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Analysis of spectral points of the operators $T^{[*]}T$ and $TT^{[*]}$ in a Krein space.

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Preliminaries

Let A be a selfadjoint operator in $\mathcal{K}$. We call A *definitizable* if $\rho(A) \neq \emptyset$ and there exists a (real or complex) polynomial p such that $[p(A)f, f] \geq 0$ for $f \in \mathcal{D}(p(A))$.
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We define the set of *critical points* of a definitizable operator A as $c(A) := c_0(A) \cap \sigma(A) \cap \mathbb{R}$, where

$$c_0(A) := \bigcap_{p \text{ definitizing for } A} p^{-1}(0).$$

It is well known that $c(A) = c_0(A) \cap \mathbb{R}$.

Regular and singular critical points

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A critical point λ_0 of A is called *regular* if

$$\lim_{x \downarrow \lambda_0} E(x, \lambda] \quad \text{exists for some} \quad \lambda > \lambda_0,$$
$$\lim_{x \uparrow \lambda_0} E[\lambda, x) \quad \text{exists for some} \quad \lambda < \lambda_0.$$

In Π_κ-space, equivalently the root subspace $\mathcal{R}(\lambda_0, A)$ is non-degenerate.

A critical point that is not regular is called *singular*. In Π_κ space, in that case the root subspace is degenerate and the spectral function is unbounded at that point.

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An isolated point of the real spectrum is never a singular critical point.

$\sigma_+(A), \sigma_-(A)$

A point λ_0 of the real spectrum which is not a critical point is in $\sigma_+(A)$ (if $E(\Delta)$ has positive definite range for a neighbourhood Δ of λ_0) or in $\sigma_-(A)$ (if $E(\Delta)$ has negative definite range for a neighbourhood Δ of λ_0).

Infinity as a critical point

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If infinity is a critical point we call it *regular* if the limits $\lim_{x\uparrow+\infty} E([\lambda, x])$ and $\lim_{x\downarrow-\infty} E([x, \lambda])$ exist in the strong operator topology for any (some) not critical $\lambda \in \mathbb{R}$, otherwise we call it *singular* .

Definitizability

Assume that $T^{[*]}T$ and $TT^{[*]}$ are (densely defined and) selfadjoint operators in $\mathcal{K}$; and that $T^{[*]}T$ and $TT^{[*]}$ have nonempty resolvent sets. Then $T^{[*]}T$ is definitizable if and only if $TT^{[*]}$ is.

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If $p(x)$ is a definitizing polynomial for $T^{[*]}T$, then $xp(x)$ is a definitizing polynomial for $TT^{[*]}$.

Indeed, for $f \in \mathcal{D}((TT^{[*]})p(TT^{[*]}))$ we have

$$\begin{aligned}[(TT^{[*]})p(TT^{[*]})f, f] &= [T^{[*]}p(TT^{[*]})f, T^{[*]}f] = \\ &= [p(T^{[*]}T)T^{[*]}f, T^{[*]}f] \geq 0.\end{aligned}$$

Standing assumption

Standing assumption from now on:

$T^{[*]}T$ and $TT^{[*]}$ are:

- densely defined, and selfadjoint,
- definitizable,
- have nonempty resolvent sets.

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Nonzero spectral points

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Proposition *Infinity can be a critical point of at most one of the operators $T^{[*]}T$ and $TT^{[*]}$.*

This uses the definitizability: if $p(x)$ is definitizing for $T^{[*]}T$ and p is of odd degree then ∞ is critical point for $T^{[*]}T$. Then $xp(x)$ is of even degree, and is definitizing for $TT^{[*]}$. So ∞ is not a critical point for $TT^{[*]}$.

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Moreover, if $T^{[*]}Tx = \lambda_0 x$, $[x, x] > 0$, then
$TT^{[*]}(Tx) = \lambda_0(Tx)$, $[Tx, Tx] = \lambda_0[x, x]$.

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Infinity as a spectral point

$TT^{[*]} \setminus T^{[*]}T$	$\infty \in \rho$	$\infty \in \sigma_+ \cup \sigma_-$	∞ reg. crit.	∞ sing. crit.
$\infty \in \rho$	+	-	-	-

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$\infty \in \sigma_+ \cup \sigma_-$		+	+	+
∞ reg. crit.			-	-
∞ sing. crit.				-

Table 1: Infinity as a spectral point

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Zero as a spectral point

$TT^{[*]} \setminus T^{[*]}T$	$0 \in \rho$	$0 \in \sigma_+ \cup \sigma_-$	0 reg. crit.	0 sing. crit.
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Table 2: Zero as a spectral point

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0 reg. crit.			+	+
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Table 2: Zero as a spectral point

All examples in Π_1.

Example 1: $0 \in \rho(T^{[*]}T)$, regular c.p. for $TT^{[*]}$.

On $\ell^2(\mathbb{Z})$ define

$$J = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

$$T = \begin{pmatrix} \ddots & \ddots & & & & \\ & \ddots & 1 & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & 1 & \ddots & \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Example 1 continued

Computation:

$$T^{[*]}T = I \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I,$$

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Example 1 continued

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\begin{aligned}
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\end{aligned}
$$

$0 \notin \sigma(T^{[*]}T)$, and 0 is a regular critical point for $TT^{[*]}$ as the algebraic root subspace corresponding to zero is nondegenerate.

Example 2: $0 \in \sigma_+(T^{[*]}T)$, and a singular c.p. for $TT^{[*]}$.

$\mathcal{K} = L^2[0,1] \oplus \mathbb{C}^2 \oplus \ell^2$.

$$J = I_{L^2[0,1]} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{\ell^2}.$$

Obviously, $\mathcal{K}$ with the J-inner product is a Π_1-space.

Example 2: the operator

Consider the operator

$$T := \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(\mathbf{1}) & 0 \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \pi(e_1) & 0 & S \end{pmatrix}$$

M_ϕ denotes the multiplication operator by a bounded function ϕ,

S is the shift operator in ℓ^2 ($Se_j = e_{j+1}$,),

$\pi(g)$ (where g is an element of some Hilbert space) maps $x \in \mathbb{C}$ to xg

and $\mathbf{1} \in L^2[0, 1]$ is a function constantly equal one.

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Example 2 continued

$$T^{[*]} = \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(\mathbf{1}) & 0 \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & \langle \cdot, e_1 \rangle \\ 0 & 0 & 0 & S^* \end{pmatrix},$$

$$T^{[*]}T = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & 0 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell^2} \end{pmatrix},$$

$$TT^{[*]} = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & \langle \cdot, e_1 \rangle \mathbf{1} \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \langle \cdot, \mathbf{1} \rangle e_1 & 0 & 0 & SS^* \end{pmatrix}.$$

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Zero is not in the point spectrum of $T^{[*]}T$, and as the root subspace of $TT^{[*]}$ corresponding to zero is degenerate, zero is a singular critical point for $TT^{[*]}$.

Jordan chains

Focus on the Π_κ case here.

Recall: Jordan chains of a selfadjoint operator A in Π_κ can not be longer than $2\kappa + 1$.

For each eigenvalue λ the root subspace $\mathcal{R}(\lambda, A)$ is well-defined.

$\mathcal{R}(\lambda, A) = \mathcal{K}_0 \dot{+} \mathcal{K}_1$, where $\mathcal{K}_0$ is finite dimensional, $A\mid_{\mathcal{K}_1} = \lambda \cdot I$.

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Let $n_0, \ldots, n_k$ be the lengths of Jordan chains of $A \mid_{\mathcal{K}_0}$ in decreasing order.

Define the *Segre characteristic*: $(n_j)_{j=0}^{\infty}$ for A at λ as

$$n_0, \ldots, n_k, \underbrace{1, \ldots, 1}_{\dim \mathcal{K}_1}, 0, 0, \ldots .$$

Jordan chains for $T^{[*]}T$ and $TT^{[*]}$

Known already (van der Mee, R.,Rodman):

the negative part of the spectrum of $T^{[*]}T$ is finite,

thus there are no singular critical points on the negative part of the real axes.

Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional.

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Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional.

Theorem *Let λ be a complex number, and denote by $(n_j)_{j=1}^{\infty}$ and $(m_j)_{j=1}^{\infty}$ the Segre characteristics for $T^{[*]}T$ and $TT^{[*]}$ respectively, corresponding to λ. If $\lambda \neq 0$ then $n_j = m_j$ for all $j \in \mathbb{N}$. If $\lambda = 0$ then $|n_j - m_j| \leq 1$ for $j \in \mathbb{N}$.*

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Follows from a result of Flanders.

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Finite dimensional case

In principle completely solved in several ways: results of Mehl, Mehrmann, Xu give a complete solution.

Results of Ran, Wojtylak give a reduction procedure.

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