Stability of invariant subspaces of matrices

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LQ-optimal control in discrete time. System is given

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad t \in \mathbb{N}_0, \\ x(0) &= x_0, \end{aligned}$$

with a cost function

$$J(x_0, u) = \sum_{t=0}^{\infty} x(t)^* Q x(t) + u(t)^* R u(t).$$

Here we assume (A, B) controllable, $Q \geq 0$, $R > 0$. Goal is to find the minimum of $J(x_0, u)$ over all stabilizing input sequences $\{u(t)\}_{t=0}^{\infty}$.

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Riccati equation

The answer involves a quadratic matrix equation; the discrete algebraic Riccati equation

$$X = Q + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA.$$

If X is the solution for which $A - B(R + B^*XB)^{-1}B^*XA$ has all its eigenvalues in the open unit disc (when this exists), then

$$u(t) = -(R + B^*XB)^{-1}B^*XAx(t)$$

is the input achieving the minimum in $J(x_0, u)$, and the minimum is then given by $x_0^*Xx_0$ for this solution X.
So the problem reduces to finding X.

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Riccati and invariant subspace

$$X = Q + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA.$$

Invariant subspace problem. For sake of simplicity assume A invertible. Put

$$T = \begin{pmatrix} A + BR^{-1}B^*(A^*)^{-1}Q & -BR^{-1}B^*(A^*)^{-1} \\ -(A^*)^{-1}Q & (A^*)^{-1} \end{pmatrix}, \; J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Also put $M = \operatorname{Im} \begin{pmatrix} I \\ X \end{pmatrix}$. Then $T^*JT = J$, and

- X solves the discrete algebraic Riccati equation if and only if $TM \subset M$,

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- X solves the discrete algebraic Riccati equation if and only if $TM \subset M$,
- $X = X^*$ if and only if $JM = M^\perp$,

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Also put $M = \operatorname{Im} \begin{pmatrix} I \\ X \end{pmatrix}$. Then $T^*JT = J$, and

- X solves the discrete algebraic Riccati equation if and only if $TM \subset M$,
- $X = X^*$ if and only if $JM = M^\perp$,
- $T|M$ is similar to $A - B(R + B^*XB)^{-1}B^*XA$.

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Quadratic eigenvalue problems

$$M\ddot{x} + D\dot{x} + Sx = F$$

with $M = M^T$, $S = S^T$ and $D = D^T$.
Concrete example: model for vibrations of a railtrack. Finite element model of a sleeperbay leads to such a model, where M, D and S are large block tri-diagonal matrices. To find the eigenmodes, use also the periodicity: need to find λ so that

$$\det\left(\frac{1}{\lambda}A_1^T + A_0 + \lambda A_1\right) = 0.$$

Here $A_0 = A_0^T$ is complex symmetric, and A_1 is highly rank deficient.
Palindromic eigenvalue problem.

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Translate to invariant subspace

There is a structure preserving way to reduce this to a linear eigenvalue problem. Eventually one ends up with having to find the eigenvalues and the stable eigenspace of a matrix that is similar to a complex symplectic one.
So, structure is now: $J = -J^T$ invertible, S a complex matrix such that $S^T JS = J$, and we are looking for a subspace M such that $SM \subset M$, $JM = M^\perp$, and $S|M$ has all its eigenvalues in the closed unit disc.

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Other problems that involve finding an invariant subspace of a matrix arise in:

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Other problems that involve finding an invariant subspace of a matrix arise in:

- radiative transfer in stellar atmospheres (transport theory),

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Other problems that involve finding an invariant subspace of a matrix arise in:

- radiative transfer in stellar atmospheres (transport theory),
- factorizations of rational matrix functions,

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Other problems that involve finding an invariant subspace of a matrix arise in:

- radiative transfer in stellar atmospheres (transport theory),
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- problems in systems and control theory.

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- radiative transfer in stellar atmospheres (transport theory),
- factorizations of rational matrix functions,
- problems in systems and control theory.

In many cases there is some symmetry involved. Today we will only consider the general case.

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Problem statement

Given an $n \times n$ complex matrix A and a subspace M of $\mathbb{C}^n$, we say that M is A-invariant if

$$AM \subset M.$$

Problem: if we perturb A what happens to an invariant subspace? Is it also perturbed slightly? Which ones are, which ones are not?

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Problem: if we perturb A what happens to an invariant subspace? Is it also perturbed slightly? Which ones are, which ones are not?

To answer this we need a metric on the set of subspaces of $\mathbb{C}^n$. This will be the gap metric:

$$\operatorname{gap}(M, N) = \|P_M - P_N\|$$

where P_M is the orthogonal projection on M, likewise for N.

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Matrix facts

Fact 1. $AM \subset M$ if and only $S^{-1}AS(S^{-1}M) \subset S^{-1}M$ for any invertible matrix S.

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So: to study invariant subspaces we can reduce to the Jordan canonical form.

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Fact 2. A matrix A is called non-derogatory if for any eigenvalue λ $\dim\ \ker(A - \lambda \cdot I) = 1$, otherwise it is called derogatory.

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So: to study invariant subspaces we can reduce to the Jordan canonical form.
Fact 2. A matrix A is called non-derogatory if for any eigenvalue λ $\dim\ \ker(A - \lambda \cdot I) = 1$, otherwise it is called derogatory.
This has a bearing on the *number* of invariant subspaces that A has.

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Examples 1

1. $A = J_4(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

There are only five invariant subspaces:

$$\{0\}, \operatorname{span}\{e_1\}, \operatorname{span}\{e_1, e_2\}, \operatorname{span}\{e_1, e_2, e_3\}, \mathbb{C}^4.$$

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Examples 2

2. $A = J_2(0) \oplus J_2(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

There is now an uncountable number of invariant subspaces:
any vector of the form $\begin{pmatrix} \alpha & 0 & \gamma & 0 \end{pmatrix}$ spans a one-dimensional invariant subspace;
two dimensional invariant subspaces are given for instance by the span of the vectors $\begin{pmatrix} \alpha & 0 & \gamma & 0 \end{pmatrix}$ and $\begin{pmatrix} \beta & \alpha & \delta & \gamma \end{pmatrix}$,
and there is also an uncountable number of three-dimensional subspaces.

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Given: A an $n \times n$ complex matrix and M an A-invariant subspace. M is called stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|A - B\| < \delta$ implies the existence of a B-invariant subspace N such that $\|P_M - P_N\| < \varepsilon$.

M is called Lipschitz stable if there are positive constants δ and K such that $\|A - B\| < \delta$ implies that B has an invariant subspace N with $\mathrm{gap}\,(M, N) \leq K\|A - B\|$.

back

Examples 3

Back to examples.

1. $A = J_4(0)$. Any invariant subspace is stable.

Sketch of proof: We prove it for $M = \operatorname{span}\{e_1, e_2\}$.

Let $A_n \to A$. Take any 2-dimensional A_n-invariant subspace M_n.

Compactness of the set of subspaces in the gap metric implies: there is a convergent subsequence. Now use uniqueness of the 2-dimensional A-invariant subspace.

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Examples 4

2. $A = J_4(0)$. No non-trivial invariant subspace is Lipschitz stable. Idea of proof: consider

$$A(\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}.$$

Any $A(\varepsilon)$-invariant subspace is spanned by a combination of its eigenvectors. For one-dimensional and three dimensional subspaces the gap is at least of the order of $\varepsilon^{\frac{1}{4}}$, for two-dimensional ones the gap is at best of the order of $\varepsilon^{\frac{1}{3}}$.

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3. $A = J_2(0) \oplus J_2(0)$. No non-trivial invariant subspace is stable. Idea of proof: we show this for the one-dimensional invariant subspaces, i.e., for eigenvectors.

Consider

$$A(\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The unique one-dimensional $A(\varepsilon)$-invariant subspace does not depend on ε and is given by $\operatorname{span}\{e_1\}$.

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Consider

$$A(\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}.$$

The unique one-dimensional $A(\varepsilon)$-invariant subspace does not depend on ε and is given by $\operatorname{span}\{e_3\}$.

Main results: preparations

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The examples can be expanded to give proofs of the following statements.

We need notation: the spectral subspace of A corresponding to an eigenvalue λ will be denoted by $R_\lambda(A)$.

Recall: $R_\lambda(A) = \mathrm{Ker}\,(A - \lambda I)^n$.

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Main results

Theorem (Kaashoek-van der Mee- Rodman) *Let* A *be an* $n \times n$ *complex matrix. An invariant subspace* M *is Lipschitz stable if and only if it is a spectral subspace, i.e., for all eigenvalues* λ *of* A *we have* $R_\lambda(A) \cap M$ *is either* $\{0\}$ *or* $R_\lambda(A)$.

Main results

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Theorem (Bart-Gohberg-Kaashoek/Campbell-Daughtry) *Let A be an $n \times n$ complex matrix. An invariant subspace M is a stable A-invariant subspace if and only if for every eigenvalue λ of A for which the geometric multiplicity is larger than one, i.e., for which $\dim \operatorname{Ker} (\lambda I - A) > 1$, we have $R_\lambda(A) \cap M$ is either $\{0\}$ or $R_\lambda(A)$.*
For each eigenvalue λ of A for which the geometric multiplicity is equal to one $R_\lambda(A) \cap M$ is an arbitrary A-invariant subspace of $R_\lambda(A)$.

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We now introduce two closely related notions that are weaker than Lipschitz stability and stronger than stability.

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Problem statement expanded

We now introduce two closely related notions that are weaker than Lipschitz stability and stronger than stability.

An A-invariant subspace M is called *α-stable* if there exist constants $K > 0,\ \delta > 0$, such that every $n \times n$ matrix B with $\|A - B\| < \delta$ has an invariant subspace N with the property that

$$\operatorname{gap}(M, N) \leq K\|A - B\|^{\frac{1}{\alpha}}.$$

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An A-invariant subspace M is called *α-stable* if there exist constants $K > 0,\ \delta > 0$, such that every $n \times n$ matrix B with $\|A - B\| < \delta$ has an invariant subspace N with the property that

$$\operatorname{gap}(M, N) \leq K\|A - B\|^{\frac{1}{\alpha}}.$$

An A-invariant subspace M is called *strongly α-stable* if for every sequence $A_n \to A$ and every sequence $M_n \to M$, where M_n is A_n-invariant, we have

$$\operatorname{gap}(M_n, M) \leq K\|A - A_n\|^{\frac{1}{\alpha}}.$$

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Examples 6

$A = J_4(0)$, M is the two-dimensional invariant subspace. Then:

- M is 3-stable,
- M is strongly 4-stable,
- M not strongly α-stable for any $\alpha < 4$.

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Results for strong α-stability

Theorem (R-Roozemond) *Let* A *be an* $n \times n$ *matrix, and let* M *be a nontrivial* A*-invariant subspace. Let* $\alpha > 0$ *be given. The subspace* M *is strongly* α*-stable if and only if either* $M \cap R_\lambda(A) = (0)$*, or* $M \cap R_\lambda(A) = R_\lambda(A)$*, for every eigenvalue* λ *of* A *that satisfies one of the following two conditions:*

(i) λ *has geometric multiplicity larger than one,*
(ii) λ *has geometric multiplicity one and* $\dim R_\lambda(A) > \alpha$*.*

For all other eigenvalues λ *of* A *there is no restriction on* $M \cap R_\lambda(A)$ *other than that it is* A*-invariant.*

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Before stating the result on α-stability, let us introduce the following notation.
For two natural numbers k and n, with $k < n$, we introduce a number $\gamma(k, n)$, as follows:
$\gamma(k, n) = n$, whenever there is no set of k distinct n-th roots of unity that sum to zero,
while
$\gamma(k, n) = n - 1$ if such a set of k distinct n-th roots of unity does exist.

For example $\gamma(2, 4) = 3$ because there is a set of two roots of unity that sum to zero (i and $-i$, or 1 and -1 will do).

As another example, if n is prime then $\gamma(k, n) = n$ for all k.

α-stability result

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Theorem (R-Rodman-A. Rubin) *Let A be an $n \times n$ matrix, and let M be a nontrivial A-invariant subspace. The subspace M is α-stable for some positive α if and only if M is stable.*
In that case M is α-stable if and only if

$$\gamma(\dim (M \cap R_\lambda(A)), \dim R_\lambda(A)) \leq \alpha,$$

for all eigenvalues λ of A such that

$$(0) \neq M \cap R_\lambda(A) \neq R_\lambda(A).$$

If there are no such eigenvalues, then M is Lipschitz stable.

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- The real case: A real, M a real invariant subspace.

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- The real case: A real, M a real invariant subspace.
- All kinds of symmetries. Mathematics involved: spaces with an indefinite inner product. Still many interesting open problems.

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- Infinite dimensional cases.
- Connections with numerical analysis.

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As a sample of applications:

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- a computational procedure that is used by astrophysicists is shown to be reliable,

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- Infinite dimensional cases.
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As a sample of applications:

- a computational procedure that is used by astrophysicists is shown to be reliable,
- the results provide an explanation for several phenomena in H^∞-control theory that were numerically observed.

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Problem statement
Matrix facts
Examples 1
Examples 2
Problem statement again
Examples 3
Examples 4
Examples 5
Main results
Problem statement expanded
Examples 6
Strong stability
Preparation for main theorem
α-stability result
Further developments

Further developments

- The real case: A real, M a real invariant subspace.
- All kinds of symmetries. Mathematics involved: spaces with an indefinite inner product. Still many interesting open problems.
- Infinite dimensional cases.
- Connections with numerical analysis.

As a sample of applications:

- a computational procedure that is used by astrophysicists is shown to be reliable,
- the results provide an explanation for several phenomena in H^∞-control theory that were numerically observed.

Thank you for your attention