}, Q \rangle - \sum_{i < j}^n J_{ij} P D_x^{-1} \langle J_{ij} \mathbf{u}_1, Q \rangle,
\end{aligned}$$

where we use the relation $\sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} P_x, Q \rangle \equiv \langle \mathbf{u}, P \rangle Q$, which can be proved by applying formula (16). Notice that

$$\mathfrak{R}P = P_{xx} + \langle \mathbf{u}, \mathbf{u} \rangle P + \mathbf{u}_1 D_x^{-1} \langle \mathbf{u}, P \rangle - \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} \mathbf{u}_1, P \rangle$$

and the terms in $\Re D_{\Re}[P](Q) - D_{\Re}[\Re P](Q)$ are \mathbf{u} -degree of 1 or 3. Its first degree terms are

$$\begin{aligned}
& \langle \mathbf{u}_2, P \rangle Q + 2\langle \mathbf{u}_1, P_x \rangle Q + \langle \mathbf{u}, P_{xx} \rangle Q \\
& + 2\langle \mathbf{u}_1, P \rangle Q_x + 2\langle \mathbf{u}, P_x \rangle Q_x + \langle \mathbf{u}, P \rangle Q_{xx} \\
& + P_x \langle \mathbf{u}, Q_x \rangle + P_x \langle \mathbf{u}_1, Q \rangle + 2P_{xx} \langle \mathbf{u}, Q \rangle + P_{xxx} D_x^{-1} \langle \mathbf{u}, Q \rangle \\
& - \sum_{i < j} J_{ij} P \langle J_{ij} \mathbf{u}_2, Q \rangle - \sum_{i < j} J_{ij} P \langle J_{ij} \mathbf{u}_1, Q_x \rangle \\
& - 2 \sum_{i < j} J_{ij} P_x \langle J_{ij} \mathbf{u}_1, Q \rangle - \sum_{i < j} J_{ij} P_{xx} D_x^{-1} \langle J_{ij} \mathbf{u}_1, Q \rangle \\
& - 2 \langle P_{xx}, \mathbf{u} \rangle Q - P_{xxx} D_x^{-1} \langle \mathbf{u}, Q \rangle - \mathbf{u}_1 D_x^{-1} \langle P_{xx}, Q \rangle \\
& + \sum_{i < j} J_{ij} P_{xx} D_x^{-1} \langle J_{ij} \mathbf{u}_1, Q \rangle + \sum_{i < j} J_{ij} \mathbf{u} D_x^{-1} \langle J_{ij} P_{xxx}, Q \rangle \\
& \equiv 0.
\end{aligned}$$

Similarly, we can check its third degree terms are also equivalent to zero. Therefore,

$$(\Re D_{\Re}[P](Q))^j - (D_{\Re}[\Re P](Q))^j \equiv 0,$$

that is, the operator \Re is hereditary. \square

B Generalized Hasimoto transformation

In this Appendix we complete the proof of Theorem 1. The formula contain possibly non-existing angles. If this is the case the convention is to take the first term in the product to be equal to one.

Lemma 1. *The matrix T gauges $\underline{\omega}$ into zero, cf. the proof of Theorem 1 for the notations of T and $\underline{\omega}$, that is $\underline{\omega}T = D_x T$.*

Proof. Notice that $R = R_{n-1,n} \cdots R_{2n} R_{n-2,n-1} \cdots R_{34} R_{24} R_{23}$ since R_{ij} commutes R_{kl} when $\{i, j\} \cap \{k, l\} = \emptyset$. Let $R^{(i)} = R_{i-1,i} \cdots R_{3i} R_{2i}$ and the part without the first row and the first column is denoted by $T^{(i)}$. Then one writes $R = R^{(n)} R^{(n-1)} \cdots R^{(4)} R^{(3)}$ and $T = T^{(n)} T^{(n-1)} \cdots T^{(4)} T^{(3)}$. The matrix $T^{(i)}$ is generated by

$$\begin{cases}
T_{jj}^{(i)} = 1, & j > i - 1 \\
T_{jj}^{(i)} = \cos \theta_{j+1,i}, & 1 < j < i - 1, \\
T_{1,i-1}^{(i)} = \sin \theta_{2i} \\
T_{lj}^{(i)} = \sin \theta_{l+1,i} \frac{\partial T_{i-1,j}^{(i)}}{\partial \theta_{l,i}}, & j < l < i - 1 \\
T_{i-1,j}^{(i)} = \frac{\partial T_{i-2,j}^{(i)}}{\partial \theta_{i-1,i}}.
\end{cases}$$

and the rest entries are equal to zero.

We first perform the gauge transformation of $T^{(n)}$ on $\underline{\omega}$. Notice the last column of $\underline{\omega}T^{(n)} - D_x T^{(n)}$ has to be zero since the others do not affect it. This implies that

$$\bar{u}^{(2)}T_{2,n-1}^{(n)} - D_x T_{1,n-1}^{(n)} = 0; \quad (17)$$

$$\bar{u}^{(k)}T_{k,n-1}^{(n)} - \bar{u}^{(k-1)}T_{k-2,n-1}^{(n)} - D_x T_{k-1,n-1}^{(n)} = 0, \quad 2 < k \leq n-1; \quad (18)$$

$$\bar{u}^{(n-1)}T_{n-2,n-1}^{(n)} + D_x T_{n-1,n-1}^{(n)} = 0, \quad (19)$$

where $T_{j,n-1}^{(n)} = \sin \theta_{j+1,n} \cos \theta_{j,n} \cdots \cos \theta_{2n}$ and $1 \leq j \leq n-1$. The formula (17) immediately leads to

$$\bar{u}^{(2)} = \frac{D_x \theta_{2n}}{\sin \theta_{3n}}.$$

Now we prove the following formula by induction on i .

$$\bar{u}^{(i)} = \frac{D_x \theta_{i,n}}{\sin \theta_{i+1,n}} + \frac{\cos \theta_{i,n}}{\sin \theta_{i+1,n}} \frac{\sin \theta_{i-1,n}}{\cos \theta_{i-1,n}} \bar{u}^{(i-1)}, \quad 2 < i \leq n-1. \quad (20)$$

Using (18) and assuming that (20) is true for $2 < l \leq k-1$, we compute

$$\begin{aligned} \bar{u}^{(k)}T_{k,n-1}^{(n)} &= \bar{u}^{(k)} \sin \theta_{k+1,n} \cos \theta_{k,n} \cdots \cos \theta_{2,n} \\ &= \bar{u}^{(k-1)}T_{k-2,n-1}^{(n)} + D_x T_{k-1,n-1}^{(n)} \\ &= \bar{u}^{(k-1)} \sin \theta_{k-1,n} \cos \theta_{k-2,n} \cdots \cos \theta_{2,n} \\ &\quad + \cos \theta_{k,n} \cos \theta_{k-1,n} \cdots \cos \theta_{2,n} D_x \theta_{k,n} \\ &\quad - \sum_{l=2}^{k-1} \sin \theta_{k,n} \cos \theta_{k-1,n} \cdots \sin \theta_{l,n} \cdots \cos \theta_{2,n} D_x \theta_{l,n} \\ &= \bar{u}^{(k-1)} \cos^2 \theta_{k,n} \sin \theta_{k-1,n} \cos \theta_{k-2,n} \cdots \cos \theta_{2,n} \\ &\quad + \cos \theta_{k,n} \cos \theta_{k-1,n} \cdots \cos \theta_{2,n} D_x \theta_{k,n}. \end{aligned}$$

Thus, we proved the formula is valid for $\bar{u}^{(k)}$. By directly computation, we can check the identity (19).

Next, we prove $\underline{\omega}T^{(n)} - D_x T^{(n)} = T^{(n)}C$, where C is an anti-symmetric matrix with

$$C_{k-1,k} = -C_{k,k-1} = \sum_{l=2}^k \frac{\cos \theta_{k+1,n}}{\sin \theta_{k+1,n}} \frac{1}{\sin \theta_{k,n}} \frac{\sin \theta_{l,n}}{\cos \theta_{l,n}} D_x \theta_{l,n}, \quad 2 \leq i \leq n-2,$$

and the remaining entries are equal to zero.

To do so, we first rewrite (20) as

$$\bar{u}^{(i)} = \sum_{l=2}^i \frac{1}{\sin \theta_{i+1,n}} \frac{\cos \theta_{i,n}}{\sin \theta_{i,n}} \frac{\sin \theta_{l,n}}{\cos \theta_{l,n}} D_x \theta_{l,n}, \quad 2 \leq i \leq n-1. \quad (21)$$

Let us check that

$$\begin{aligned} 0 &= \bar{u}^{(i+1)}T_{i+1,j}^{(n)} - \bar{u}^{(i)}T_{i-1,j}^{(n)} - D_x T_{i,j}^{(n)} \\ &\quad - T_{i,j-1}^{(n)}C_{j-1,j} + T_{i,j+1}^{(n)}C_{j,j+1}. \end{aligned} \quad (22)$$

It is easy to see that this formula is valid for $j > i + 1$ since $T_{l,k}^{(n)} = 0$ when $l < k$ and $k \neq n - 1$. When $j = i + 1$, we have

$$\bar{u}^{(i+1)}T_{i+1,i+1}^{(n)} - T_{i,i}^{(n)}C_{i,i+1} = \bar{u}^{(i+1)}\cos\theta_{i+2,n} - \cos\theta_{i+1,n}C_{i,i+1} = 0.$$

Similarly, for $j = i$, we can check $\bar{u}^{(i+1)}T_{i+1,i}^{(n)} - D_x T_{i,i}^{(n)} - T_{i,i-1}^{(n)}C_{i-1,i} = 0$. Now we concentrate on the case $j < i$. Notice that

$$T_{i,j}^{(n)} = -\sin\theta_{i+1,n}\cos\theta_{i,n}\cdots\cos\theta_{j+2,n}\sin\theta_{j+1,n}, \quad i > j.$$

So, we have

$$\begin{aligned} &\bar{u}^{(i+1)}T_{i+1,j}^{(n)} - \bar{u}^{(i)}T_{i-1,j}^{(n)} - D_x T_{i,j}^{(n)} - T_{i,j-1}^{(n)}C_{j-1,j} + T_{i,j+1}^{(n)}C_{j,j+1} \\ &= \sum_{l=2}^{i+1} \frac{1}{\sin\theta_{i+2,n}} \frac{\cos\theta_{i+1,n}}{\sin\theta_{i+1,n}} \frac{\sin\theta_{l,n}}{\cos\theta_{l,n}} D_x \theta_{l,n} (-\sin\theta_{i+2,n} \prod_{k=j+2}^{i+1} \cos\theta_{k,n} \sin\theta_{j+1,n}) \\ &\quad - \sum_{l=2}^i \frac{1}{\sin\theta_{i+1,n}} \frac{\cos\theta_{i,n}}{\sin\theta_{i,n}} \frac{\sin\theta_{l,n}}{\cos\theta_{l,n}} D_x \theta_{l,n} (-\sin\theta_{i,n} \prod_{k=j+2}^{i-1} \cos\theta_{k,n} \sin\theta_{j+1,n}) \\ &\quad - D_x \left(-\sin\theta_{i+1,n} \prod_{k=j+2}^i \cos\theta_{k,n} \sin\theta_{j+1,n} \right) \\ &\quad - (-\sin\theta_{i+1,n} \prod_{k=j+1}^i \cos\theta_{k,n} \sin\theta_{j,n}) \sum_{l=2}^j \frac{\cos\theta_{j+1,n}}{\sin\theta_{j+1,n}} \frac{1}{\sin\theta_{j,n}} \frac{\sin\theta_{l,n}}{\cos\theta_{l,n}} D_x \theta_{l,n} \\ &\quad + (-\sin\theta_{i+1,n} \prod_{k=j+3}^i \cos\theta_{k,n} \sin\theta_{j+2,n}) \sum_{l=2}^{j+1} \frac{\cos\theta_{j+2,n}}{\sin\theta_{j+2,n}} \frac{1}{\sin\theta_{j+1,n}} \frac{\sin\theta_{l,n}}{\cos\theta_{l,n}} D_x \theta_{l,n} \\ &= 0. \end{aligned}$$

Now we apply induction procedure on the matrix C since it is of the form of Frenét frame. Using (21) for the $n - 1$ case, we have

$$C_{k-1,k} = \sum_{l=2}^k \frac{1}{\sin\theta_{k+1,n-1}} \frac{\cos\theta_{k,n-1}}{\sin\theta_{k,-1n}} \frac{\sin\theta_{l,n-1}}{\cos\theta_{l,n-1}} D_x \theta_{l,n-1}, \quad 2 \leq k \leq n - 2.$$

Form these, we can solve $D_x \theta_{k,n}$ for $2 \leq k \leq n - 2$.

$$\begin{aligned} D_x \theta_{k,n} &= \sum_{l=2}^k \frac{\sin\theta_{k+1,n}}{\cos\theta_{k+1,n}} \frac{\cos\theta_{k,n}}{\sin\theta_{k+1,n-1}} \frac{\cos\theta_{k,n-1}}{\sin\theta_{k,n-1}} \frac{\sin\theta_{l,n-1}}{\cos\theta_{l,n-1}} D_x \theta_{l,n-1} \\ &\quad - \sum_{l=2}^{k-1} \frac{\cos\theta_{k,n}}{\sin\theta_{k,n}} \frac{\sin\theta_{l,n}}{\cos\theta_{l,n}} D_x \theta_{l,n} \quad 2 \leq k \leq n - 2. \end{aligned}$$

Assume that the formula (9) is valid for $j = n - 1$, and $2 \leq i \leq n - 3$. When $k = 2$, we obtain

$$\begin{aligned} D_x \theta_{2,n} &= \frac{\sin \theta_{3,n} \cos \theta_{2,n}}{\cos \theta_{3,n} \sin \theta_{3,n-1}} \prod_{l=4}^{n-1} \frac{\cos \theta_{2,l}}{\cos \theta_{3,l}} \sin \theta_{3,n-1} A_2 \\ &= \prod_{l=4}^n \frac{\cos \theta_{2,l}}{\cos \theta_{3,l}} \sin \theta_{3,n} A_2. \end{aligned}$$

Now assuming that the formula (9) is valid for $j = n$, and $2 \leq i \leq k - 1$, we compute

$$\begin{aligned} D_x \theta_{k,n} &= \sum_{l=2}^k \frac{\sin \theta_{k+1,n} \cos \theta_{k,n} \cos \theta_{k,n-1} \sin \theta_{l,n-1}}{\cos \theta_{k+1,n} \sin \theta_{k+1,n-1} \sin \theta_{k,n-1} \cos \theta_{l,n-1}} \\ &\quad \left(\prod_{r=l+2}^{n-1} \frac{\cos \theta_{lr}}{\cos \theta_{l+1,r}} \sin \theta_{l+1,n-1} A_l - \prod_{r=l+1}^{n-2} \frac{\cos \theta_{l-1,r}}{\cos \theta_{l,r}} \sin \theta_{l-1,n-1} A_{l-1} \right) \\ &\quad - \sum_{l=2}^{k-1} \frac{\cos \theta_{k,n} \sin \theta_{l,n}}{\sin \theta_{k,n} \cos \theta_{l,n}} \\ &\quad \left(\prod_{r=l+2}^n \frac{\cos \theta_{lr}}{\cos \theta_{l+1,r}} \sin \theta_{l+1,n} A_l - \prod_{r=l+1}^{n-1} \frac{\cos \theta_{l-1,r}}{\cos \theta_{l,r}} \sin \theta_{l-1,n} A_{l-1} \right) \\ &= \prod_{r=k+2}^n \frac{\cos \theta_{kr}}{\cos \theta_{k+1,r}} \sin \theta_{k+1,n} A_k - \prod_{r=k+1}^{n-1} \frac{\cos \theta_{k-1,r}}{\cos \theta_{k,r}} \sin \theta_{k-1,n} A_{k-1}. \end{aligned}$$

Therefore, (9) is also valid for $j = n$ and $i = k$.

Substituting (9) into (21), we have

$$\begin{aligned} \bar{u}^{(i)} &= \sum_{l=2}^i \frac{1}{\sin \theta_{i+1,n}} \frac{\cos \theta_{i,n} \sin \theta_{l,n}}{\sin \theta_{i,n} \cos \theta_{l,n}} \\ &\quad \left(\prod_{r=l+2}^n \frac{\cos \theta_{lr}}{\cos \theta_{l+1,r}} \sin \theta_{l+1,n} A_l - \prod_{r=l+1}^{n-1} \frac{\cos \theta_{l-1,r}}{\cos \theta_{l,r}} \sin \theta_{l-1,n} A_{l-1} \right) \\ &= \prod_{r=i+2}^n \frac{\cos \theta_{ir}}{\cos \theta_{i+1,r}} A_i. \end{aligned}$$

Finally, we prove (8) for $i = n - 1$. Taking $i = n - 1$ in (20), we have

$$\bar{u}^{(n-1)} = D_x \theta_{n-1,n} + \cos \theta_{n-1,n} \frac{\sin \theta_{n-2,n}}{\cos \theta_{n-2,n}} \bar{u}^{(n-2)}$$

On the other hand, we have $\bar{u}^{(n-1)} = A_{n-1}$ and $\bar{u}^{(n-2)} = \frac{\cos \theta_{n-2,n}}{\cos \theta_{n-1,n}} A_{n-2}$. Therefore

$$A_{n-1} = D_x \theta_{n-1,n} + \sin \theta_{n-2,n} A_{n-2}.$$

By now we completely proved the statement. \square

C Integrable systems in 3 dimensional Riemannian geometry

In this appendix we specialize to $n = 3$. We show that the operator \mathcal{Q} in theorem 3 can be explicitly constructed. We make a comparison with known results, [MBSW02] and [DS94]. The equations are (cf. [DS94])

$$\bar{\mathbf{u}}_t = \bar{\Psi}_1 \bar{h} - \varkappa \bar{\Psi}_2 \bar{h},$$

where, with $\bar{\mathbf{u}}^\top = (\kappa, \tau)$,

$$\bar{\Psi}_1 = \begin{pmatrix} D_x^2 + D_x \kappa D_x^{-1} \kappa - \tau^2 & -D_x \tau - \tau D_x \\ D_x \frac{\tau}{\kappa} D_x + D_x \tau D_x^{-1} \kappa + D_x \frac{1}{\kappa} D_x \tau + \kappa \tau & D_x \frac{1}{\kappa} D_x^2 - D_x \frac{\tau^2}{\kappa} + \kappa D_x \end{pmatrix}.$$

and

$$\bar{\Psi}_2 = \begin{pmatrix} 1 & 0 \\ 0 & D_x \frac{1}{\kappa} \end{pmatrix}.$$

We know from the Hasimoto transformation (6) that

$$\begin{aligned} \gamma_t &= (\bar{h}_1, \bar{h}_2, \bar{h}_3) \begin{pmatrix} T \\ N \\ B \end{pmatrix} = (h_1, h_2, h_3) \begin{pmatrix} T \\ N^1 \\ N^2 \end{pmatrix} \\ &= (h_1, h_2, h_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \end{aligned}$$

or

$$(\bar{h}_2, \bar{h}_3) = (h_2, h_3) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = (h_2, h_3) T$$

In this case, we know that $\mathcal{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a Hamiltonian operator of (10).

$$\begin{aligned} \mathcal{Q} &= T \mathcal{H} T^\top \Psi_2^* \\ &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\kappa} D_x \end{pmatrix} \\ &= - \begin{pmatrix} 0 & \frac{1}{\kappa} D_x \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Using the notation in [MBSW02] we have $(\bar{\Psi}_1 + \varkappa \bar{\Psi}_2) \mathcal{Q} = -\mathcal{P}$. This explains the rather mysterious result that \mathcal{P} is a compatible Hamiltonian operator.

D Integrable systems in 4 dimensional Riemannian geometry

In this appendix we specialize to $n = 4$. The equations are

$$\bar{\mathbf{u}}_t = \bar{\Psi}_1 \bar{h} - \varkappa \bar{\Psi}_2 \bar{h},$$

where, with $\bar{\mathbf{u}}^\top = (u, v, w)$,

$$\bar{\Psi}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D_x \frac{1}{u} & -\frac{w}{u} \\ 0 & D_x \frac{w}{uv} & D_x \frac{1}{v} D_x \frac{1}{u} + \frac{v}{u} \end{pmatrix}.$$

By taking $(\bar{h}^2, \bar{h}^3, \bar{h}^4) = (u_1, uv, 0)$, we have the following integrable system:

$$\begin{cases} u_t = u_3 + \frac{3}{2}u_1u^2 - 3u_1v^2 - 3uvv_1 \\ v_t = 3\frac{vu_3}{u} + v_3 + 3uu_1v + \frac{3}{2}u^2v_1 - 3v^2v_1 + 3\frac{u_1v_2}{u} + 6\frac{u_2v_1}{u} \\ \quad - 3\frac{u_1u_2v}{u^2} - 3\frac{u_1^2v_1}{u^2} - 3vww_1 - 3v_1w^2 - 3\frac{u_1vw^2}{u} \\ w_t = 6\frac{wu_3}{u} + 3\frac{wv_3}{v} + w_3 + 3uu_1w + \frac{3}{2}u^2w_1 - 3w^2w_1 \\ \quad + 9\frac{u_2w_1}{u} + 3\frac{u_1w_2}{u} + 6\frac{v_2w_1}{v} + 3\frac{v_1w_2}{v} + 3\frac{u_1v^2w}{u} \\ \quad - 12\frac{u_1u_2w}{u^2} - 3\frac{v_1v_2w}{v^2} + 6\frac{u_2v_1w}{uv} + 6\frac{u_1v_2w}{uv} + 6\frac{u_1v_1w_1}{uv} \\ \quad - 6\frac{u_1^2w_1}{u^2} - 3\frac{v_1^2w_1}{v^2} + 6\frac{u_1^3w}{u^3} - 6\frac{u_1^2v_1w}{u^2v} - 6\frac{u_1v_1^2w}{uv^2}. \end{cases} \quad (23)$$

First let us write out Hasimoto transformation for $n = 4$ using Theorem 1.

$$\begin{aligned} v &= \frac{\cos \theta_{24}}{\cos \theta_{34}} D_x \theta_{23}, \\ w &= A_3 = D_x \theta_{34} + \sin \theta_{24} D_x \theta_{23}, \\ D_x \theta_{24} &= \frac{\cos \theta_{24}}{\cos \theta_{34}} \sin \theta_{34} D_x \theta_{23}, \end{aligned}$$

which, together with the Euler transformation

$$u^{(1)} = u \cos \theta_{23} \cos \theta_{24}, \quad u^{(2)} = u \sin \theta_{23} \cos \theta_{24}, \quad u^{(3)} = u \sin \theta_{24},$$

transform equation (23) into the vector mKdV, i.e., $\mathbf{u}_t = \mathbf{u}_3 + \frac{3}{2}\langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u}_1$, where $\mathbf{u} = (u^{(1)}, u^{(2)}, u^{(3)})^\top$. The matrix T equals

$$\begin{pmatrix} \cos \theta_{23} \cos \theta_{24} & \sin \theta_{23} \cos \theta_{24} & \sin \theta_{24} \\ -\sin \theta_{34} \sin \theta_{24} \cos \theta_{23} - \sin \theta_{23} \cos \theta_{34} & -\sin \theta_{34} \sin \theta_{24} \sin \theta_{23} + \cos \theta_{23} \cos \theta_{34} & \sin \theta_{34} \cos \theta_{24} \\ -\cos \theta_{34} \sin \theta_{24} \cos \theta_{23} + \sin \theta_{23} \sin \theta_{34} & -\cos \theta_{34} \sin \theta_{24} \sin \theta_{23} - \cos \theta_{23} \sin \theta_{34} & \cos \theta_{34} \cos \theta_{24} \end{pmatrix}$$

with the property $TT^\top = I$ and

$$D_x T = \begin{pmatrix} 0 & v & 0 \\ -v & 0 & w \\ 0 & -w & 0 \end{pmatrix} T = \underline{\omega} T$$

Now we compute the operator \mathcal{Q} in Theorem 3

$$\begin{aligned}
\mathcal{Q} &= T\mathcal{H}T^\top\bar{\Phi}_2^* \\
&= D_x\bar{\Phi}_2^* - \underline{w}\bar{\Phi}_2^* + \sum_{i=1}^2 \sum_{j=2}^3 TJ_{ij}UD_x^{-1}(J_{ij}U)^\top T^\top\bar{\Phi}_2^* \\
&= \begin{pmatrix} D_x & \frac{v}{u}D_x & \frac{w}{u}D_x \\ v & -D_x\frac{1}{u}D_x - u + \frac{w^2}{u} & -D_x\frac{w}{uv}D_x - \frac{w}{u}D_x\frac{1}{v}D_x - \frac{vw}{u} \\ 0 & -\frac{w}{u}D_x - D_x\frac{w}{u} & \frac{u}{v}D_x - \frac{w^2}{uv}D_x + D_x\frac{1}{u}D_x\frac{1}{v}D_x + D_x\frac{v}{u} \end{pmatrix}.
\end{aligned}$$

Equation (23) is bi-Hamiltonian with Hamiltonian function $\frac{1}{2}(\frac{1}{4}u^4 - u_1^2 - u^2v^2)$ corresponding to Hamiltonian operator $\bar{\Phi}_2\mathcal{Q}$ and Hamiltonian function $\frac{1}{2}u^2$ corresponding to Hamiltonian operator $\bar{\Phi}_1\mathcal{Q}$.

According to the order of D_x , we can write $\bar{\Phi}_2\mathcal{Q}$ as the sum of three operators, namely,

$$\begin{aligned}
\mathcal{H}_1 &= \begin{pmatrix} D_x & \frac{v}{u}D_x & \frac{w}{u}D_x \\ D_x\frac{v}{u} & -D_x + 3\frac{w}{u}D_x\frac{w}{u} & -\frac{w}{v}D_x + \frac{w^3}{u^2v}D_x - D_x\frac{vw}{u^2} - \frac{w}{u}D_x\frac{v}{u} \\ D_x\frac{w}{u} & -D_x\frac{w}{v} + D_x\frac{w^3}{u^2v} - \frac{vw}{u^2}D_x - \frac{v}{u}D_x\frac{w}{u} & -2\frac{w}{u}D_x\frac{w}{u} + D_x + \frac{v}{u}D_x\frac{v}{u} \end{pmatrix}; \\
\mathcal{H}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -D_x\frac{1}{u}D_x\frac{1}{u}D_x & \mathcal{J}_{23} \\ 0 & -\mathcal{J}_{23}^* & \mathcal{J}_{33} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_{23} &= -D_x\frac{1}{u}D_x\frac{w}{uv}D_x - D_x\frac{w}{u^2}D_x\frac{1}{v}D_x - \frac{w}{u}D_x\frac{1}{u}D_x\frac{1}{v}D_x, \\
\mathcal{J}_{33} &= D_x\frac{1}{v}D_x\frac{1}{v}D_x - D_x\frac{1}{v}D_x\frac{w^2}{u^2v}D_x - D_x\frac{w^2}{u^2v}D_x\frac{1}{v}D_x - D_x\frac{w}{uv}D_x\frac{w}{uv}D_x \\
&\quad + D_x\frac{1}{v}D_x\frac{1}{u}D_x\frac{v}{u} + \frac{v}{u}D_x\frac{1}{u}D_x\frac{1}{v}D_x
\end{aligned}$$

and

$$\mathcal{H}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_x\frac{1}{v}D_x\frac{1}{u}D_x\frac{1}{u}D_x\frac{1}{v}D_x \end{pmatrix}.$$

Surprisingly, for any constants a and b , the operator $a^2\mathcal{H}_1 + ab\mathcal{H}_2 + b^2\mathcal{H}_3$ is Hamiltonian.

References

- [AF87] Chris Athorne and Allan Fordy. Generalised KdV and MKdV equations associated with symmetric spaces. *J. Phys. A*, 20(6):1377–1386, 1987.

- [BE00] Alexander I. Bobenko and Ulrich Eitner. *Painlevé equations in the differential geometry of surfaces*, volume 1753 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
- [Bia27] L. Bianchi. *Lezioni di Geometria Differenziale 1-4*, volume 1. Zanichelli, 1923-1927.
- [Bia53] L. Bianchi. Applicazioni geometriche del metodo delle approssimazione successive di Picard. In *Opere. Vol. I, Parte seconda*, page 137. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1953. A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche.
- [Bis75] Richard L. Bishop. There is more than one way to frame a curve. *Amer. Math. Monthly*, 82:246–251, 1975.
- [Cal00] Annalisa Calini. Recent developments in integrable curve dynamics. In *Geometric approaches to differential equations (Canberra, 1995)*, pages 56–99. Cambridge Univ. Press, Cambridge, 2000.
- [Dor93] Irene Dorfman. *Dirac structures and integrability of nonlinear evolution equations*. John Wiley & Sons Ltd., Chichester, 1993.
- [DS94] A. Doliwa and P. M. Santini. An elementary geometric characterization of the integrable motions of a curve. *Phys. Lett. A*, 185(4):373–384, 1994.
- [FF82] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D*, 4(1):47–66, 1981/82.
- [FK83] Allan P. Fordy and Peter P. Kulish. Nonlinear Schrödinger equations and simple Lie algebras. *Comm. Math. Phys.*, 89(3):427–443, 1983.
- [For90] A. P. Fordy. Prolongation structures of nonlinear evolution equations. In Allan P. Fordy, editor, *Soliton theory: a survey of results*, pages 403–426. Manchester University Press, Manchester, 1990.
- [Gue97] Martin A. Guest. *Harmonic maps, loop groups, and integrable systems*, volume 38 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.
- [Has72] R. Hasimoto. A soliton on a vortex filament. *J. Fluid Mechanics*, 51:477–485, 1972.
- [Hél01] Frédéric Hélein. *Constant mean curvature surfaces, harmonic maps and integrable systems*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001. Notes taken by Roger Moser.

- [Hél02] Frédéric Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 2002. Translated from the 1996 French original, With a foreword by James Eells.
- [Lio53] J. Liouville. Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$. *Journal de Mathématiques Pures et Appliquées*, 18:71–72, 1853.
- [LP91] Joel Langer and Ron Perline. Poisson geometry of the filament equation. *J. Nonlinear Sci.*, 1(1):71–93, 1991.
- [LP96] Joel Langer and Ron Perline. The planar filament equation. In *Mechanics day (Waterloo, ON, 1992)*, pages 171–180. Amer. Math. Soc., Providence, RI, 1996.
- [LP00] Joel Langer and Ron Perline. Geometric realizations of Fordy-Kulish nonlinear Schrödinger systems. *Pacific J. Math.*, 195(1):157–178, 2000.
- [Mag78] F. Magri. A simple model of integrable Hamiltonian equation. *Journal of Mathematical Physics*, 19(5):1156–1162, 1978.
- [MBSW02] G. Marí Beffa, J. A. Sanders, and Jing Ping Wang. Integrable systems in three-dimensional Riemannian geometry. *J. Nonlinear Sci.*, 12(2):143–167, 2002.
- [MN01] A. Ya. Maltsev and S. P. Novikov. On the local systems Hamiltonian in the weakly non-local Poisson brackets. *Phys. D*, 156(1-2):53–80, 2001.
- [MW83] Jerrold Marsden and Alan Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Phys. D*, 7(1-3):305–323, 1983. Order in chaos (Los Alamos, N.M., 1982).
- [Olv93] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.
- [Pal97] Richard S. Palais. The symmetries of solitons. *Bull. Amer. Math. Soc. (N.S.)*, 34(4):339–403, 1997.
- [RS02] C. Rogers and W. K. Schief. *Bäcklund and Darboux transformations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002. Geometry and modern applications in soliton theory.
- [Sha97] R. W. Sharpe. *Differential geometry*. Springer-Verlag, New York, 1997. Cartan's generalization of Klein's Erlangen program, With a foreword by S. S. Chern.

- [SW01a] Jan A. Sanders and Jing Ping Wang. Integrable systems and their recursion operators. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 8 (Catania, 2000)*, volume 47, pages 5213–5240, 2001.
- [SW01b] Vladimir V. Sokolov and Thomas Wolf. Classification of integrable polynomial vector evolution equations. *J. Phys. A*, 34(49):11139–11148, 2001.
- [TT01] Chuu-Lian Terng and Gudlaugur Thorbergsson. Completely integrable curve flows on adjoint orbits. *Results Math.*, 40(1-4):286–309, 2001. Dedicated to Shiing-Shen Chern on his 90th birthday.
- [Wan03] Jing Ping Wang. Generalized Hasimoto transformation and vector Sine-Gordon equation. In S. Abenda, G. Gaeta, and S. Walcher, editors, *Symmetry and Perturbation Theory, SPT 2002*. World Scientific, 2003.
- [YS98] Yukinori Yasui and Norihito Sasaki. Differential geometry of the vortex filament equation. *J. Geom. Phys.*, 28(1-2):195–207, 1998.