Iterative methods: introduction and background

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The problem

Consider the set of linear equations

$$(I + \alpha T)u = v,$$

where v is given, $\alpha > 0$ and T is the symmetric tridiagonal Toeplitz matrix

$$T = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

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This time we will discuss an iterative method.

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Rewrite the problem as follows:

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After dividing through by $1 + 2\alpha$:

$$(I + M)u = \frac{1}{1+2\alpha} v = y,$$

where $M = \frac{\alpha}{1+2\alpha} T_1$.

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Defining $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(u) = y - Mu$, we see that a solution of the original matrix-vector equation can be viewed as a *fixed point* of f, that is, a point for which $f(u) = u$.

Banach's fixed point theorem

A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a *contraction* if there is a constant $0 \leq c < 1$ such that

$$\|f(x_1) - f(x_2)\| < c \cdot \|x_1 - x_2\|$$

for all vectors x_1 and x_2.

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A very important theorem is Banach's fixed point theorem (also called the *contraction mapping principle*):

Theorem *If f is a contraction on $\mathbb{R}^n$, then f has a unique fixed point $\hat{x}$. Moreover, for every vector x_0 in $\mathbb{R}^n$ we have that the sequence of iterations of f on x_0 converges to the unique fixed point, that is if $x_n = f^{(n)}(x_0)$, then $\lim x_n = \hat{x}$.*

Back to the original problem

Recall the problem:

$$u = y - Mu;$$

where

$$M = \frac{\alpha}{1+2\alpha} T_1 = \frac{\alpha}{1+2\alpha} \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

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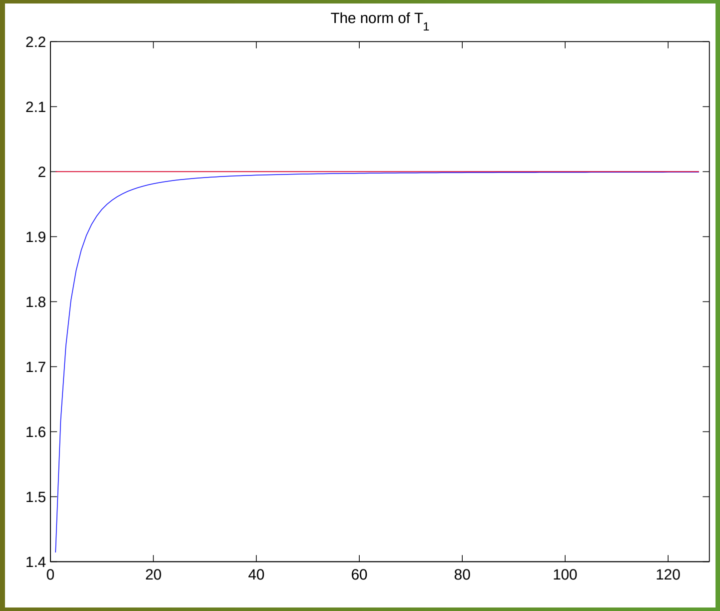

Jacobi's method

Hence, for every $x_0 \in \mathbb{R}^n$ the iteration

$$x_{k+1} = y - Mx_k$$

will converge to the unique solution of $u = y - Mu$. Recall also that

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This is what amounts to *Jacobi's method* for solving the original equation

$$(I + \alpha T)\, u = v.$$

Jacobi's method for the heat equation

Denote by $u_n^{m,k}$ the value of $u_n^m = u(n\delta x, m\delta \tau)$ at the k-th iteration.
Recall $\alpha = \frac{\delta \tau}{\delta x^2}$.

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Jacobi's method in coordinates can be written as

$$u_n^{m+1,k+1} = \frac{1}{1+2\alpha}(b_n^m + \alpha(u_{n-1}^{m+1,k} + u_{n+1}^{m+1,k}))$$

(in the notation of the book).

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Then iterate untill $\|u^{m+1,k+1} - u^{m,k}\| < \varepsilon$, where $\varepsilon > 0$ is a given tolerance.

Convergence rate

The rate of convergence depends on $\|M\| \approx \frac{2\alpha}{1+2\alpha}$. We have for every k:

$$\|u_{k+1} - u_k\| \leq \|M\| \cdot \|u_k - u_{k-1}\| \leq \|M\|^k \|u_1 - u_0\|.$$

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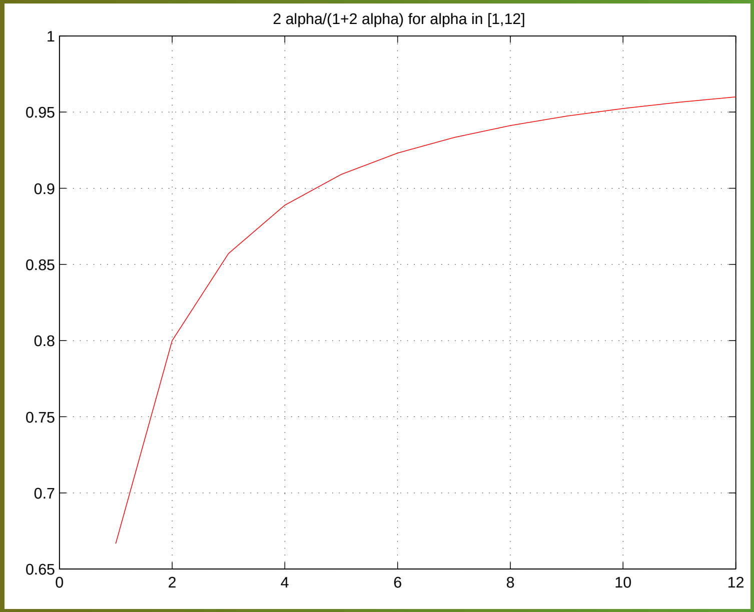
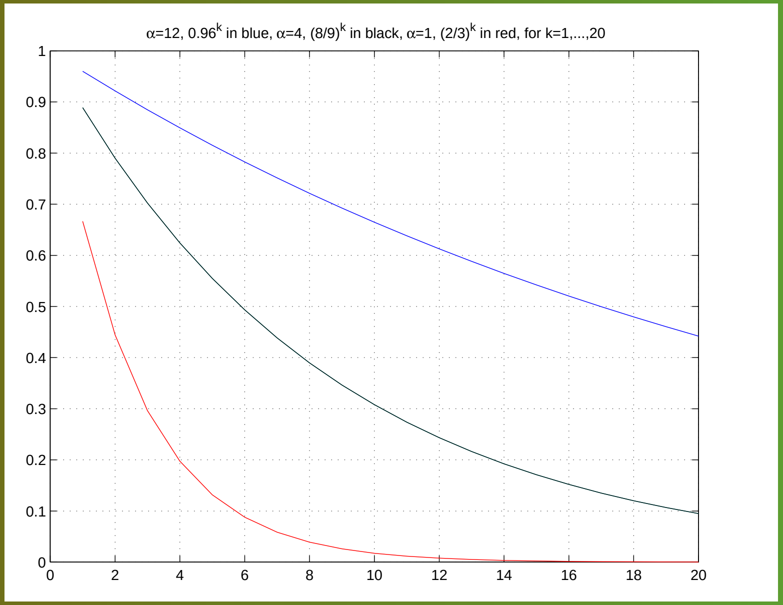
For large α this gets close to one, so then convergence is very slow.
For moderate values of α, convergence can still be fairly good: $\alpha = 12$ gives $\frac{2\alpha}{1+2\alpha} = \frac{24}{25} = 0.96$.



Convergence rate as function of α



Conclusion: when applying Jacobi's method it pays to keep α moderate (near 1, say), otherwise the iteration will convergence very slowly.