Something that is not in the book: separation of variables

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The problem

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $[0, 1]$ with boundary conditions $u(0, t) = u(1, t) = 0$ for all t and initial condition

$$u(x, 0) = \sin^2(\pi x).$$

We also want $\lim_{t\to\infty} u(x, t) = 0$.

The equation is known as the heat equation.

Separation of variables

Separating the variables, we try first solutions of the form

$$u(x,t) = X(x)T(t).$$

Insert in the equation to obtain

$$X(x)T'(t) = X''(x)T(t),$$

resulting in

$$X''(x) + cX(x) = 0,$$
$$T'(t) = -cT(t).$$

Simple solutions

Use condition at $t \to \infty$ to see $c > 0$, put $c = \omega^2$.

Solutions are

$$\begin{aligned} X(t) &= \quad a\cos(\omega x) + b\sin(\omega x), \qquad a, b \in \mathbb{R}, \\ T(t) &= \qquad\qquad d\exp(-\omega^2 t), \qquad d \in \mathbb{R}. \end{aligned}$$

So $u(x,t) = e^{-\omega^2 t}(a\cos(\omega x) + b\sin(\omega x))$.

Boundary conditions

$u(x,t) = e^{-\omega^2 t}(a\cos(\omega x) + b\sin(\omega x))$

Use $u(0,t) = 0$ for all t, to obtain $a = 0$, use $u(1,t) = 0$ for all t to obtain $\omega = k\pi$ for some $k \in \mathbb{N}$.

Equation is linear: linear combinations of solutions are also solutions. So

$$u(x,t) = \sum_{k=1}^{m} e^{-(k\pi)^2 t} b_k \sin(k\pi x)$$

is a solution satisfying the boundary conditions, and satisfying $\lim_{t\to\infty} u(x,t) = 0$.

Initial condition

$$u(x,t) = \sum_{k=1}^{m} e^{-(k\pi)^2 t} b_k \sin(k\pi x)$$

But such a function for $t = 0$ will not satisfy just any given initial condition $u(x,0) = f_0(x)$. However, consider

$$u(x,t) = \sum_{k=1}^{\infty} e^{-(k\pi)^2 t} b_k \sin(k\pi x).$$

One can prove: such a function is again a solution, provided that the numbers $b_1, b_2, b_3, \cdots$ are sufficiently well-behaved.

Fourier coefficients

$$u(x,t) = \sum_{k=1}^{\infty} e^{-(k\pi)^2 t} b_k \sin(k\pi x)$$

Then $u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) f_0(x)$. From Fourier theory:

$$b_k = 2\int_0^1 f_0(s)\sin(k\pi s)ds.$$

The numbers b_k are “sufficiently well-behaved” if and only if f_0 is.
With $f_0(x) = \sin^2(\pi x) = \frac{1}{2}(1-\cos(2\pi x))$ we get $b_k = 0$ for even k, and

$$b_{2k-1} = \frac{8}{\pi(2k-1)(4-(2k-1)^2)}.$$

Matlab code

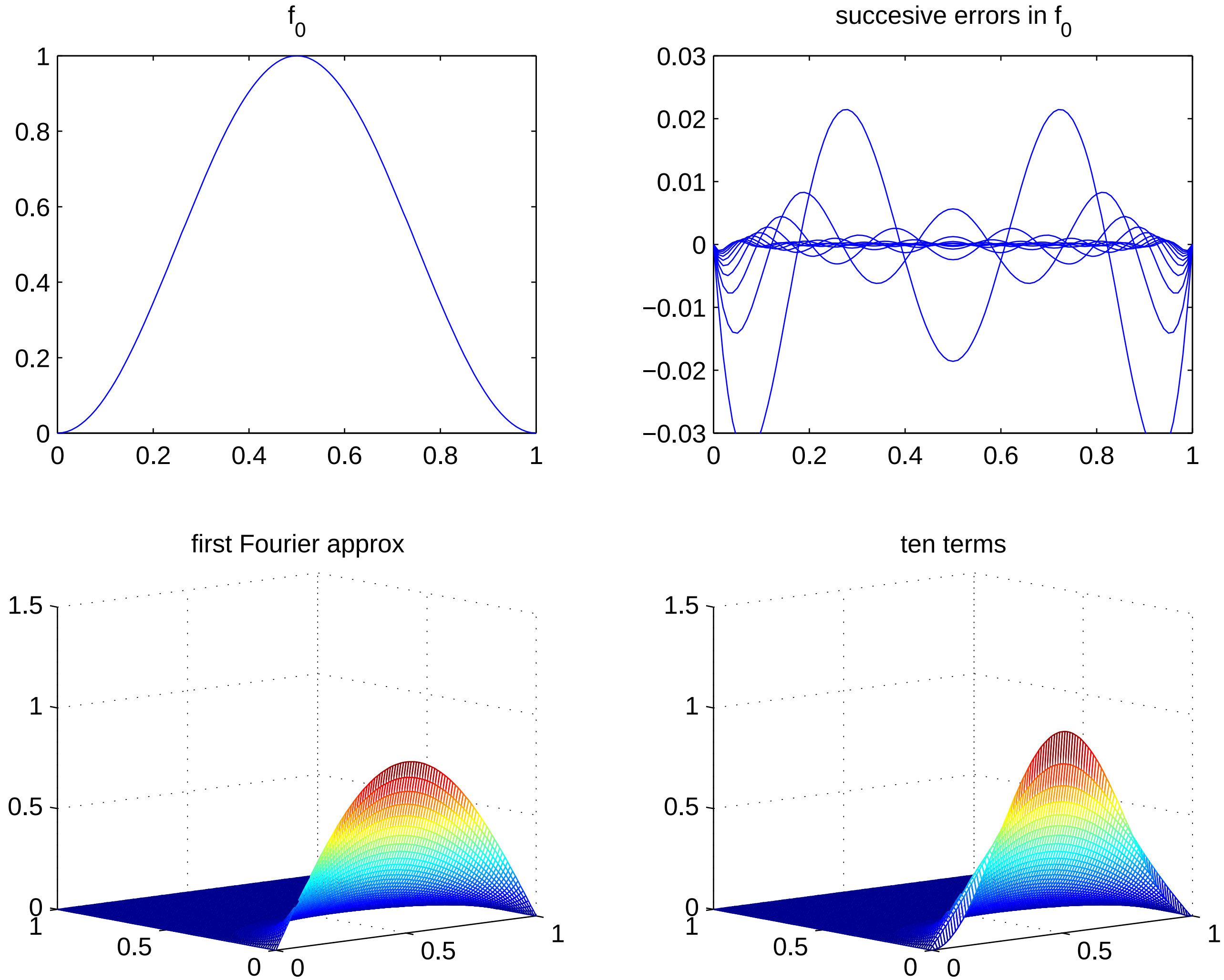

```
x=[0:0.01:1]; t=[0:0.01:1];
u0=(sin(pi*x)).^2;
subplot(2,2,1), plot(x,u0)

[X,T]=meshgrid(x,t);
u1=(8/(3*pi))*(sin(pi*X)).*(exp(-(pi)^2*T));
subplot(2,2,3), mesh(X,T,u1), axis([0 1 0 1 0 1.5])

for j=0:1:9
view(-j*10,10) pause(1)
end
view(-40,10)
```

```
u2=u1;
for k=1:1:10
u2oud=u2;
u2=u2+ (8/(pi*(2*k+1)*(4-(2*k+1)^2)))*
  *(sin((2*k+1)*pi*X)).*(exp(-(pi)^2*T*(2*k+1)^ 2));
subplot(2,2,4),mesh(X,T,u2), axis([0 1 0 1 0 1.5]), view(-40,10)
u20=u2(1,:);
subplot(2,2,2), plot(x,u0-u20), axis([0 1 -0.03 0.03]), hold on
max(u0-u20) pause(4)
end

diff_of_last_two_iterations =max(max(u2-u2oud))
```



f0
succesive errors in f0
first Fourier approx
ten terms