Right invertible Toeplitz operators and stable rational solutions to an associate Bezout equation

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Problem statement

Given a *stable* rational $m \times p$ matrix function $G(z)$.

Stable: G is in H^∞, that is, analytic on the closed unit disc.

Generally $m \geq p$ so G has more columns than rows.

Looking for a stable $p \times m$ rational matrix function $X(z)$ such that

$$G(z)X(z) = I_m, \qquad z \in \mathbb{D}.$$

For general H^∞ functions this is connected to the corona theorem (Carleson) and its matrix valued extension (due to Fuhrmann).

For rational matrix functions there is also a connection to coprime factorizations (see, e.g., Vidyasagar's book).

Connection to operator theory

Consider the vector valued Hardy spaces $H^2(\mathbb{C}^p)$ and $H^2(\mathbb{C}^m)$ of vector valued functions that are (norm-wise) square integrable on the unit circle and analytic in $\mathbb{D}$.

Let $M_G : H^2(\mathbb{C}^p) \to H^2(\mathbb{C}^m)$ be the operator of multiplication by G: $(M_G f)(z) = G(z)f(z)$.

The operator version of the corona theorem implies there exists an H^∞ solution $X(z)$ to $G(z)X(z) = I_m$ if and only if M_G is right invertible.

Necessity of right invertibility of M_G is trivial: $M_G M_X = I$. Sufficiency can be proved in several ways, for instance using the commutant lifting theorem.

For rational G existence of just any H^∞ solution implies existence of a rational stable solution by an (involved) approximation argument.

Intermezzo: right invertible operators and right inverses

Let $T : H_1 \to H_2$ be a right invertible operator acting between two Hilbert spaces. Then T is onto, and T^* is 1–1.

Claim. TT^* is invertible when $T = M_G$ is onto.

Put $X = T^*(TT^*)^{-1}$. Then $TX = TT^*(TT^*)^{-1} = I$. So, X is a right inverse.

Moreover, $XT = T^*(TT^*)^{-1}T$ is an orthogonal projection onto $\operatorname{Im} X = \operatorname{Im} T^* = (\ker T)^{\perp}$.

Intermezzo: right invertible operators and right inverses

Finally, if V is any other right inverse, then $T(V - X) = 0$, that is, $\text{Im}\,(V - X) \subset \ker T$. So, for all $x \in H_2$ we have

$$\langle (V - X)x, Xx \rangle = 0, \text{ i.e., } \langle Vx, Xx \rangle = \|Xx\|^2.$$

Then

$$0 \leq \langle (V - X)x, (V - X)x \rangle = \|Vx\|^2 - \|Xx\|^2,$$

which means

$$\|Xx\|^2 \leq \|Vx\|^2.$$

X is the *Moore-Penrose* right inverse.

Back to our problem

$G(z)X(z) = I_m$ if and only if M_G is right invertible.

Right inverse of M_G is $M_G^*(M_G M_G^*)^{-1}$.

Let $E : \mathbb{C}^m \to H^2(\mathbb{C}^m)$ be the canonical embedding (Ey is the constant function y).

Put $X(\cdot)y = M_G^*(M_G M_G^*)^{-1}Ey$. Then:

- $G(z)X(z) = I_m$,
- $X(z)$ is a rational function,
- $\|X(\cdot)u\|_{H^2(\mathbb{C}^p)} \leq \|V(\cdot)u\|_{H^2(\mathbb{C}^p)}$ for any other stable rational V for which $G(z)V(z) = I_m$, with equality if and only if $V = X$. *Least squares solution.*

Connection with Toeplitz operators

Fourier transform $\mathcal{F}$ takes $\ell^2_+(\mathbb{C}^m)$ unitarily onto $H^2(\mathbb{C}^m)$.

For an $m \times p$ function F, entries in $L^\infty(\mathbb{T})$, with $F(z) = \sum_{j=-\infty}^{\infty} z^j F_j$, $|z| = 1$, the *Toeplitz* operator T_F is

$$T_F = \begin{bmatrix} F_0 & F_{-1} & F_{-2} & \cdots \\ F_1 & F_0 & F_{-1} & \cdots \\ F2 & F_1 & F_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m)$$

and the *Hankel* operator H_F is

$$H_F = \begin{bmatrix} F_1 & F_2 & F_3 & \cdots \\ F_2 & F_3 & F_4 & \cdots \\ F_3 & F_4 & F_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m).$$

$M_G \mathcal{F}_{\mathbb{C}^p} = \mathcal{F}_{\mathbb{C}^m} T_G.$

Laurent operators

Laurent operator L_F connected to F

$$L_F = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & F_1 & F_0 & F_{-1} & F_{-2} & \cdots \\ \cdots & F_2 & F_1 & F_0 & F_{-1} & \cdots \\ \cdots & F_3 & F_2 & F_1 & F_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2(\mathbb{C}^p) \to \ell^2(\mathbb{C}^m)$$

Define $J : \ell^2 \to \ell^2_+ \oplus \ell^2_+$ by

$$J\begin{pmatrix} \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots \end{pmatrix} = \begin{pmatrix} x_{-1} & x_{-2} & \cdots & | & x_0 & x_1 & x_2 & \cdots \end{pmatrix}.$$

Then J is unitary. Further:

$$JL_FJ^* = \begin{bmatrix} T_F^\# & H_{F_-} \\ H_F & T_F \end{bmatrix}.$$

Product of two Toeplitz operators

Recall, we need for the Moore Penrose inverse of M_G also $M_G M_G^*$. So we are interested in $T_G T_G^*$. Since G is stable L_G is lower triangular, and $L_G^* = L_{G^*}$ is upper triangular.

$$G(z) = G_0 + zG_1 + z^2 G_2 + \cdots \quad G^*(z) = G_0 + z^{-1}G_1^* + z^{-2}G_2^* + \cdots .$$

Then

$$JL_G J^* = \begin{bmatrix} T_G^{\#} & 0 \\ H_G & T_G \end{bmatrix}, \quad JL_{G^*}J^* = \begin{bmatrix} T_{G^*}^{\#} & H_G^* \\ 0 & T_G^* \end{bmatrix}.$$

Observe also that $T_G^* = T_{G^*}$ and $L_G^* = L_{G^*}$. Since $L_G L_{G^*} = L_{GG^*}$ we can compute T_{GG^*} by considering the product of L_G and L_G^*.

$$JL_G J^* JL_{G^*} J^* = \begin{bmatrix} \star & \star \\ \star & T_{GG^*} \end{bmatrix} = \begin{bmatrix} T_G^{\#} & 0 \\ H_G & T_G \end{bmatrix} \begin{bmatrix} T_{G^*}^{\#} & H_G^* \\ 0 & T_G^* \end{bmatrix} = \begin{bmatrix} \star & \star \\ \star & T_G T_G^* + H_G H_G^* \end{bmatrix}.$$

In conclusion:

$$T_{GG^*} = T_G T_G^* + H_G H_G^*.$$

Properties of Toeplitz and Hankel

With $\widetilde{E} : \mathbb{C}^p \to \ell^2_+(\mathbb{C}^p)$ the embedding into the first coordinate we have $\mathcal{F}_{\mathbb{C}^p}\widetilde{E} = E$.

$M_G^*(M_G M_G^*)^{-1}\mathcal{F}_{\mathbb{C}^m} = \mathcal{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}$.

We will need also $R(z) = G(z)G^*(z)$. We have:

$$T_R = T_{GG^*} = T_G T_G^* + H_G H_{G^*},$$

and if all inverses in the next formula exist, then

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1}.$$

Operator theory background of main result

Proposition *The following statements are equivalent.*

(a) *The equation* $G(z)X(z) = I_m$ *has a stable rational matrix solution.*

(b) *The Toeplitz operator* T_G *is right invertible.*

(c) *The Toeplitz operator* T_R *is invertible and the same holds true for the operator* $I - H_G^* T_R^{-1} H_G$.

Moreover, if one of these conditions is satisfied, then $T_G T_G^*$ *is invertible, its inverse is given by*

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1},$$

and the function $X = \mathcal{F}_{\mathbb{C}^p} T_G^* (T_G T_G^*)^{-1} \widetilde{E}$ *is a stable rational matrix function satisfying* $G(z)X(z) = I_m$.

Realization

To actually compute things, we need a *realization* of G

$$G(z) = D + zC(I - zA)^{-1}B.$$

We assume A is stable, that is, all eigenvalues of A are in $\mathbb{D}$. Then $(I - zA)^{-1} = I + zA + z^2A^2 + \cdots$, so

$$G(0) = D, G_1 = CB, G_2 = CAB, G_3 = CA^2B, \cdots .$$

Then the Hankel operator H_G is given by

$$H_G = \begin{bmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & CA^3B & \cdots \\ CA^2B & CA^3B & CA^4B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}$$
$$= W_{obs} \quad W_{con}.$$

Computation of T_R

$G(z) = D + zC(I - zA)^{-1}B$, and $R(z) = G(z)G^*(z)$. Can compute a realization for R. Need the solution P of

$$P - APA^* = BB^*.$$

Note: $P = \sum_{j=0}^{\infty} A^j BB^*(A^*)^j = W_{con}W_{con}^*$.

Set $R_0 = DD^* + CPC^*, \Gamma = BD^* + APC^*$, then

$$R(z) = zC(I_n - zA)^{-1}\Gamma + R_0 + \Gamma^*(zI_n - A^*)^{-1}C^*.$$

Invertibility of T_R

Earlier result of Frazho, Kaashoek, R.: T_R is invertible if and only if there is a solution Q to the discrete algebraic Riccati equation

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$$

with the additional properties that
(a) $R_0 - \Gamma^*Q\Gamma$ is positive definite,
(b) $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable.

Moreover, the solution Q is unique and hermitian. In fact

$$Q = W_{obs}^* T_R^{-1} W_{obs}.$$

In addition, formulas for the inverse of T_R are given in terms of A, B, C, D and P and Q (also involving Γ and R_0).

Next step

Recall:

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1},$$

So, we need $H_G(I - H_G^* T_R^{-1} H_G)^{-1} H_G^*$. Use $H_G = W_{obs} W_{con}$:

$$I - H_G^* T_R^{-1} H_G = I - W_{con}^* W_{obs}^* T_R^{-1} W_{obs} W_{con} = I - W_{con}^* Q W_{con}.$$

Observe that this is invertible if and only if $I_n - W_{con} W_{con}^* Q = I - PQ$ is invertible. In addition:

$$H_G(I - H_G^* T_R^{-1} H_G)^{-1} H_G^* = W_{obs} W_{con} (I - W_{con}^* Q W_{con})^{-1} W_{con}^* W_{obs}^*$$
$$= W_{obs} (I - PQ)^{-1} P W_{obs}^*.$$

So,

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} W_{obs} (I - PQ)^{-1} P W_{obs}^* T_R^{-1}.$$

....after a lot of manipulations

Theorem $G(z)X(z) = I_m$ *has a stable rational matrix solution if and only if*

(i) *the discrete algebraic Riccati equation*

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$$

has a solution Q*,with the properties*

(a) $R_0 - \Gamma^*Q\Gamma$ *is positive definite,*

(b) *the matrix* $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ *is stable;*

(ii) *the matrix* $I_n - PQ$ *is non-singular.*

Moreover, (i) *and* (ii) *are equivalent to* M_G *being right invertible.*

Formula for the solution $X(z)$

If (i) and (ii) hold, then the $p \times m$ matrix-valued function X defined by $X(\cdot)y = M_G^*(M_G M_G^*)^{-1}Ey$ is a stable rational matrix solution, and X is given by

$$X(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)D_1,$$

where

$$A_0 = A - \Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$C_1 = D^* C_0 + B^* QA_0,$$
$$\text{with } C_0 = (R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$D_1 = (D^* - B^* Q\Gamma)(R_0 - \Gamma^* Q\Gamma)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*.$$

Finally, X is the least squares solution, the McMillan degree of X is less than or equal to the McMillan degree of G, and

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{it})^* X(e^{it})\,dt = D_1^*\Big(I_p + B^*Q(I_n - PQ)^{-1}B\Big)D_1.$$

... and some more...

- There is also a parametrization of all right inverses, explicitly in terms of the matrices occuring in the realization of G, and the matrices introduced in the previous theorem.

- Connection with coprime factorizations, with Tolokonnikov's lemma are all discussed in detail in the papers.

- Everything is explicitly computable using Matlab, even for examples with large degrees.

- Analogue for Wiener-Hopf operators is in preparation.

References

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Thank you for your attention