

First series of Exercises for Mathematical Systems Theory

This series of exercises consists of two types: there are exercises just for practice, marked as “Exercises”, and some which you have to hand in. The latter ones are marked out as “Assignments”. The practice sessions are meant to give you a small introduction to Matlab and some of its uses in mathematics.

Exercise 1.

1. Construct in Matlab the 10×10 matrix $A = \left(\frac{1}{i+j-1} \right)_{i,j=1}^{10}$. Do this in three ways. First by use of a for-LOOP, and let Matlab do the calculating. Study the way a for-LOOP works in Matlab. (Consult Matlab’s Help Desk: “Getting Started”, and then “Flow Control”.)

Next by

```
>> B= ones(6,1) * [1:1:6]
>> A= ones(6,6) ./ (B' + B -ones(6,6) )
% and finally, as follows
>> help hilb
>> A= hilb(6)
```

2. Construct the inverse of A . Again, this can be done in at least three ways in Matlab. First, by using the function that determines the inverse of a matrix. Find that function in the list. Secondly, by dividing the matrix on the left by the identity matrix. So, do $A \setminus \text{eye}(10)$. Use `help \` if you do not fully comprehend what happens here. (‘Matlab functions’ and then ‘Operators and Special Characters’). Thirdly, you can also do $\text{eye}(\text{size}(A))/A$. Check to see if the answers are the same.

Also determine the relative errors in the answers, for instance:

```
>> A1 = A-1;  
>> A2 = A \ eye(10);  
>> max(abs(A1-A2))/max(abs(A1))
```

What do you think about this relative error? What is `eye(size(A))\A`?

3. Find the eigenvalues and eigenvectors of A . (Use `help eig`.) Is A invertible? What does that mean for the eigenvalues of A ?
4. Let D be the diagonal matrix with on its diagonal the eigenvalues of A , and let S be the matrix with on its columns the corresponding eigenvectors. Compute $A - SDS^{-1}$ and interpret the answer.
5. Determine the trace of A , its determinant, and its characteristic polynomial (Matlab command: `poly(A)`.) What is the constant term of p_A ? Do you understand how Matlab represents a polynomial? Do you recognize the (hopefully well-known) relations between p_A , the trace and the determinant of A ?

Assignment 1. Let A be a 3×3 matrix and x_0 a vector in \mathbf{R}^3 . The matrix A and the vector x_0 are obtained from the web page

"<http://www.cs.vu.nl/~ran>"

(Scroll down to "Onderwijs", then click Mathematical System Theory and then 'First series, first assignment')

Put those in a file `exercise_08_1.1.m` in your matlab-directory. Then do

```
>> Snummer = [n1; n2] % Here n1 and n2 are your student numbers.  
>> exercise_08_1.1
```

You now have the matrix A .

Let $x_k = Ax_{k-1}$ for $k = 1, 2, \dots, 63$. We are going to draw the points x_k in space. This can be done as follows. Do in matlab (make sure you understand what happens!):

```
>> Y=A, X = X0 ;  
>> for tel = 1:6;  
    X = [X Y*X] ;  
    Y = Y * Y;
```

```
end;
```

```
% this makes in X the columns  $x_k$ . Don't forget ';' !
```

```
% Without ';' you get all the intermediate results on your screen.
```

```
>> plot3(X(1,:),X(2,:),X(3,:),'+');
```

```
>> hold on ; plot3(X(1,:),X(2,:),X(3,:),'-');grid;hold off;
```

```
% it can also be done with
```

```
plot3(X(1,:),X(2,:),X(3,:),'+',X(1,:),X(2,:),X(3,:),'-');grid;
```

Explain what you see in the plot, using the eigenvalues and eigenvectors of A . By rotating the plot to get a better view of how the sequence evolves. The plot can be rotated by clicking on the rotate-icon in the figure-window of Matlab. In the older student editions of Matlab this cannot be done in this way. You can then look at the projections onto the coordinate plains (using a two-dimensional plot).

Exercise 2. Let $x = \left(1 \quad \frac{1}{2} \quad \cdots \quad \frac{1}{7} \quad \frac{1}{8} \right)$ and $y = \left(1 \quad -\frac{1}{2} \quad \frac{1}{3} \quad \cdots \quad \frac{1}{7} \quad -\frac{1}{8} \right)$.

1. Enter these vectors in Matlab. This can for instance be done as follows:

```
>> x = 1 ./ [1 : 8]'      % Take transpose, and divide into one element-  
wise.
```

```
>> y = [ ones(1,4) ; -ones(1,4) ]
```

```
>> y = y(:)      % put everyting in one vector
```

```
>> y = y .* x    % multiply elementwise
```

2. Compute the inner product of x and y . You can do this by using tranpose, or by array-multiplication.
3. Determine the matrix of rank one which has x as a basis for its column-space and y as a basis for its row-space.
4. As we have seen earlier, x and y also represent polynomials of degree 7. Determine the roots of these polynomials. Obviously, Matlab has a function for this ('roots')! Explain the relation between the two series of roots.
5. Determine a polynomial q such that its roots are the elements of x . What is the constant term of this polynomial exactly? In the representation of the polynomial it looks like there is a zero. That happens because the initial 'format' of matlab is 5 decimals. By the way, the printed number of decimals is unrelated to the arithmetical accuracy. There are other possibilities. To invesigate those, do 'help format' and try the effect on the outcome of your computations. In the sequel choose the 'format' that fits the question best.

6. In Matlab polynomials can be multiplied. To understand this operation you have to realize that if a_1, \dots, a_{n+1} are the coefficients of a polynomial of degree at most n and b_1, \dots, b_{m+1} are the coefficients of a polynomial of degree at most m , then the coefficients c_1, \dots, c_{m+n+1} of the product are given by the formula

$$c_\ell = \sum_{k=0}^N a_{k+1} b_{\ell-k}.$$

Here N is the biggest number such that all a_{k+1} and all $b_{\ell-k}$ exist. Thus, N depends on n , m and ℓ . The operation expressed by this formula is called convolution. That is why polynomial multiplication is called `conv` in Matlab. Now compute the product of the given polynomials x and y . What do the zeros you find in the answer mean, and does this fit with your expectations?

7. Of course, you can also divide polynomials in Matlab. Division with rest is called `deconv`. Construct in Matlab a vector z with 16 elements that contains the elements of x and y in that order. Make a polynomial p that has the elements of z as its roots, and a polynomial q that has the elements of x as its roots. Now compute $p + 1$ divided by q .

Assignment 2. Make a vector `Snummer` with your studentnumbers as its components. Get the data for 'First series, second assignment' from the webpage and put those in a file `exercise_08_1_2.m` in your matlab-directory. Then do in matlab

```
>> Snummer= [n1; n2] % where n1 and n2 are your studentnumbers
>> exercise_08_1_2
```

Now 'teller' is the representation in matlab of a polynomial of degree 4, which is the numerator of a rational function f and 'noemer' is the denominator of f . (Teller and noemer are the terms in Dutch for numerator and denominator.)

1. Use the function `polyval`, `plot`, `subplot` and `axis` to produce *clear* graphs of f on intervals of the real axis, from which the zeros and poles (that is, the points a for which $\lim_{x \rightarrow a} |f(x)| = \infty$) are clearly identifiable. Give the numerical values of the zeros and poles. Check that these values agree with your graphs.

2. We can write f in the form

$$f(x) = a_0 + a_1x^{-1} + a_2x^{-2} + \cdots + a_nx^{-n} + x^{-n}R_n(x)$$

where R_n is a rational function with $\lim_{x \rightarrow \infty} R_n(x) = 0$. Use matlab to determine the values of a_0, a_1, \dots, a_6 and $R_6(x)$.

3. Is it also possible to write

$$f(x) = b_0 + b_1x^1 + b_2x^2 + \cdots + b_nx^n + x^nS_n(x)$$

with S_n a rational function with $\lim_{x \rightarrow 0} S_n(x) = 0$? If so, explain how to do this with matlab.

Exercise 3. In this exercise it is necessary to choose 'format short e'.

1. Use the functions `eye` and `zeros` to construct the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Use the fact that A is a 2×2 block matrix consisting of zero and identity blocks. Now change A_{23} to 2. Construct a vector a_3 being the third row of A . Construct the submatrix

$$B = \begin{pmatrix} a_{23} & a_{25} \\ a_{43} & a_{45} \end{pmatrix}.$$

of A .

We are now going to work with the 'singular value decomposition' of a matrix. Usually this is abbreviated to `svd`. There is a Matlab function with that name. Read its help, recalling that in Matlab A' stand for the transpose A^T . The theory of `svd` can be found for instance in Chapter 4 of Lay's book on Linear Algebra. First read that, or some other source on the `svd` (any book on advanced linear algebra will have a section on this). After that continue with the next part of the exercise.

2. Recall that if $A = USV^T$ with U and V orthogonal matrices, then the rank of A is equal to the rank of S . Use that to find the rank of the matrix

$$A = \begin{pmatrix} -1 & 2 & -1 & 1 \\ 11 & 18 & 1 & 19 \\ 2 & 0 & 1 & 1 \\ -8 & 12 & -7 & 5 \\ -9 & 14 & -8 & 6 \end{pmatrix}.$$

Use the svd to find the singular values of A . Now, what is actually the rank of S , is it 2 or 3? Next, use Matlab's function rank to determine the rank of A .

Matlab uses the svd to determine the rank of A , so apparently, Matlab is of the opinion that $S_{33} = 0$. Why is this reasonable? What happens with the rank if we multiply A with 10^{16} ? Does this matter for the rank determined by Matlab?

Assignment 3. In this assignment it is useful to choose 'format short e'. On the webpage of the course you will find the matlab-command that constructs the matrix A of this exercise. This matrix is in fact a matrix that has been perturbed by round off to three digits after the decimal point. Determine the rank of the original matrix. Determine, *starting from the matrix A* a matrix that differs in all entries at most 10^{-3} from the corresponding entries in A , and which has the rank of the original matrix. Remember to show that the method you used has the desired result. You can choose to show that your method works for this specific case, or prove that your method works in general, independent of this particular matrix A .

As a hint: recall that a matrix of rank r only has r non-zero singular values.

Exercise 4. The following commands will plot a circle:

```
>> t = [-pi : pi/100 : pi];  
>> plot( cos(t),sin(t));
```

Perhaps you want to scale the axes. This can be done with the function axis. For instance, by choosing axis('square') you obtain rectangular axes, and by choosing axis('equal') you get the same scale on both axes. See help graph2d for more possibilities.

A branch of a hyperbola can be drawn using

```

>> t = [ -2 : 4/125 :2 ];
>> plot( 2* cosh(t) , sinh(t));
The full hyperbola is given by
>> plot( 2* cosh(t) , sinh(t)); hold on ; plot(-2*cosh(t),-sinh(t));
>> grid; hold off

```

What are the asymptotes in this case?

An example of a nice three-dimensional plot is give by the following:

```

>> x=[-1: .01 : 1] ; y=x;
>> [X,Y] = meshgrid(x,y);
>> Z = (2 * X.* Y)./( X.^ 2 + Y.^ 2);
>> mesh(X,Y,Z);
>> surf(X,Y,Z); % this is a different way to give the surface;
>> shading flat;
>> surf(X,Y,Z); % yet another way;
>> colormap(gray);
>> % and this way is especially suited for a black-and-white printer.

```

See also 'help graph3d' or 'helpwin'. For a complete overview use 'helpdesk' with your webbrowser.

Assignment 4. Surprise us with a nice application of Matlab. For instance, you can make a nice graph or 3D-plot, illustrate an interesting theorem, or come up with an interesting application.