Discrete Riccati equations and block Toeplitz matrices

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Discrete algebraic Riccati equation

$$X = AXA^* + Q - AXB(R + B^*XB)^{-1}B^*XA^*.$$

Standing assumptions: A is invertible, R is Hermitian and invertible, Q is Hermitian.

No definitiness conditions on Q and R.

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Goal: describe the set of all solutions in terms of two given solutions X_1 and X_0, when $X_1 - X_0$ is invertible.

Second goal: to apply this to an interesting special case.

Reduction to the case $Q = 0$.

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Let X_0 be a fixed solution and let X be any other solution. Denote by $A_0 = A - AX_0B(R + B^*X_0B)^{-1}B^*$, and $R_0 = R + B^*X_0B$. The latter matrix is invertible by assumption, since X_0 is a solution.

Then $Y = X - X_0$ satisfies the equation

$$Y = A_0YA_0^* - A_0YB(R_0 + B^*YB)^{-1}B^*YA_0^*.$$

So, without loss of generality $Q = 0$, supposing we know a solution.

Reduction to Lyapunov equation

Assume that A is invertible, and let X be an invertible solution of DARE:

$$X = AXA^* - AXB(R + B^*XB)^{-1}B^*XA^*.$$

Then $Y = X^{-1}$ solves the Stein equation

$$Y - A^*YA = -BR^{-1}B^*$$

and conversely, if Y is an invertible solution of the Stein equation, then $X = Y^{-1}$ solves the discrete algebraic Riccati equation above.

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Note: the zero matrix is an obvious solution of the algebraic Riccati equation.

Non-invertible solutions

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Theorem *Let N be an A^*-invariant subspace which is X_+-nondegenerate, that is, $\mathbb{C}^r = (X_+N)^\perp \dot{+} N$. Let P_N be the projection onto $(X_+N)^\perp$ along N, and put $X_N = X_+P_N$. Then X_N is a Hermitian solution of the algebraic Riccati equation for which $\ker X_N = N$.*

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For converse we need an additional assumption.

Description of all solutions

$$X = AXA^* - AXB(R + B^*XB)^{-1}B^*XA^*.$$

A is invertible, so

$$X(A^*)^{-1} = \left(A - AXB(R + B^*XB)^{-1}B^*\right) X.$$

Consequently, for any solution X the subspace $N = \ker X$ is invariant under A^*.

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Would like to conclude that for any solution X its kernel is X_+-nondegenerate.
This is true under an extra assumption on A.

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Main theorem I

Theorem *Assume that* A *and* $(A^*)^{-1}$ *have no common eigenvalues. Then, if* X *solves DARE* $\ker X$ *is* X_+*-nondegenerate.*

There is a one-to-one correspondence between solutions of DARE and subspaces N *that are* A^**-invariant and* X_+*-nondegenerate, which is given by* $X = X_+ P_N$*, where* P_N *is the projection onto* $(X_+ N)^\perp$ *along* N*.*

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In particular, there can be only one invertible solution of DARE.

Observe that the uniqueness of the solution of the Stein equation is in fact equivalent to A^* and A^{-1} having no common eigenvalues.

Main theorem II

Assume that A and $(A^*)^{-1}$ have no common eigenvalues, and let Y be the unique solution of the Stein equation

$$Y - A^*YA = -BR^{-1}B^*.$$

Theorem *There is a one-to-one correspondence between solutions of DARE and subspaces* N *that are* A^**-invariant and* Y*-nondegenerate which is given by* $X = Y^{-1}P_N$*, where* P_N *is the projection onto* $YN^\perp$ *along* N*.*

Special case: companion type

$$A = K = \begin{pmatrix} K_1 & I & 0 & \cdots & 0 \\ K_2 & 0 & I & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & I \\ K_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Define $X_{00}^{(n)} = -R, X_{j0}^{(n)} = -K_j R$.
Assume K is invertible.

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Stein equation:

$$Y - K^* Y K = \operatorname{diag} \begin{pmatrix} -(X_{00}^{(n)})^{-1} & 0 & \cdots & 0 \end{pmatrix}.$$

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Connection to block Toeplitz matrices

The invertible solution X of the Riccati equation

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is congruent to the inverse of a block Toeplitz matrix.

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Use Gohberg-Lerer result: $Y = X^{-1} = T_{n-1}$, where T_n is the block Toeplitz matrix given by

$$T_n = \widehat{K}^* \begin{pmatrix} X_{00}^{(n)} & 0 \\ 0 & X^{-1} \end{pmatrix} \widehat{K},$$

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$$\widehat{K} = \begin{pmatrix} (X_{00}^{(n)})^{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ K_1 & I & 0 & & & 0 \\ K_2 & 0 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & I & 0 \\ K_n & 0 & \cdots & \cdots & 0 & I \end{pmatrix}.$$

Discussion of the extra assumption

Gohberg-Lerer: If $R = -X_{00} > 0$, K and $(K^*)^{-1}$ are even allowed to have common eigenvalues!!!

To ensure existence of an invertible solution to the Stein equation

$$Y - K^*YK = \operatorname{diag}\begin{pmatrix} -(X_{00}^{(n)})^{-1} & 0 & \cdots & 0 \end{pmatrix}.$$

$$P(\lambda) = (\sum_{j=0}^{n} \lambda^{n-j} K_j) R^{1/2}$$

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Theorem[Gohberg-Lerer] *The Stein equation has an invertible solution if and only if for every symmetric pair of eigenvalues* λ_0, $\overline{\lambda_0}^{-1}$ *and for any right Jordan chains* $x_1, \cdots, x_k$ *and* $y_1, \cdots, y_l$ *of* P *corresponding to* λ_0 *and* $\overline{\lambda_0}^{-1}$*, respectively, there holds, with* $\nu = \max\{k, l\}$*:*

$$(\star) \qquad \sum_{j=1}^{\nu} \langle x_{\nu-j}, y_j \rangle = 0.$$

Putting it all together

K, B and R as above, assume K invertible, $R > 0$. If ($\star$) holds whenever $\lambda_0, \overline{\lambda_0}^{-1}$ are both eigenvalues of K, then:

- there is an invertible block Toeplitz Hermitian matrix $Y = T_{n-1}$ solving the Stein equation,

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- there is an invertible block Toeplitz Hermitian matrix $Y = T_{n-1}$ solving the Stein equation,
- the solutions of DARE are in 1-1 correspondence to K-invariant subspaces N which are Y-nondegenerate:

$$X_N = P_N^* Y^{-1} P_N$$

where P_N is the projection onto $YN^\perp$ along N.

Example

$$K = \begin{pmatrix} 5 & 0 & 1 & 0 \\ 0 & 5/6 & 0 & 1 \\ -6 & 0 & 0 & 0 \\ 0 & -1/6 & 0 & 0 \end{pmatrix}, \qquad R = I_2.$$

$$\sigma(K) = \{2, 1/2, 3, 1/3\}.$$

There are pairs of eigenvalues symmetrically placed with respect to the unit circle.

The corresponding eigenvectors are, respectively,

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

Condition ($\star$) is satisfied.

Solution to Stein equation

The block Toeplitz invertible solution to the Stein equation:

$$Y = \begin{pmatrix} 7/120 & 0 & 5/120 & 0 \\ 0 & -21/10 & 0 & -3/2 \\ 5/120 & 0 & 7/120 & 0 \\ 0 & -3/2 & 0 & -21/10 \end{pmatrix}$$

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14 nontrivial K-invariant subspaces: four one-dimensional subspaces, six two-dimensional ones, and again four three-dimensional ones. All of them are Y-nondegenerate.

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So 14 solutions in total.

Invertible and rank 3 solutions

Subspace	Solution
$N = (0)$	$Y^{-1} = \begin{pmatrix} 35 & 0 & -25 & 0 \\ 0 & -35/36 & 0 & 25/36 \\ -25 & 0 & 35 & 0 \\ 0 & 25/36 & 0 & -35/36 \end{pmatrix}$
$N = \operatorname{span}\{x_1\}$	$\begin{pmatrix} 45/4 & 0 & 15/4 & 0 \\ 0 & -35/36 & 0 & 25/36 \\ 15/4 & 0 & 5/4 & 0 \\ 0 & 25/36 & 0 & -35/36 \end{pmatrix}$
$N = \operatorname{span}\{x_2\}$	$\begin{pmatrix} 35 & 0 & -25 & 0 \\ 0 & -1/30 & 0 & -1/10 \\ -25 & 0 & 35 & 0 \\ 0 & -1/10 & 0 & -3/10 \end{pmatrix}$
$N = \operatorname{span}\{x_3\}$	$\begin{pmatrix} 96/11 & 0 & 48/11 & 0 \\ 0 & -35/36 & 0 & 25/36 \\ 48/11 & 0 & 24/11 & 0 \\ 0 & 25/36 & 0 & -35/36 \end{pmatrix}$
$N = \operatorname{span}\{x_4\}$	$\begin{pmatrix} 35 & 0 & -25 & 0 \\ 0 & -2/33 & 0 & -4/33 \\ -25 & 0 & 35 & 0 \\ 0 & -4/33 & 0 & -8/33 \end{pmatrix}$

Rank 2 solutions

Subspace	Solution
$N = \operatorname{span}\{x_1, x_3\}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -35/36 & 0 & 25/36 \\ 0 & 0 & 0 & 0 \\ 0 & 25/36 & 0 & -35/36 \end{pmatrix}$
$N = \operatorname{span}\{x_2, x_4\}$	$\begin{pmatrix} 35 & 0 & -25 & 0 \\ 0 & 0 & 0 & 0 \\ -25 & 0 & 35 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$N = \operatorname{span}\{x_1, x_2\}$	$\begin{pmatrix} 45/4 & 0 & 15/4 & 0 \\ 0 & -1/30 & 0 & -1/10 \\ 15/4 & 0 & 5/4 & 0 \\ 0 & -1/10 & 0 & -3/10 \end{pmatrix}$
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$N = \operatorname{span}\{x_3, x_4\}$	$\begin{pmatrix} 96/11 & 0 & 48/11 & 0 \\ 0 & -2/33 & 0 & -4/33 \\ 48/11 & 0 & 24/11 & 0 \\ 0 & -4/33 & 0 & -8/33 \end{pmatrix}$

Rank 1 solutions

Subspace	Solution
$N = \operatorname{span}\{x_1, x_2, x_3\}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/30 & 0 & -1/10 \\ 0 & 0 & 0 & 0 \\ 0 & -1/10 & 0 & -3/10 \end{pmatrix}$
$N = \operatorname{span}\{x_1, x_2, x_4\}$	$\begin{pmatrix} 45/4 & 0 & 15/4 & 0 \\ 0 & 0 & 0 & 0 \\ 15/4 & 0 & 5/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$N = \operatorname{span}\{x_1, x_3, x_4\}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2/33 & 0 & -4/33 \\ 0 & 0 & 0 & 0 \\ 0 & -4/33 & 0 & -8/33 \end{pmatrix}$
$N = \operatorname{span}\{x_2, x_3, x_4\}$	$\begin{pmatrix} 96/11 & 0 & 48/11 & 0 \\ 0 & 0 & 0 & 0 \\ 48/11 & 0 & 24/11 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$N = \mathbb{C}^4$	$0_{4\times 4}$

Finally

Thank you for your attention!