# Controllability concepts for coordinated linear systems

André Ran and Pia Kempker
Vrije Universiteit Amsterdam
Jan H. van Schuppen
CWI Amsterdam

VU University Amsterdam

CWI Centrum Wiskunde & Informatica

# Outline

- Refresher on controllability for linear systems
- Coordinated systems
- Several reachability concepts
- Decomposition of the system
- Several controllability concepts
- Concluding remarks

# Refresher on controllability for linear systems

Linear system, only input-to-state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

$A$ and $n \times n$ matrix, $B$ an $n \times m$ matrix.

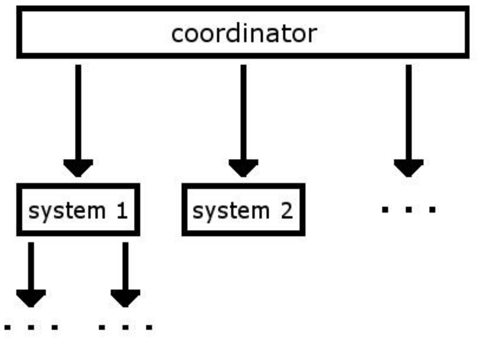
*Controllable* if for every two states $x_0$ and $x_1$ there is a time $T > 0$ and an input $u : [0, T] \to \mathbb{R}^m$ so that $x(T, x_0, u) = x_1$.

The state $x_1$ is called *reachable* from $x_0$.

Equivalent condition: Im $\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = \mathbb{R}^n$. Shall say: the pair $(A, B)$ is controllable.

*Pole placement theorem*: if $(A, B)$ is controllable then for every $n$-tuple of complex numbers $\Lambda$ there is an $F$ so that $\sigma(A+BF) = \Lambda$. In particular $(A, B)$ is stabilizable.

# Coordinated linear systems



What are coordinated linear systems:

- Hierarchical systems with top-down information structure

Motivation:

- Control synthesis can be done in a top-down manner

Applications:

- Mission control for groups / platoons of autonomous vehicles
- Control of road networks
- Control of electric power networks

# Definition

Given a linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and given decompositions

$$X = X_1 \dot{+} X_2 \dot{+} X_c,$$
$$U = U_1 \dot{+} U_2 \dot{+} U_c,$$

we call the system a *coordinated linear system* if

- $X_1$ and $X_2$ are $A$-invariant,
- $BU_1 \subseteq X_1$ and $BU_2 \subseteq X_2$.

Two subsystems, with state spaces $X_1$ and $X_2$, one coordinator, state space $X_c$.

## Matrix structure for coordinated systems

- $X_1$ and $X_2$ are $A$-invariant,
- $BU_1 \subseteq X_1$ and $BU_2 \subseteq X_2$.

Coordinated linear systems are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}.$$

# Some remarks

- Structure of the matrices is kept under addition and multiplication as well as under inversion.

# Some remarks

- Structure of the matrices is kept under addition and multiplication as well as under inversion.

- Dynamics of the system: $e^{At}B$ has the same type of structure.

## Some remarks

- Structure of the matrices is kept under addition and multiplication as well as under inversion.
- Dynamics of the system: $e^{At}B$ has the same type of structure.
- Extension to many subsystems easy.

# Some remarks

- Structure of the matrices is kept under addition and multiplication as well as under inversion.

- Dynamics of the system: $e^{At}B$ has the same type of structure.

- Extension to many subsystems easy.

- Extension to mutiple layers possible.

# Some remarks

- Structure of the matrices is kept under addition and multiplication as well as under inversion.

- Dynamics of the system: $e^{At}B$ has the same type of structure.

- Extension to many subsystems easy.

- Extension to mutiple layers possible.

- There is an explicit construction from an interconnected system to a coordinated control system, identifying the part of the interconnected system that acts as coordinator.

## Controllability decompositions

Coordinated linear system $\dot{x} = Ax + Bu$,

$$X = X_1 \dot{+} X_2 \dot{+} X_c,$$
$$U = U_1 \dot{+} U_2 \dot{+} U_c,$$

with $AX_i \subseteq X_i$ and $BU_i \subseteq X_i$ $(i = 1, 2)$.

$$A = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix}.$$

Denote by $\mathfrak{R}$ the controllable subspace

$$\mathfrak{R} = \mathrm{Im} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

## Problem

Which part of the system is controllable by using **which input**?

# Definitions

- For $i = 1, 2$ a state $\bar{x}_i \in X_i$ is called *$u_i$-reachable* if there is a time $T$ and an input function $u_i : [0, T] \to U_i$ such that the subsystem $\dot{x}_i = Ax_i + Bu_i, \quad x_i(0) = 0$ satisfies $x_i(T) = \bar{x}_i$.

- A state $\bar{x} \in X$ is called *$u_c$-reachable* if there is a time $T$ and an input function $u_c : [0, T] \to U_c$ such that the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} u_c, \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

satisfies $x(T) = \bar{x}$.

$$\mathfrak{R}_i = \{x_i \mid x_i \text{ is } u_i \text{ reachable}\} \subseteq X_i,$$
$$\mathfrak{R}_c = \{x \mid x \text{ is } u_c \text{ reachable}\} \subseteq X.$$

## Example

$$\dot{x}(t) = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] x(t) + \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] u(t),$$

with $X_1 = \text{span}\,\{e_1, e_2\}$, $X_2 = \text{span}\,\{e_3, e_4\}$, $X_c = \text{span}\,\{e_5\}$.

This system has $\mathfrak{R}_1 = \text{span}\,\{e_1\} \subset X_1$ and $\mathfrak{R}_2 = \text{span}\,\{e_3\} \subset X_2$.

To find $\mathfrak{R}_c$ consider the controllability matrix of

$$\left( \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix}, \begin{bmatrix} B_{1c} \\ B_{2c} \\ B_{cc} \end{bmatrix} \right)$$

## Example continued

$$\mathfrak{R}_c = \text{Im}\begin{bmatrix} B_{1c} & A_{11}B_{1c} + A_{1c}B_{cc} & A_{11}^2B_{1c} + A_{11}A_{1c}B_{cc} + A_{1c}A_{cc}B_{cc} & \dots \\ B_{2c} & A_{22}B_{2c} + A_{2c}B_{cc} & A_{22}^2B_{2c} + A_{22}A_{2c}B_{cc} + A_{2c}A_{cc}B_{cc} & \dots \\ B_{cc} & A_{cc}B_{cc} & A_{cc}^2B_{cc} & \dots \end{bmatrix}$$

$$= \text{Im}\begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \end{bmatrix} = \text{span}\,\{e_1 + e_3, e_2 + e_4, e_5\}.$$

- $X_c \subset \mathfrak{R}_c$, so the coordinator is $u_c$-reachable.

- $X_1 \cap \mathfrak{R}_c = \{0\}$ and $X_2 \cap \mathfrak{R}_c = \{0\}$, so no part of either of the subsystems is $u_c$-reachable.

- However, $(X_1 \dot{+} X_2) \cap \mathfrak{R}_c = \text{span}\,\{e_1 + e_3, e_2 + e_4\} \neq \{0\}$, so $X_1 \dot{+} X_2$ has a non-trivial $u_c$-reachable subspace. In fact, any state in $X_1$ can be reached via $u_c$, but then subsystem 2 will arrive at the same state, and vice versa.

# More definitions...

Lemma

$$\mathfrak{R} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 \dot{+} \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 \dot{+} \mathfrak{R}_c$$

Proof is straightforward.

Split $\mathfrak{R}_c$ according to $X = X_1 \dot{+} X_2 \dot{+} X_c$: define

- $X_i \cap \mathfrak{R}_c$ the *independently $u_c$-reachable states* in $X_i$,
- $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$ the *jointly $u_c$-reachable states* in $X_1$,
- $\begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c$ the *jointly $u_c$-reachable states* in $X_2$.

Observe:

$$(X_1 \cap \mathfrak{R}_c) \dot{+} (X_2 \cap \mathfrak{R}_c) \dot{+} (X_c \cap \mathfrak{R}_c) \subseteq \mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathfrak{R}_c \dot{+} \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c.$$

# Decomposition of Subsystem 1

$\mathfrak{R}_1 \subseteq X_1$ and $X_1 \cap \mathfrak{R}_c \subseteq \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c \subseteq X_1$.
Define the following subspaces:

- $X_1^1 = \mathfrak{R}_1 \cap (X_1 \cap \mathfrak{R}_c)$,
- $X_1^2$ is a complement of $X_1^1$ in $\mathfrak{R}_1 \cap \begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$,
- $X_1^3$ is a complement of $X_1^1 \dotplus X_1^2$ in $\mathfrak{R}_1$,
- $X_1^4$ is a complement of $X_1^1$ in $X_1 \cap \mathfrak{R}_c$,
- $X_1^5$ is a complement of $X_1^1 \dotplus X_1^2 \dotplus X_1^4$ in $\begin{bmatrix} I & 0 & 0 \end{bmatrix} \mathfrak{R}_c$,
- $X_1^6$ is a complement of $X_1^1 \dotplus X_1^2 \dotplus X_1^3 \dotplus X_1^4 \dotplus X_1^5$ in $X_1$.

| subspace | $u_1$-reachable | $u_c$-reachable |
|---|---|---|
| $X_1^1$ | yes | independently |
| $X_1^2$ | yes | only jointly |
| $X_1^3$ | yes | no |
| $X_1^4$ | no | independently |
| $X_1^5$ | no | only jointly |
| $X_1^6$ | no | no |

# The controllability decomposition for $(A, B)$

$$\left[\begin{array}{cccccc|cccccc|ccc}
A_{11}^{11} & A_{11}^{12} & A_{11}^{13} & A_{11}^{14} & A_{11}^{15} & A_{11}^{16} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{11} & A_{1c}^{12} & A_{1c}^{13} \\
0 & A_{11}^{22} & A_{11}^{23} & 0 & A_{11}^{25} & A_{11}^{26} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{21} & A_{1c}^{22} & A_{1c}^{23} \\
0 & A_{11}^{32} & A_{11}^{33} & 0 & A_{11}^{35} & A_{11}^{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{32} & A_{1c}^{33} \\
0 & 0 & 0 & A_{11}^{44} & A_{11}^{45} & A_{11}^{46} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{41} & A_{1c}^{42} & A_{1c}^{43} \\
0 & 0 & 0 & 0 & A_{11}^{55} & A_{11}^{56} & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{51} & A_{1c}^{52} & A_{1c}^{53} \\
0 & 0 & 0 & 0 & A_{11}^{65} & A_{11}^{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1c}^{62} & A_{1c}^{63} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{11} & A_{22}^{12} & A_{22}^{13} & A_{22}^{14} & A_{22}^{15} & A_{22}^{16} & A_{2c}^{11} & A_{2c}^{12} & A_{2c}^{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{22} & A_{22}^{23} & 0 & A_{22}^{25} & A_{22}^{26} & A_{2c}^{21} & A_{2c}^{22} & A_{2c}^{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{32} & A_{22}^{33} & 0 & A_{22}^{35} & A_{22}^{36} & 0 & A_{2c}^{32} & A_{2c}^{33} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{44} & A_{22}^{45} & A_{22}^{46} & A_{2c}^{41} & A_{2c}^{42} & A_{2c}^{43} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{55} & A_{22}^{56} & A_{2c}^{51} & A_{2c}^{52} & A_{2c}^{53} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^{65} & A_{22}^{66} & 0 & A_{2c}^{62} & A_{2c}^{63} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{11} & A_{cc}^{12} & A_{cc}^{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{21} & A_{cc}^{22} & A_{cc}^{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{cc}^{33}
\end{array}\right], \quad
\left[\begin{array}{c|c|c}
B_{11}^{1} & 0 & B_{1c}^{1} \\
B_{11}^{2} & 0 & B_{1c}^{2} \\
B_{11}^{3} & 0 & 0 \\
0 & 0 & B_{1c}^{4} \\
0 & 0 & B_{1c}^{5} \\
0 & 0 & 0 \\
\hline
0 & B_{22}^{1} & B_{2c}^{1} \\
0 & B_{22}^{2} & B_{2c}^{2} \\
0 & B_{22}^{3} & 0 \\
0 & 0 & B_{2c}^{4} \\
0 & 0 & B_{2c}^{5} \\
0 & 0 & 0 \\
\hline
0 & 0 & B_{cc}^{1} \\
0 & 0 & B_{cc}^{2} \\
0 & 0 & 0
\end{array}\right]$$

# Local controllability

We call a CLS *locally controllable* if all parts of the system are controllable via their local inputs, i.e. if

$$\mathfrak{R}_1 = X_1, \mathfrak{R}_2 = X_2, \text{ and } X_c \subseteq \begin{bmatrix} 0 & 0 & I \end{bmatrix} \mathfrak{R}_c.$$

Control synthesis can be done in a fully decentralized manner:

- $u_c$ can be found independently of the subsystems
- once $u_c$ is fixed, the subsystems can do control synthesis locally.

Useful in large systems as it reduces complexity of control synthesis.

Necessary and sufficient for pole placement.

# Joint controllability

We call a CLS *jointly controllable* if each part of the system is controllable via either of the control inputs, i.e. if

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathfrak{R}_1 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathfrak{R}_2 + \mathfrak{R}_c = X.$$

Equivalent to usual controllability.

This concept is not very useful:

Suppose $\mathfrak{R}_c = \mathrm{Im} \begin{bmatrix} I \\ I \\ I \end{bmatrix}$, then reaching $\begin{bmatrix} 0 \\ 0 \\ x_c \end{bmatrix}$ means reaching $\begin{bmatrix} x_c \\ x_c \\ x_c \end{bmatrix}$ via $u_c$ , and then steering the subsystem states back from $x_c$ to 0 via the local inputs.

# Independent controllability

More useful: We call a CLS *independently controllable* if each part of the system is independently controllable via either of the control inputs, i.e. if

$$(\mathfrak{R}_1 + (X_1 \cap \mathfrak{R}_c)) \dot{+} (\mathfrak{R}_2 + (X_2 \cap \mathfrak{R}_c)) \dot{+} (X_c \cap \mathfrak{R}_c) = X.$$

Each part of the system can be controlled in such a way that no other part of the system is influenced.

# Remarks

- The three concepts just introduced can be characterized in terms of the decomposition.

- Analogous decomposition for observability is much more involved 54 subspaces in the state space!!

- Application to LQ-optimal control for coordinated linear systems.

- Applications: Automated Underwater Vehicles, Unmanned Aerial Vehicles and coordinated ramp metering for highways.

# Acknowledgments

Research partially funded by
European Community's Seventh Framework Programme
(FP7/2007-2013) under grant agreement no. INFSO-ICT-223844.

C4C project. Gratefully acknowledge discussions with and support from other team members of the C4C team, in particular with respect to real-life applications.

# References

A.C.M. Ran and J.H. van Schuppen,
Control for Coordination of Linear Systems.
In Proc. International Symposium on the Mathematical Theory of Networks and Systems (MTNS.2008), Blacksburg, 2008.

P.L. Kempker, A.C.M. Ran and J.H. van Schuppen,
Construction of a Coordinator for Coordinated Linear Systems.
In Proc. European Control Conference (ECC.2009), Budapest, 2009.

P.L. Kempker, A.C.M. Ran and J.H. van Schuppen.
Controllability and observability of coordinated linear systems.
Submitted.

P.L. Kempker, A.C.M. Ran and J.H. van Schuppen.
A formation flying algorithm for autonomous underwater vehicles.
In Proc. CDC/ECC 2011.

# Thank you for your attention

# Questions?