Some examples for the obstacle problem

André Ran
Vrije Universiteit Amsterdam

The problem

As a first example we consider one that can be solved explicitly.

Consider

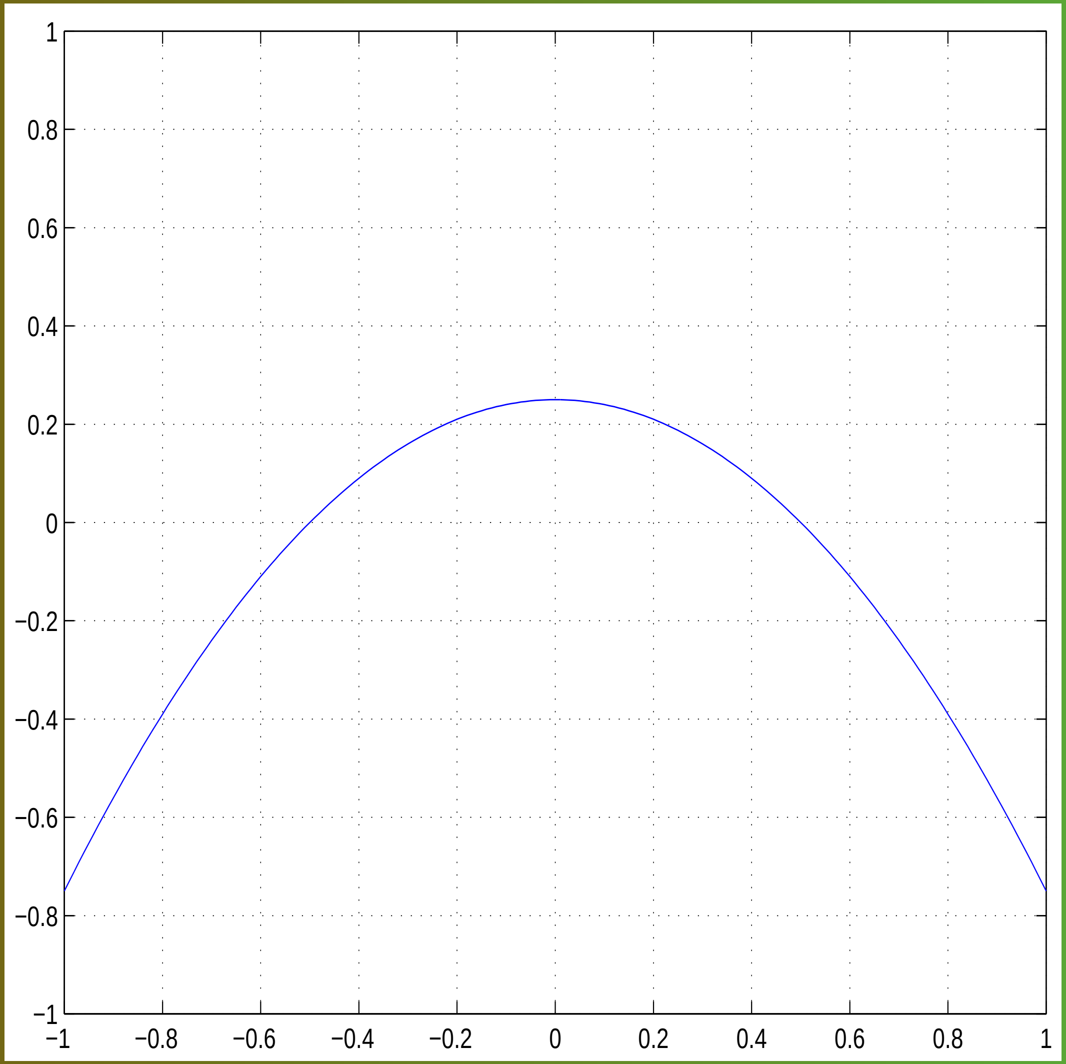
$$f(x) = \frac{1}{4} - x^2$$

on the interval $[-1, 1]$. The goal is to find the form of the curve that is determined by

$$\begin{aligned} u''(x) \cdot (u(x) - f(x)) &= 0, \\ -u''(x) &\geq 0, \\ (u(x) - f(x)) &\geq 0, \end{aligned}$$

subject to $u(-1) = u(1) = 0$, and u, u' continuous.

Figure 1



General observations

We know that the curve is a straight line before it hits the graph of f.
Also, the problem is symmetric, we only look at the left hand side.
Consider $x_0 < 0$. The line through $(x_0, f(x_0))$ and $(-1, 0)$ is given by

$$y = \frac{f(x_0)}{x_0 + 1}(x + 1).$$

Since we know that at the point of contact we want the derivatives of the line and the graph of f to be the same, the contact point is determined by

$$f'(x_0) = \frac{f(x_0)}{x_0 + 1}.$$

Solution in case $f(x) = \frac{1}{4} - x^2$

The contact point x_0 is determined by

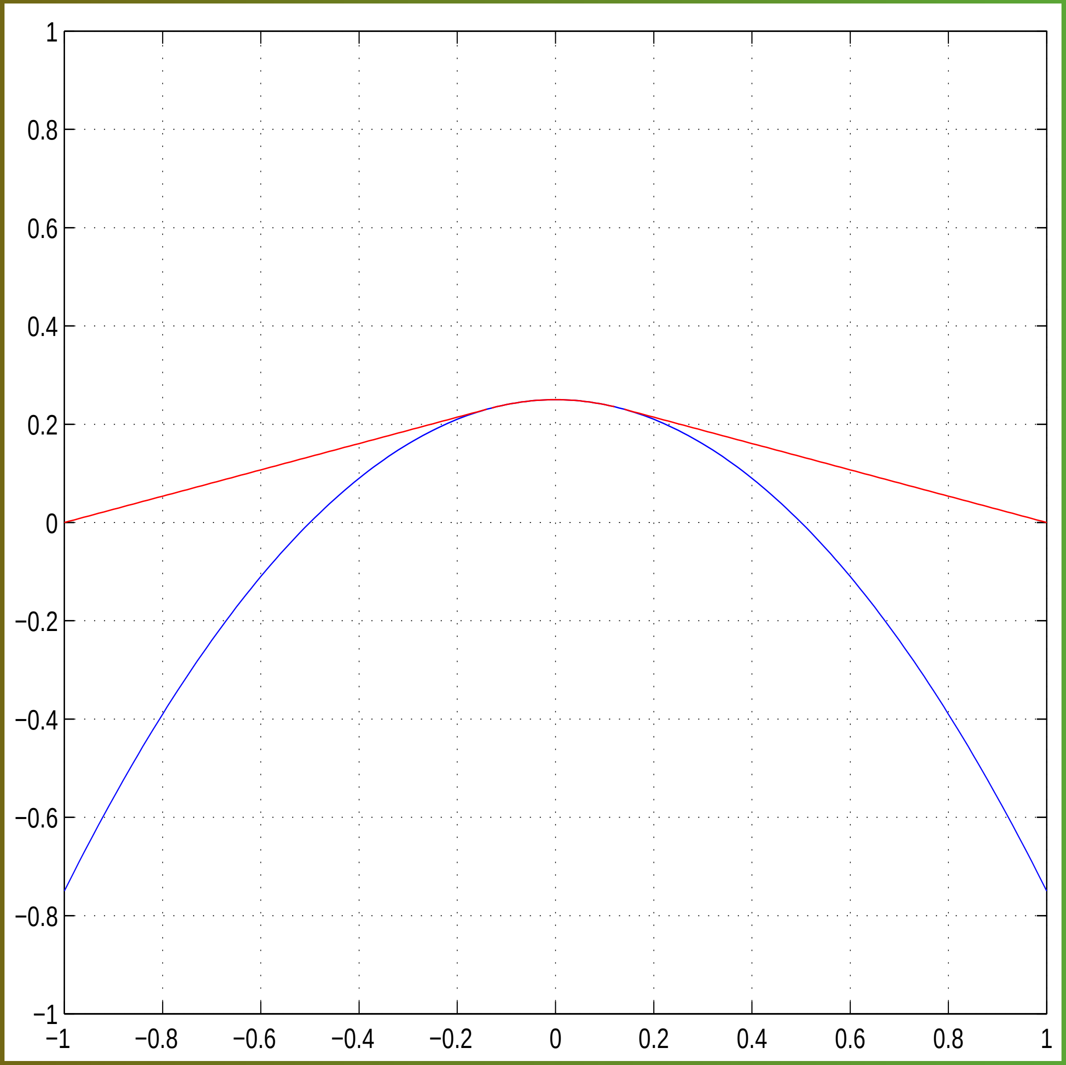
$$-2x_0 = \frac{\frac{1}{4} - x_0^2}{x_0 + 1},$$

i.e., by the quadratic equation

$$x_0^2 + 2x_0 + \frac{1}{4} = 0.$$

Moreover, we need the solution in the interval $[-1, 0]$. That solution is then $x_0 = -1 + \frac{1}{2}\sqrt{3} \approx -0.134$.

Figure 2



Second example

Take $f(x) = \cos(\pi x) - \frac{1}{2}$. Then $f'(x) = -\pi \sin(\pi x)$.
The contact point will be determined by the equation

$$-\pi \sin(\pi x_0)(x_0 + 1) = \cos(\pi x_0) - \frac{1}{2}.$$

Matlab gives, via the symbolic toolbox and

solve('-pi*sin(pi*x)*(x+1) -cos(pi*x)+1/2=0')

the answer

$$-.52250822154212187092330565189710e - 1,$$

so approximately 0.05.

Figure 3