Symmetric factorization Riccati equations and Bezoutians

collaboration with

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Symmetric factorization

Given is a $m \times m$ rational matrix valued function $\Phi(\lambda)$ with

$$\Phi(\lambda) \geq 0 \qquad \lambda \in i\mathbb{R}$$

A rational matrix function $W(\lambda)$ (possibly nonsquare) is called a *symmetric factor* of $\Phi(\lambda)$ if

$$\Phi(\lambda) = W(\lambda)W(-\bar{\lambda})^*$$

The symmetric factorization problem plays a role in stochastic realization theory and in several optimal or robust control problems.

Realization

$W(\lambda)$ a rational matrix function with $W(\infty) = D$.

A *realization* of $W(\lambda)$ is a representation

$$W(\lambda) = D + C(\lambda I_n - A)^{-1}B$$

This always exists. It is called a

minimal realization

if the number n is as small as possible. In that case n is called the

McMillan degree

of W. Notation: $\delta(W)$.

Poles and zeros

If D is invertible, then

$$W(\lambda)^{-1} = D^{-1} - D^{-1}C(\lambda - A^{\times})^{-1}BD^{-1},$$

where

$$A^{\times} = A - BD^{-1}C.$$

If the realization is minimal then:

- Eigenvalues of A: poles of $W(\lambda)$
- Eigenvalues of $A^{\times}$: zeros of $W(\lambda)$

Realization for selfadjoint functions

Assume $\Phi(\lambda)$ takes selfadjoint values on the imaginary line, and suppose $\Phi(\infty) = I$.

Let $\Phi(\lambda) = I + C(\lambda I_n - A)^{-1}B$ be a minimal realization.

Then there is a unique skew-hermitian

$$H = -H^*$$

such that

$$HA = -A^*H, \qquad HB = C^*.$$

Minimal factorization

In general if $W(\lambda) = W_1(\lambda)W_2(\lambda)$ then

$$\delta(W) \leq \delta(W_1) + \delta(W_2)$$

In case equality holds we say that the factorization is *minimal*

Minimal factorization and realization

Let $W(\lambda) = D + C(\lambda I_n - A)^{-1}B$ be a minimal realization. Assume D is invertible.

Put $A^\times = A - BD^{-1}C$.

Let M be A-invariant, $M^\times$ $A^\times$ invariant and

$$M \oplus M^\times = \mathbb{C}^n$$

Let Π be the projection onto M along $M^\times$ and let $D = D_1 D_2$. Put

$$W_1(\lambda) = D_1 + C|_M (\lambda - A|_M)^{-1} (I - \Pi) B D_2^{-1}$$

$$W_2(\lambda) = D_2 + D_1^{-1} C \Pi (\lambda - \Pi A \Pi)^{-1} B$$

Then $W = W_1 W_2$ is a minimal factorization with square factors and all minimal factorizations with square factors are obtained this way.

(Bart, Gohberg, Kaashoek, VanDooren)

Specialization to square symmetric factors

Now consider $\Phi(\lambda)$ taking positive semidefinite values on $i\mathbb{R}$, $\Phi(\infty) = I_m$. The McMillan degree of such a rational matrix function is always even, and we denote it by $\delta(\Phi) = 2n$.

We have a minimal realization

$$\Phi(\lambda) = I_m + C(\lambda I_{2n} - A)^{-1}B$$

and the corresponding $H = -H^*$ for which $HA = -A^*H$ and $HB = C^*$.

What conditions should we put on M and $M^\times$ to get a minimal factorization

$$\Phi(\lambda) = W(\lambda)W(-\bar{\lambda})^*$$

with square factor $W(\lambda)$?

Let M be A-invariant and H-Lagrangian, i.e., $HM = M^\perp$. Also, let $M^\times$ be $A^\times$-invariant and H-Lagrangian. Then

- *automatically* we have

$$\mathbb{C}^{2n} = M \oplus M^\times.$$

  (Automatic matching.)

- The corresponding minimal factorization is a symmetric factorization.

- All minimal square symmetric factors are obtained this way.

Pole pair

A pair of matrices (C, A) is called a *pole pair* for the rational matrix function $W(\lambda)$ if

- there is a matrix B such that

  $$W(\lambda) - C(\lambda I - A)^{-1}B$$

  is analytic over the whole complex plane,

and

- (A, B, C) is minimal.

Special case

In many cases Φ arises as a product:

$$\Phi(\lambda) = W_1(\lambda) W_1(-\bar{\lambda})^*,$$

where this is minimal, and we have a minimal realization for $W_1(\lambda)$:

$$W_1(\lambda) = I_m + C(\lambda I_n - A)^{-1} B.$$

We are then looking for all $W(\lambda)$ such that

- $\Phi(\lambda) = W(\lambda) W(-\bar{\lambda})^*$ minimally,
- (C, A) is a pole pair for $W(\lambda)$.

A minimal realization for Φ:

$$\Phi(\lambda) = I_m + \begin{pmatrix} C & -B^* \end{pmatrix} \left(\lambda I_{2n} - \begin{pmatrix} A & -BB^* \\ 0 & -A^* \end{pmatrix} \right)^{-1} \begin{pmatrix} B \\ C^* \end{pmatrix}.$$

Then $M = \text{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\mathcal{A}^\times = \begin{pmatrix} A - BC & 0 \\ -CC^* & -A^* + C^*B^* \end{pmatrix}$, $H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. An H-Lagrangian $\mathcal{A}^\times$-invariant subspace is of the form $M^\times = \text{Im} \begin{pmatrix} X \\ I \end{pmatrix}$, where $X = X^*$ satisfies the algebraic Riccati equation

$$XCC^*X + X(A - BC) + (A^* - C^*B^*)X = 0.$$

And conversely. The corresponding $W(\lambda)$ is given by:

$$W(\lambda) = I_m + C(\lambda I_n - A)^{-1}(B - XC^*).$$

Bezoutians

Consider

$$\frac{W_1(\mu)W_1(-\bar{\lambda})^* - W(\mu)W(-\bar{\lambda})^*}{\mu - \lambda}$$

General theory gives: this can be written as

$$C(\mu - A)^{-1}\mathbf{T}(\lambda + A^*)^{-1}C^*$$

for some hermitian $\mathbf{T}$.

This is called the *Bezoutian*. Turns out:

$$\mathbf{T} = X.$$

Moreover, $\operatorname{Ker}\mathbf{T}$ describes common right zero structure of W and W_1.

Lerer and Rodman

Minimal realizations for nonsquare minimal symmetric factors

Consider again positive semidefinite $\Phi(\lambda)$, with minimal realization

$$\Phi(\lambda) = I_m + C(\lambda I_{2n} - A)^{-1}B$$

and the corresponding $H = -H^*$ for which $HA = -A^*H$ and $HB = C^*$.

Now we are looking for all possibly nonsquare minimal symmetric factorizations:

$$\Phi(\lambda) = W(\lambda)W(-\bar{\lambda})^*.$$

First step

Proposition 1. *If W is a minimal (possibly nonsquare) symmetric factor, then there is an A-invariant H-Lagrangian subspace M such that $(C|_M, A|_M)$ is a pole pair for W.*

Main result

If W is a minimal symmetric factor of Φ then $W(\infty) = V$ is a co-isometry, i.e., $VV^* = I_m$.

With respect to appropriate bases we can take $V = \begin{pmatrix} I_m & 0 \end{pmatrix}$.

We describe all minimal symmetric factors for which $W(\infty) = \begin{pmatrix} I_m & 0 \end{pmatrix}$, thereby describing all minimal symmetric factors up to choice of bases.

Theorem 1. *There is a one-to-one correspondence between the set of minimal symmetric factors* $W(\lambda)$ *of* $\Phi(\lambda)$ *such that* $W(\infty) = \begin{pmatrix} I_m & 0 \end{pmatrix}$ *and the set of triples* $\{M, X, \widehat{B}_1\}$ *described below.*

- M *is an* A*-invariant* H*-Lagrangian subspace.*

To describe X *and* $\widehat{B}_1$*, let* A_1 *and* C_1 *be given by* $A_1 = A|_M$ *and* $C_1 = C|_M$*. Furthermore, suppose that* $M^\times$ *is the* $A^\times = (A - BC)$*-invariant,* H*-Lagrangian subspace such that* $\sigma(A^\times|_{M^\times}) \subset \overline{\mathbb{C}}_-$.

Let π *be the projection onto* M *along* $M^\times$ *and denote a matrix representation for* πB *by* $\widetilde{B}_1$.

- *Then X solves the Riccati inequality*

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X \leq 0$$

- $\widehat{B}_1$ *satisfies*

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X = -\hat{B}_1\hat{B}_1^*.$$

This correspondence is given by

$$W(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} + C_1(\lambda I - A_1)^{-1} \begin{pmatrix} XC_1^* + \widetilde{B}_1 & \hat{B}_1 \end{pmatrix}.$$

Remarks

1. Proof uses that we have a minimal square symmetric factor

$$W_1(\lambda) = I + C_1(\lambda I - A_1)^{-1}\tilde{B}_1.$$

2. Observe that we can write

$$W(\lambda) = \begin{pmatrix} W_1(\lambda) & 0 \end{pmatrix} + C_1(\lambda - A_1)^{-1}\begin{pmatrix} XC_1^* & B_{12} \end{pmatrix}$$

3. Thus the co-isometry

$$U(\lambda) = W_1(\lambda)^{-1}W(\lambda)$$

is given by

$$U(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} + C_1(\lambda - (A_1 - \widetilde{B}_1 C_1))^{-1}\begin{pmatrix} XC_1^* & B_{12} \end{pmatrix}.$$

J-symmetric factorization

Given $\Phi(\lambda)$ with only selfadjoint (not nonnegative) values on $i\mathbb{R}$. Given also an invertible hermitian matrix J.

A rational matrix function $W(\lambda)$ is called a *J-symmetric factor* of $\Phi(\lambda)$ if

$$\Phi(\lambda) = W(\lambda) J W(-\bar{\lambda})^*.$$

Obviously necessary: the number of positive and negative eigenvalues of the matrix $\Phi(\lambda)$ does not depend on $\lambda \in i\mathbb{R}$ (at least outside of poles and zeros). That is:

$\Phi(\lambda)$ has *constant signature*.

W.l.o.g. we can take $W(\infty) = J$.

Let $\Phi(\lambda) = J + C(\lambda I_n - A)^{-1}B$ be a minimal realization. Recall

$$H = -H^*, \quad HA = -A^*H, \quad HB = C^*.$$

$$A^\times = A - BJ^{-1}C.$$

A necessary condition for existence of *minimal* J-symmetric factorization:

- there exists an A-invariant H-Lagrangian M
- there exists an $A^\times$-invariant H-Lagrangian $M^\times$

Not any longer automatic matching!

Example

$$\Phi(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & \lambda^{-2} \end{pmatrix}$$

A minimal realization is

$$\Phi(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left(\lambda - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^{\times} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There is a unique invariant H-Lagrangian subspace $M = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

All necessary conditions are satisfied, yet no minimal factorization exists.

Completeness of the set of minimal J-symmetric factorizations

$\Phi(\lambda)$ has a *complete set of minimal J-symmetric factorizations* if for any A-invariant H-Lagrangian subspace M and any $A^\times$-invariant H-Lagrangian subspace $M^\times$ we have

$$M \oplus M^\times = \mathbb{C}^n.$$

Theorem 2. *The following are equivalent:*

- *there is a complete set of J-symmetric factorizations*

- *for every A-invariant H-Lagrangian subspace M, and for every nonzero vector $x \in M$ we have*

$$\langle H(\lambda - A^{\times})^{-1}x, x\rangle \not\equiv 0, \quad \lambda \in \rho(A^{\times}),$$

- *for every $A^{\times}$-invariant H-Lagrangian subspace $M^{\times}$, and for every nonzero vector $x \in M^{\times}$ we have*

$$\langle H(\lambda - A)^{-1}x, x\rangle \not\equiv 0, \quad \lambda \in \rho(A).$$

Minimal nonsquare J-symmetric factorization

Start from square minimal factor

$$\Phi(\lambda) = W_1(\lambda) J W_1(-\bar{\lambda})^*$$

Looking for $W(\lambda)$ such that

- $\Phi(\lambda) = W(\lambda) \begin{pmatrix} J & 0 \\ 0 & J_{22} \end{pmatrix} W(-\bar{\lambda})^*$,
  for some $J_{22} = J_{22}^*$, and this is a minimal factorization,
- W has the same pole pair as W_1.

Results

- As in the case of nonnegative $\Phi(\lambda)$ algebraic Riccati equations are involved.

- Again solutions of it correspond to Bezoutians.

- Under the condition $J_{22} \geq 0$ the kernel of the Bezoutian again describes common right zero structure of W and W_1.