

Integral operators with semi-separable kernel and with symmetries

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Preliminaries and introduction

Integral operator with semi-separable kernel

$$T : L_2([a, b], \mathcal{U}) \rightarrow L_2([a, b], \mathcal{Y})$$

$$y(t) = (Tu)(t) = D(t)u(t) + \int_a^b k(t, s)u(s) ds$$

where

$$k(t, s) = \begin{cases} F_1(t)G_1(s), & a \leq s < t \leq b, \\ -F_2(t)G_2(s), & a \leq t < s \leq b, \end{cases}$$

Here $F_i(t) : \mathcal{X}_i \rightarrow \mathcal{Y}$ and $G_i(t) : \mathcal{U} \rightarrow \mathcal{X}_i$, $i = 1, 2$ are square integrable.

All spaces $\mathcal{U}, \mathcal{Y}, \mathcal{X}_i$ are finite dimensional.

Realization

Gohberg-Kaashoek, mid eighties Systems with boundary values

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), & a \leq t \leq b, \\ y(t) = C(t)x(t) + D(t)u(t), & a \leq t \leq b, \\ (I - P)x(a) + PU(b)^{-1}x(b) = 0. \end{cases}$$

where $A(t)$ is integrable, $B(t)$ and $C(t)$ are square integrable, P is a projection and $U(t)$ is the fundamental operator corresponding to $A(t)$, i.e., $\dot{U}(t) = A(t)U(t)$, $U(a) = I$.

The system is well-posed, that is $u(t) \equiv 0$ implies the existence of a unique solution (which is the zero solution).

In that case

$$y(t) = D(t)u(t) + \int_a^b k(t, s)u(s) ds$$

where

$$k(t, s) = \begin{cases} C(t)U(t)(I - P)U(s)^{-1}B(s), & a \leq s < t \leq b, \\ -C(t)U(t)PU(s)^{-1}B(s), & a \leq t < s \leq b, \end{cases}$$

So, the input-output operator of the system with boundary values is of the semi-separable type.

Conversely, any such operator admits a realization of this type:

$$A(t) = 0, B(t) = \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix}, C(t) = (F_1(t) \quad F_2(t)), P = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{X}_2} \end{pmatrix},$$

on the state space $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$.

A realization of T of the form

$$\Theta = (A(t), B(t), C(t), D(t); I - P, PU(b)^{-1})$$

with P a projection is called an *SB-realization* of the integral operator T .

Problem

Characterize in terms of the SB-realization when such integral operators will be

- selfadjoint,
- J -unitary for some invertible $J = J^*$,
- positive definite,
- contractive,
- positive real or dissipative.

Intermezzo: time invariant case

Let $b = +\infty$, and $k(t, s) = k(t - s)$. Let

$$k(t) = Ce^{tA}B$$

be a *minimal* realization.

Then the corresponding integral operator is selfadjoint if and only if $k(t) = k(-t)^*$ for real t .

In terms of the minimal realization, the integral operator is self-adjoint if and only if there exists an invertible skew-hermitian matrix $S = -S^*$ such that

$$SA = -A^*S, \quad SB = -C^*.$$

Elements in the proof

(A, B, C) is a minimal realization if and only if $(-A^*, C^*, -B^*)$ is a minimal realization.

The proof is now based on Kalman's state space isomorphism theorem: two *minimal* realizations are *similar* and the similarity is *unique*. That similarity is the matrix S .

Remark iA is selfadjoint in the indefinite inner product given by iS . That fact, and a lot of Krein space theory was used in the study of the time invariant case.

Time-Varying Similarity

Two SB-realizations

$$\Theta_i = \left(A_i(t), B_i(t), C_i(t), D(t); (I - P_i), P_i U_i(b)^{-1} \right)$$

are called *similar* if there exists an invertible operator $E : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ and an absolutely continuous function $S(t) : \mathcal{X}_1 \rightarrow \mathcal{X}_2, a \leq t \leq b$, with invertible operator values, such that a.e. on $[a, b]$

$$\begin{aligned} A_2(t) &= S(t)A_1(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1}, \\ B_2(t) &= S(t)B_1(t), \\ C_2(t) &= C_1(t)S(t)^{-1}, \\ (I - P_2)S(a) &= E(I - P_1), \\ P_2U_2(b)^{-1}S(b) &= EP_1U_1(b)^{-1}. \end{aligned}$$

Then $U_2(t) = S(t)U_1(t)S(a)^{-1}$.

Minimality

An SB-realization is called *SB-minimal* if among all SB-realizations with the same input-output operator the dimension of the state space is as small as possible.

Big problem: this is not unique up to similarity.

USB-class

Def. 1. *We say that an integral operator with semi-separable kernel is in the USB-class if up to similarity it has a unique SB-minimal realization.*

Example: if F_1, F_2, G_1 and G_2 are analytic then T is in the USB-class.

So far all Gohberg-Kaashoek

Uniqueness of similarity

Let T be in the USB-class and let for $i = 1, 2$

$$\Theta_i = \left(A_i(t), B_i(t), C_i(t), D(t); (I - P_i), P_i U_i(b)^{-1} \right)$$

be two SB-minimal realizations of T . Then the two are similar.

Theorem 1. (GPR, 2002) *The similarity between two SB-minimal SB-realizations of an integral operator in the USB-class is unique.*

The adjoint

If $\Theta = (A(t), B(t), C(t), D(t); I - P, PU(b)^{-1})$ is an SB-realization of T , then

$$\Theta^* = (-A(t)^*, C(t)^*, -B(t)^*, D(t)^*; P^*, (I - P^*)U(b)^*)$$

is an SB-realization of T^* .

The fundamental operator for Θ^* is $U(t)^{-*}$.

Selfadjointness in the USB-class

Theorem 2. *Let T be in the USB-class and let*

$$\Theta = \left(A(t), B(t), C(t), D(t); I - P, PU(b)^{-1} \right)$$

be an SB-minimal SB-realization of T . Then $T = T^$ if and only if $D(t) = D(t)^*$ and there is a unique invertible $S(t)$ such that*

$$\begin{aligned} \dot{S}(t) &= -A(t)^* S(t) - S(t) A(t), \\ P^* S(a) &= S(a)(I - P), \\ C(t)^* &= S(t) B(t), \\ S(t) &= -S(t)^*. \end{aligned}$$

Unitarity

Let $J = J^* = J^{-1}$ be a signature matrix.

Theorem 3. *Under the same assumptions on T and its Θ : T is J -unitary, i.e., $T^*JT = J$ if and only if there is an invertible $S(t)$ such that*

$$\begin{aligned}\dot{S}(t) &= -A(t)^*S(t) - S(t)A(t) + C(t)^*JC(t), \\ P^*S(a) &= S(a)(I - P), \\ C(t)^*J &= S(t)B(t), \\ S(t) &= S(t)^*.\end{aligned}$$

Further

We continue the paper with

- positivity: $I + T > 0$, with $D(t) \equiv 0$, this involves factorization,
- positive real: $T + T^* > 0$, with $D(t)$ constant,
- dissipative: $-i(T - T^*) > 0$, also with $D(t)$ constant,
- strictly contractive: $\|T\| < 1$, with $D(t) \equiv 0$.

All characterizations involve Riccati differential equations.

Sample Result

Let $\Theta = (A(t), B(t), C(t), 0; I - P, PU(b)^{-1})$ be an SB-minimal SB-realization of T in the USB class, assume that Θ is similar to Θ^* , and that the similarity is given by $S(t)$.

Theorem 4. *The following statements are equivalent:*

- a. $I + T$ is positive definite,
- b. the Riccati differential equation

$$\begin{aligned}\dot{Y}(t) &= (Y(t)PS(a)^{-*} - P^*)U(t)^*C(t)^*C(t)U(t)(S(a)^{-1}P^*Y(t) - P), \\ Y(a) &= 0,\end{aligned}$$

has a solution on $[a, b]$. This solution is positive semidefinite, increasing and satisfies $Y(t)(I - P) = 0$ for all $a \leq t \leq b$.