Inertia theorems for infinite dimensional operators on a Hilbert space

Leonid Lerer, Igor Margulis
Technion, Israel,
André Ran
Vrije Universiteit Amsterdam

Introduction

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$\nu(A) =$ no. of eigenvalues in the *left* half plane

$\delta(A) =$ no. of eigenvalues on the imaginary axis.

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Theorem (Carlson and Schneider) *Let A and $X = X^*$ be such that*

$$AX + XA^* = W \geq 0.$$

(i) If $\delta(A) = 0$, then $\pi(X) \leq \pi(A)$ and $\nu(X) \leq \nu(A)$.

(ii) If X is nonsingular, then $\pi(A) \leq \pi(X)$ and $\nu(A) \leq \nu(X)$.

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(ii) If X is nonsingular, then $\pi(A) \leq \pi(X)$ and $\nu(A) \leq \nu(X)$.

(iii) From (i) and (ii) it follows that if $\delta(A) = \delta(X) = 0$, then $\pi(X) = \pi(A)$ and $\nu(X) = \nu(A)$.

Goal of the talk

Goal of this talk: present infinite dimensional generalizations of this theorem, and some applications.

Matrix case

Theorem (Ostrowski-Schneider, M. Krein, O. Taussky) *Let A and $X = X^*$ be such that*

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with W (uniformly) positive, then X is nonsingular and

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$$\delta(A) = \delta(X) = 0,$$

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Conversely, if $\delta(A) = 0$*, then there exists a Hermitian matrix* X *such that the equations above hold.*

Enters controllability

A pair of operators (A, B), where $A \in L(\mathcal{H})$ and $B \in L(\mathcal{G}, \mathcal{H})$, is called *almost exactly controllable* if for some $p \in \mathbb{N}$ the linear set

$$\mathrm{Im}\,[B, AB, A^2B, \ldots, A^{p-1}B] = \sum_{j=0}^{p-1} \mathrm{Im}\,A^jB$$

is closed and has finite codimension in $\mathcal{H}$.

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A pair (A, B) is called *approximately controllable* if

$$\overline{\mathrm{Im}\,[B, AB, A^2B, A^3B \ldots]} = \mathcal{H}$$

Chen-Wimmer theorem

Theorem (Chen-Wimmer) *Let* A *and* X *be* $n \times n$ *complex matrices such that* $X = X^*$,

$$AX + XA^* = W \geq 0$$

and let the pair (A, W) *be exactly controllable. Then*

$$\delta(A) = \delta(X) = 0, \quad \pi(A) = \pi(X), \quad \nu(A) = \nu(X).$$

Notation

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Define

$$\nu(A) = \sum_{\lambda_0 \in \sigma(A) \cap \mathbb{C}_-} \dim \operatorname{Im} P_{\{\lambda_0\}}(A)$$

if the intersection $\sigma(A) \cap \mathbb{C}_-$ consists only of a finite number of regular eigenvalues of A.

Otherwise, put $\nu(A) = \infty$.

Similarly for $\pi(A)$.

Infinite dimensional case

$$AX + XA^* = W \geq 0$$

Infinite dimensional results for bounded operators by:

- J.W. Bunce, assuming (A, W) exactly controllable,
- L. Lerer and L. Rodman, assuming almost exact controllability.

Infinite dimensional case continued

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Theorem (Sasane-Curtain) *Let A be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Suppose that*

- $\sigma = \sigma(A) \cap \mathbb{C}^+$ *is a bounded spectral set of* A,
- $\dim \operatorname{Im} P_\sigma(A) < \infty$,
- $H = H^* \in L(\mathcal{H})$ *such that* $0 \notin \sigma_p(H)$, $\nu(H) < \infty$

and

$$\langle (A^*H + HA)x, x \rangle \leq 0 \quad \text{for all} \quad x \in D(A)$$

Then $\pi(A) \leq \nu(H)$.

Main results I

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Theorem *Let* $A, H \in L(\mathcal{H})$ *be bounded linear operators, such that* H *is self-adjoint and invertible,* $\nu(H) < \infty$*, the spectrum of* A *does not contain eigenvalues which lie on the imaginary axis, and* $\sigma(A) \cap \mathbb{C}^-$ *is a spectral set. If*

$$A^*H + HA \geq 0$$

then $\nu(H) = \nu(A)$.

Main results II

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Full generalization of part (iii) of the Carlson-Schneider result for the unbounded case:

Theorem *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear, densely defined closed operator with domain $\mathcal{D}(A)$. Suppose, $H \in L(\mathcal{H})$ is a self-adjoint invertible operator such that $\nu(H) < \infty$ and*

$$\langle (A^* H + HA)x, x \rangle \geq 0, \qquad \forall x \in \mathcal{D}(A).$$

Assume, in addition, that A is boundedly invertible, the spectrum of A does not contain eigenvalues which lie on the imaginary axis, and $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set.
Then $\nu(H) = \nu(A)$.

Indefinite inner product spaces

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An operator A on X is called H-dissipative if

$$\text{Im}\,[Ax, x] \geq 0.$$

H-dissipative operators

Now recall the main result on H-dissipative operators. It is important in the proof of the main theorems.

Theorem *Let A be a bounded, dissipative operator on the Pontryagin space Π_κ.*
Assume that $\sigma(A) \cap \mathbb{R}$ consists entirely of points λ_0 for which $\dim \ker(A - \lambda_0 I) = 0$.
Then there is a κ-dimensional A-invariant subspace M which is maximal nonpositive, and is such that $\sigma(A|_M)$ is in the open lower half plane.

Sketch of proof I

$H = H^*$ invertible and $\kappa = \nu(H) < \infty$ implies that $[x, y] = \langle Hx, y \rangle$ induces on $\mathcal{H}$ an indefinite inner product and makes it into a Π_κ space.

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We conclude that there exists a κ-dimensional, A-invariant, maximal H-non-positive subspace M, such that $\sigma(A|_M)$ is entirely in $\mathbb{C}^-$. On the other hand, since $\sigma = \sigma(A) \cap \mathbb{C}^-$ is a spectral set, $\mathrm{Im}\, P_\sigma(A)$ is a non-positive A-invariant subspace. From the maximality of M, it follows that $\mathrm{dim} \mathrm{Im}\, P_\sigma(A) = \kappa$.

Sketch of proof II

From the fact that A is boundedly invertible it follows that

$$(A^{-1})^*(A^*H + HA)A^{-1} \geq 0.$$

Hence

$$(A^{-1})^*H + HA^{-1} \geq 0.$$

Sketch of proof II cont.

$\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set so $\sigma(A^{-1}) \cap \mathbb{C}^-$ is a spectral set.

$0 \neq \lambda \in \sigma_p(A)$ if and only if $\frac{1}{\lambda} \in \sigma_p(A^{-1})$

A boundedly invertible gives $0 \notin \sigma_p(A) \cup \sigma_p(A^{-1})$.

So A^{-1} does not have eigenvalues on imaginary axis.

So, applying the first theorem to the operator A^{-1}, we conclude that

$$\nu(H) = \nu(A^{-1})$$

But $\nu(A^{-1}) = \nu(A)$, and the theorem follows.

Special case

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Theorem *Let K be a compact selfadjoint operator such that 1 is not an eigenvalue of K. Further, let A be bounded, and assume that A has no eigenvalues on the imaginary axis, and that $\sigma(A) \cap \mathbb{C}^-$ is a spectral set. If*

$$A^*(I - K) + (I - K)A \geq 0$$

then $\nu(I - K) = \nu(A)$.

Applications I

Analogue of the Chen-Wimmer result.

Theorem *Let $A, H \in L(\mathcal{H})$ be bounded linear operators, such that $\sigma(A) \cap \mathbb{C}^-$ is a spectral set, H is self-adjoint and invertible and $\nu(H) < \infty$. If*

$$A^*H + HA = BB^*$$

and the pair (A, B) is approximately controllable, then $\nu(H) = \nu(A)$.

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and the pair (A, B) is approximately controllable, then $\nu(H) = \nu(A)$.

Proof: show that the approximate controllability guarantees that A does not have eigenvalues on the imaginary axis, and then use the earlier result.

Applications II

A semigroup $T(t)$, $t \geq 0$ is called hyperbolic if the space $\mathcal{H}$ decomposes into $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm$ are invariant under $T(t)$, and the restriction $T_+(t)$, $t \geq 0$ of $T(t)$ to $\mathcal{H}_+$ is exponentially stable, while the restriction $T_-(t)$, $t \geq 0$ extends to a strongly continuous group on $\mathcal{H}_-$, defined by $T(-t) = T(t)^{-1}$, $t \geq 0$.

Applications II cont.

$E(t)$ is called a *bisemigroup* on the Hilbert space $\mathcal{H}$ if there is a bounded projection operator P, such that the restriction

$$E(t)|_{\mathrm{Ker}\,P}, \qquad t \geq 0$$

is a strongly continuous semigroup on $\mathrm{Ker}\,P$, while the restriction

$$-E(-t)|_{\mathrm{Im}\,P}, \qquad t \geq 0$$

is a strongly continuous semigroup on $\mathrm{Im}\,P$, and, moreover, both these semigroups are exponentially stable.

Applications II cont.

Let S_+, S_- be the closed, densely defined linear operators on $\mathrm{Ker}\, P$ and $\mathrm{Im}\, P$ respectively, such that
S_+ is the generator of the semigroup $E(t)|_{\mathrm{Ker}\, P}$, where $t \geq 0$,
and
$-S_-$ be the generator of the semigroup $-E(-t)|_{\mathrm{Im}\, P}$, where $t \geq 0$.

The operator S with domain $D(S) = D(S_+) \oplus D(S_-)$ defined by $S(x_+ + x_-) = S_+x_+ - S_-x_-$, where $x_+ \in D(S_+)$ and $x_- \in D(S_-)$, is called the *generator* of the bisemigroup.

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The generator S of the bisemigroup has the following spectral property: the resolvent set of S contains a strip around the imaginary axis.

Applications II cont.

Theorem *Let A be the generator of a bisemigroup. Assume that $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set. Suppose further that there exists an invertible self-adjoint operator $H \in L(\mathcal{H})$ with $\nu(H) < \infty$ such that*

$$\langle (A^*H + HA)x, x\rangle \geq 0, \qquad \forall x \in D(A).$$

Then $\nu(A) = \nu(H) < \infty$, and in particular, A generates a hyperbolic semigroup.
That is, A_- generates a strongly continuous group.

Applications II cont.

Theorem *Let A be the generator of a bisemigroup. Assume that $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set. Suppose further that there exists an invertible self-adjoint operator $H \in L(\mathcal{H})$ with $\nu(H) < \infty$ such that*

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Then $\nu(A) = \nu(H) < \infty$, and in particular, A generates a hyperbolic semigroup.
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Proof. All that is to prove are the spectral properties of A. Clearly, since A is the generator of a bisemigroup there is a strip around the imaginary axis in the resolvent set of A. This shows that A is invertible, and no point on the imaginary axis is an eigenvalue of A. ☐

Thank you for your attention