

Polar decomposition in indefinite inner product spaces

Joint work:

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Introduction

For an $n \times n$ matrix X

$$X = UA$$

with $A = \sqrt{X^*X}$ and U unitary is the **polar decomposition**.

Always exists, even in Hilbert space setting.

Indefinite inner product space case

$H = H^*$ an $n \times n$ invertible hermitian matrix

$$[x, y] = \langle Hx, y \rangle$$

defines an indefinite inner product on \mathbb{C}^n .

Classes of matrices

H -selfadjoint: $[Ax, y] = [x, Ay]$ for all x, y .

H -isometric on a subspace \mathcal{M} : $[Ux, Uy] = [x, y]$ for all $x, y \in \mathcal{M}$

H -unitary: isometry on the full space (and then U is invertible)

H -adjoint of an operator X : $X^{[*]}$ defined by

$$[Xx, y] = [x, X^{[*]}y]$$

for all x, y .

H -polar decomposition of X : $X = UA$ where

- A is H -selfadjoint
- U is an H -isometry defined on a linear set containing the range of A

First observation

Lemma 1. *X admits H -polar decomposition if and only if $X^{[*]}X = A^2$ for some H -selfadjoint A with $\text{Ker } X = \text{Ker } A$. In this case we can take U to be H -unitary.*

Idea of proof: one way is evident. The other way, define an H -isometry U on the range of A to the range of X by $UAx = Xx$. Then use Witt's theorem to extend U to an H -unitary matrix.

Witt's Theorem

Theorem 1. Let $[\cdot, \cdot]$ be the scalar product in \mathbb{C}^n defined by the invertible hermitian matrix H . Let $U_0 : V_1 \rightarrow V_2$, where V_1 and V_2 are subspaces in \mathbb{C}^n , be a nonsingular linear transformation that preserves the scalar product

$$[U_0x, U_0y] = [x, y] \quad \text{for every } x, y \in V_1.$$

Then there exists a linear transformation U on \mathbb{C}^n such that $[Ux, Uy] = [x, y]$ for every vectors x and y , and $Ux = U_0x$ for every $x \in V_1$.

Plan of attack

To characterize those matrices that allow H -polar decomposition follow the following programm:

- Study matrices of the form $X^{[*]}X$.
- Determine which of these matrices have an H -selfadjoint square root.
- Find the ones that have an H -selfadjoint square root with $\text{Ker } X = \text{Ker } A$.

Remarks

The second item puts restrictions on the eigenvalues of $X^{[*]}X$ in $(\infty, 0]$.

The third item puts additional constraints on $\text{Ker } X$.

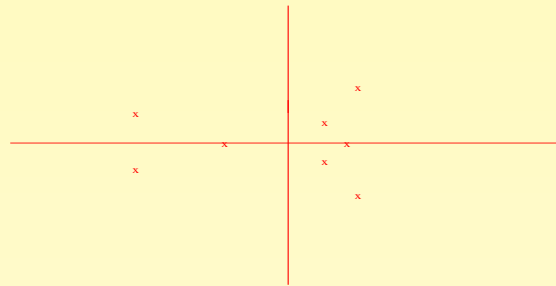
It is easily seen in a 2×2 example that H -polar decomposition need not always exist.

Intermezzo: H -selfadjoint matrices

$H = H^*$ invertible and $HA = A^*H$. Eigenvalues are symmetrically placed with respect to \mathbb{R} with same multiplicities at λ and $\bar{\lambda}$

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Examples

Define $P_k = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$ of size $k \times k$ and $J_k(\lambda)$ the $k \times k$ Jordan block $J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \cdots & \cdots & 1 \\ & & & \lambda \end{pmatrix}$

- $H = \varepsilon P_k, A = J_k(\lambda)$ with $\lambda \in \mathbb{R}, \varepsilon = \pm 1$
- $H = P_{2k}, A = J_k(\lambda) \oplus J_k(\bar{\lambda})$ with $\lambda \notin \mathbb{R}$

Canonical form

Theorem 2. *For any pair of matrices (H, A) with $H = H^*$ invertible and $HA = A^*H$ there exists an invertible matrix S such that $(S^*HS, S^{-1}AS)$ is a diagonal direct sum of blocks of the form as in the examples above.*

The signs ε , one for each block corresponding to a real eigenvalue of A , are unique (up to trivialities).

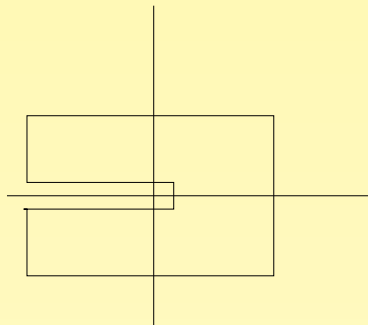
These signs are called the **sign characteristic** of the pair.

Square roots of $X^{[*]}X$

When no eigenvalues on $(-\infty, 0]$ it is simple to construct an H -selfadjoint A so that $A^2 = X^{[*]}X$: use the functional calculus.

Take a symmetric contour γ and consider $A = \int_{\gamma} \sqrt{z}(zI - X^{[*]}X)^{-1} dz$.

That is H -selfadjoint and satisfies $A^2 = X^{[*]}X$. Moreover, the eigenvalues of A will be in the open right half plane.



$$A = \int_{\gamma} \sqrt{z} (zI - X^{[*]} X)^{-1} dz$$

Negative eigenvalues of $X^{[*]}X$

If ai and $-ai$ are eigenvalues of A then $-a^2$ must be a double eigenvalue of $X^{[*]}X$.

$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} ai & 0 \\ 0 & -ai \end{pmatrix}$ then $A^2 = \begin{pmatrix} -a^2 & 0 \\ 0 & -a^2 \end{pmatrix}$ Canonical form $H \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A^2 \sim -a^2 I$. So the corresponding signs in the sign characteristic are 1 and -1 .

In general: Jordan blocks at negative eigenvalues of $X^{[*]}X$ must appear twice with opposite signs.

Zero eigenvalue

Example. $H = P_4$, $A = J_4(0)$. Then

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The canonical form is then

$$H \sim P_2 \oplus P_2, A^2 \sim J_2(0) \oplus J_2(0).$$

For H -polar decomposition one set of conditions comes from $X^{[*]}X = A^2$, another set of conditions comes from $\text{Ker } X = \text{Ker } A$.

All in all a complicated affair.....

Invariant maximal positive subspaces and polar decomposition

A subspace \mathcal{M} is called **positive** if $\langle Hx, x \rangle > 0$ for all nonzero vectors $x \in \mathcal{M}$.

It is called **maximal positive** if it is not contained in a larger **nonnegative subspace**.

Theorem 3. *Let X be an invertible $n \times n$ matrix and suppose that X has an invariant maximal H -positive subspace \mathcal{M} . Assume that $X\mathcal{M}^\perp$ is H -nonpositive. Then X allows a H -polar decomposition $X = UA$ such that the spectrum of A is contained in the open right halfplane.*

Proof 1.

We shall show that $X^{[*]}X$ has no eigenvalues in $(-\infty, 0]$.

With respect to the orthogonal decomposition $\mathbb{C}^n = \mathcal{M} \oplus (\mathcal{M})^\perp$:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}.$$

Here $H_{11} > 0$. Applying a transformation

$$X \longrightarrow P^{-1}XP, \quad H \longrightarrow P^*HP, \quad \text{where } P = \begin{pmatrix} H_{11}^{-1/2} & -H_{11}^{-1}H_{12} \\ 0 & I \end{pmatrix},$$

we have without loss of generality that $H_{11} = I$ and $H_{12} = 0$. Since \mathcal{M} is maximal H -positive, the $(2, 2)$ -block H_{22} is necessarily congruent to $-I$.

Proof 2.

Thus, we may assume that X and H have the forms

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad H = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then one easily computes that

$$X^{[*]}X = \begin{pmatrix} X_{11}^*X_{11} & X_{11}^*X_{12} \\ -X_{12}^*X_{11} & X_{22}^*X_{22} - X_{12}^*X_{12} \end{pmatrix}.$$

By the assumption we have that $X_{22}^*X_{22} - X_{12}^*X_{12}$ is nonnegative.

Proof 3.

Suppose that $X^{[*]}X \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ with $\lambda < 0$. Then

$$\begin{aligned} X_{11}^* X_{11} x + X_{11}^* X_{12} y &= \lambda x \\ -X_{12}^* X_{11} x + (X_{22}^* X_{22} - X_{12}^* X_{12}) y &= \lambda y \end{aligned}$$

Solving the first equation for x and inserting in the second we have that

$$(\lambda I - (X_{22}^* X_{22} - X_{12}^* X_{12})) + X_{12}^* X_{11} (\lambda I - X_{11}^* X_{11})^{-1} X_{11}^* X_{12} y = 0$$

Since $\lambda < 0$ and $X_{22}^* X_{22} - X_{12}^* X_{12} \geq 0$ this implies that $y = 0$, and then also x is zero.

Special classes of matrices and polar decomposition

An $n \times n$ matrix X is an H -contraction if $\langle HXx, Xx \rangle \leq \langle Hx, x \rangle$ for all vectors x , and an H -expansion if $\langle HXx, Xx \rangle \geq \langle Hx, x \rangle$ for all vectors x .

Proposition 1. *Let X be an H -contraction or an H -expansion. Then X admits a unique H -polar decomposition $X = UA$, with the additional property that $\sigma(A) \subset \{\lambda \mid \lambda \geq 0\}$. For this H -polar decomposition we also have that $\text{Ker } A = \text{Ker } A^2$.*

An $n \times n$ matrix X is called *H-normal* if $X^{[*]}X = XX^{[*]}$.

Proposition 2. *Let X be an H -normal matrix. Then X admits an H -polar decomposition.*

Already known for some time: true if X is an invertible H -normal.
Recent result: we can drop the condition of invertibility.

End of part I

Pontryagin space

A Kreĭn space is a direct sum of a Hilbert space and an anti-Hilbert space, in other words:

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$$

equipped with the indefinite inner product

$$[x, y] = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle$$

($x_i, y_i \in \mathcal{G}_i$ for $i = 1, 2$).

In case $\dim \mathcal{G}_2 < \infty$ the space is called a Pontryagin space.

Example \mathcal{G} a Hilbert space, and H an invertible selfadjoint operator, with

$\dim E_{(-\infty,0)}(H) < \infty$. Then

$$[x, y] = \langle Hx, y \rangle$$

defines an indefinite inner product on \mathcal{G} and with this indefinite inner product it becomes a Pontryagin space.

Henceforth our Pontryagin space will be of this type.

H -polar decomposition

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The H -polar decomposition is called H -unitary if we can take U to be H -unitary.

Lemma 2. *An operator X admits H -polar decomposition if and only if $X^{[*]}X = A^2$ for some H -selfadjoint A with $\text{Ker } X = \text{Ker } A$.*

But what if we want something extra??

After all, partial H -isometries are rather weak.

H-unitary *H*-polar decomposition

Theorem 4. *An operator X admits *H*-unitary *H*-polar decomposition if and only if $X^{[*]}X = A^2$ for some *H*-selfadjoint A with*

(a) $\text{Ker } A = \text{Ker } X$

(b) $C_1 \|Ax\| \leq \|Xx\| \leq C_2 \|Ax\|$ for some $C_2 \geq C_1 > 0$

(c) $\text{codim } \overline{\text{Im } A} = \text{codim } \overline{\text{Im } X}$.

Proof is based on a Pontryagin space version of Witt's theorem.

Witt's theorem for Pontryagin space

Theorem 5. *Let \mathcal{V}_1 and \mathcal{V}_2 be closed subspaces of \mathcal{G} and let U_0 be a continuous H -isometric operator mapping \mathcal{V}_1 onto \mathcal{V}_2 . Assume that $\text{codim } \mathcal{V}_1 \geq \text{codim } \mathcal{V}_2$. Then there exists a linear operator $U : \mathcal{G} \rightarrow \mathcal{G}$ such that*

- $[Ux, Uy] = [x, y]$ for all $x, y \in \mathcal{G}$
- $Ux = U_0x$ for all $x \in \mathcal{V}_1$.

Moreover, in case $\text{codim } \mathcal{V}_1 = \text{codim } \mathcal{V}_2$, U_0 can be extended to an H -unitary operator on \mathcal{G} .

Operators of the form $X^{[*]}X$

Operators of this form have a nice property: $\sigma(X^{[*]}X) \setminus [0, \infty)$ is a *finite* set consisting of eigenvalues of *finite* algebraic multiplicity.

Idea of proof: 1. $X^{[*]}X$ is a finite rank perturbation of X^*X . Hence $\sigma(X^{[*]}X) \setminus [0, \infty)$ is a discrete set in $\mathbb{C} \setminus [0, \infty)$ with possible accumulation points in $[0, \infty)$. Each of the points in this discrete set is an eigenvalue of finite algebraic multiplicity.

2. $X^{[*]}X$ is H -selfadjoint in a Pontryagin space, and hence the number of non-real points in the spectrum is finite.

3. Only to prove that the negative spectrum is a finite set.

If $\lambda < 0$ is an eigenvalue of $X^{[*]}X$ with eigenvector x then

$$\lambda[x, x] = [Xx, Xx].$$

As $\lambda < 0$ either $[x, x] = [Xx, Xx] = 0$, or $[x, x] < 0$ or $[Xx, Xx] < 0$.
Now it is a matter of dimension count, and the fact that we are working in a Pontryagin space (finite dimensional negative part).

Intermezzo: Critical points

If A is H -selfadjoint in a Pontryagin space \mathcal{G} then there is polynomial $p(\lambda)$ such that $p(A)$ is H -nonnegative, that is

$$\langle Hp(A)x, x \rangle = [p(A)x, x] \geq 0$$

for all $x \in \mathcal{G}$. (In Kreĭn space terminology: every H -selfadjoint operator is definitizable.)

The set of *critical points* is defined as

$$c(A) = \sigma(A) \cap \{\lambda \in \mathbb{R} \mid p(\lambda) = 0\}$$

H. Langer showed that there is a "spectral function" for A .

Define R_A to be the semiring of all bounded intervals with endpoints not in $c(A)$.

Then there is a map $\Delta \rightarrow E(\Delta)$ from R_A into the set of H -selfadjoint projections in \mathcal{G} satisfying the usual properties for a spectral function.

Two types of critical points

Let α be in $c(A)$, let $\lambda_0 < \alpha, \lambda_1 > \alpha$.

If for each such λ_0, λ_1 the limits

$$\lim_{\lambda \uparrow \alpha} E([\lambda_0, \lambda]), \lim_{\lambda \downarrow \alpha} E([\lambda, \lambda_1])$$

exist in the strong operator topology, then α is called a *regular* critical point.

Otherwise it is called a *singular* critical point.

Regular critical points are indeed "regular": one can define $E(\Delta)$ as long as Δ contains no singular critical points (or has them as its endpoints).

Back to the main issue
A decomposition for $X^{[*]}X$

Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Then \mathcal{G} decomposes *H-orthogonally* as follows:

$$\mathcal{G} = \mathcal{M}_{00} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$$

where each of these subspaces is invariant under $X^{[*]}X$ and has the following properties

- \mathcal{M}_{00} is finite dimensional and $X^{[*]}X$ restricted to \mathcal{M}_{00} is nilpotent,
- $\mathcal{M}_1 \subset \text{Ker } X^{[*]}X$ and it is a strictly *H*-positive subspace,

- $\sigma(X^{[*]}X|_{\mathcal{M}_2}) \subset [0, \infty)$; $\text{Ker } X^{[*]}X|_{\mathcal{M}_2} = (0)$,
- \mathcal{M}_3 is finite dimensional and $X^{[*]}X$ restricted to \mathcal{M}_3 has no spectrum in $[0, \infty)$.

H -selfadjoint square roots of $X^{[*]}X$

Suppose that $X^{[*]}X$ does not have a singular critical point at zero.

Denote $Z_i = X^{[*]}X|_{\mathcal{M}_i}$, for $i = 0, 1, 2, 3$,
let P_i be the H -orthogonal projection onto \mathcal{M}_i for $i = 0, 1, 2, 3$
along the other \mathcal{M}_j 's,
and let $H_i = P_i H|_{\mathcal{M}_i}$.

Then there exists an H -selfadjoint square root A of $X^{[*]}X$, that is $X^{[*]}X = A^2$, if and only if Z_{00} has an H_{00} -selfadjoint square root and Z_3 has an H_3 -selfadjoint square root.

Note: this reduces the problem to the finite dimensional case.

Taking the H_2 -selfadjoint square root of Z_2 involves techniques from operator theory (functional calculus). This makes heavy use of the fact that zero is not a singular critical point.

Example

$$\mathcal{G} = L_2([0, 1]) \oplus \mathbb{C}^2, \quad H = I \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let M_t be the operator of multiplication by t on $L_2([0, 1])$, take z a non-zero complex number, and define

$$Z = \begin{bmatrix} M_t & 0 & z\sqrt{t} \\ \bar{z}\langle \cdot, \sqrt{t} \rangle & 0 & |z|^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then Z is bounded H -selfadjoint, $\sigma(Z) = [0, 1]$ and zero is a critical point of Z .

Moreover, $Z = A^2$ where the H -selfadjoint A is given by

$$A = \begin{bmatrix} M_{\sqrt{t}} & 0 & z \cdot \mathbf{1} \\ \bar{z} \langle \cdot, \mathbf{1} \rangle & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

Here $a \in \mathbb{R}$ is arbitrary.

H-polar decomposition

Theorem 6. *Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Assume in addition that the range of X is closed. Then X admits *H*-polar decomposition if and only if Z_{00} has an H_{00} -polar decomposition and Z_3 has an H_3 -selfadjoint square root (and in that case automatically also an H_3 -polar decomposition), and $\mathcal{M}_1 \subset \text{Ker } X$.*

Again, this works via reduction to the finite dimensional case. The part on \mathcal{M}_2 is now much more difficult and requires a further decomposition.

Concerning *H*-unitary *H*-polar decomposition we have:

Theorem 7. *Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Assume in addition that the range of X is closed and $\dim \text{Ker } X = \text{codim Im } X$. Then X admits *H*-unitary *H*-polar decomposition if and only if it admits *H*-polar decomposition.*