Perturbations of bisemigroups

optimal control and

approximation of solutions of algebraic Riccati equations

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Motivation

LQ-optimal control.

System $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$.

A generator of an exponentially decaying C_0-semigroup acting on a separable Hilbert space $\mathcal{H}$

$B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, where also $\mathcal{U}$ is a separable Hilbert space.

Cost function

$J(u, x_0) = \int_0^\infty \langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle \, dt$

$Q = Q^* \geq 0, Q \in \mathcal{L}(\mathcal{H}),\ R = R^* > 0, R \in \mathcal{L}(\mathcal{U})$.

Concrete example

Wave equation with distributed control

$$\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + u(x,t),$$

$$w(0,t) = w(1,t) = 0.$$

Here $u \in L_2(0,1)$.

State space reformulation: put $A_0 = -\frac{d^2}{dx^2}$, and

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

on the state space

$$Z = \mathcal{D}(A_0^{\frac{1}{2}}) \oplus L_2(0,1).$$

Cost function:

$$J(z_0, u) = \int_0^\infty \int_0^1 |z_2(x,s)|^2 + |u(x,s)|^2 \, dx \, ds,$$

where the state is $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

So, $R = I$, and $Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

Back to general case: solution

$min_{u \text{ admissible}} J(u, x_0) = \langle X_+ x_0, x_0\rangle$

where X_+ satisfies

1. for $x \in \mathcal{D}(A)$ we have

$$(X_+ BR^{-1}B^* X_+ - A^* X_+ - X_+ A - Q)x = 0,$$

2. $A - BR^{-1}B^* X_+$ generates an exponentially decaying C_0-semigroup.

See, e.g.,

R.F. Curtain and H. Zwart: *An Introduction to Infinite-Dimensional Linear Systems Theory,* Springer Verlag, 1995.

Concrete example is done there as well.

Finite dimensional approach

$H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}$

Let $\mathcal{M}_+$ be the invariant subspace of H corresponding to the open left half plane

$\mathcal{M}_+ = \text{Im} \begin{pmatrix} I \\ X \end{pmatrix}$ for some $X = X^*$

and X satisfies the algebraic Riccati equation

$$XBR^{-1}B^*X - A^*X - XA - Q = 0$$

and

$$\sigma(A - BR^{-1}B^*X) = \sigma(H|_{\mathcal{M}_+})$$

Infinite dimensional analogue

Problem We would like to see the infinite dimensional result in the same way

Idea: set $H_0 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$

This generates an exponentially decaying *bisemigroup*

View H as a bounded perturbation of this.

Under what conditions does H generate an exponentially decaying bisemigroup

If so, what can we say about $\mathcal{M}_+$?

Bisemigroups

A closed and densely defined linear operator S on a Banach space $\mathcal{X}$ is called *exponentially dichotomous* if there is a projection P commuting with S such that

$-S|_{\text{Im}\, P}$ and $S|_{\text{Ker}\, P}$

generate exponentially decaying C_0-semigroups.

The *bisemigroup* generated by S

$$E(t;S) = \begin{cases} e^{tS}(I-P), & t > 0 \\ -e^{tS}P, & t < 0. \end{cases}$$

Its *separating projection* P is given by

$P = -E(0^-;S) = I_{\mathcal{X}} - E(0^+;S)$.

There is a $\varepsilon > 0$ such that

$\{\lambda \in \mathbb{C} : |\text{Re}\,\lambda| \le \varepsilon\} \subset \rho(S)$

and for every $x \in \mathcal{X}$

$(\lambda - S)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t;S)x\,dt, \qquad |\text{Re}\,\lambda| \le \varepsilon.$

Hence for every $x \in \mathcal{X}$ we have

$\|(\lambda - S)^{-1}x\| \to 0$

as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : |\text{Re}\,\lambda| \le \varepsilon'\}$ for some $\varepsilon' \in (0, \varepsilon]$.

We call the restrictions of e^{tS} to $\text{Ker}\, P$ and of e^{-tS} to $\text{Im}\, P$ the *constituent semigroups* of the exponentially dichotomous operator S.

First perturbation result: BGK

Theorem 1. (Bart, Gohberg, Kaashoek)

S exponentially dichotomous.

$\tilde{S}$ closed, densely defined with

$$\mathcal{D}(\tilde{S}^2) \subset \mathcal{D}(S^2),$$

and such that there is $\varepsilon' > 0$ with

$$\{\lambda \in \mathbb{C} : |Re\lambda| \le \varepsilon'\} \subset \rho(S) \cap \rho(\tilde{S})$$

and

$$\sup_{|Re\lambda| \le \varepsilon'} |\lambda^2| \|(\lambda - \tilde{S})^{-1} - (\lambda - S)^{-1}\| < \infty.$$

Then $\tilde{S}$ is exponentially dichotomous.

Formula for separating projection: for $x \in \mathcal{D}(\tilde{S}^2)$

$$\tilde{P}x = -\frac{1}{2\pi i} \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{1}{\lambda^2} (\lambda - \tilde{S})^{-1} \tilde{S}^2 x \, d\lambda$$

Try to apply to Hamiltonian operator

$$S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$$

$$\tilde{S} = H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}$$

The domain condition

$$\mathcal{D}(\tilde{S}^2) \subset \mathcal{D}(S^2)$$

is complicated.

Can we do better?

Two possibilities: strengthen the assumptions on S, or strengthen the assumptions on the perturbation.

Second perturbation result: LRR

Assume S generates an analytic bisemigroup (i.e., the constituent semigroups are analytic).

Then the separating projection

$$Px = -\frac{1}{2\pi i} \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{1}{\lambda^2} S(\lambda - S)^{-1} S x \, d\lambda$$

for $x \in \mathcal{D}(S)$.

Theorem 2. (Langer, Ran, van de Rotten)
S dichotomous, and for some ε

$$\{\lambda \in \mathbb{C} : |Re\lambda| \le \varepsilon\} \subset \rho(S)$$

and for some $\gamma > 0$ and $\beta > \frac{1}{2}$

$$\|(\lambda - S)^{-1}\| < \frac{\gamma}{1 + |\lambda|^\beta}, \quad |Re\lambda| \le \varepsilon$$

For B bounded let $\tilde{S} = S + B$. Assume that

$$\{\lambda \in \mathbb{C} : |Re\lambda| \le \varepsilon\} \subset \rho(\tilde{S})$$

Then $\tilde{S}$ is dichotomous as well.

Third perturbation result: MR

Theorem 3. (van der Mee, Ran)

S exponentially dichotomous

B bounded with $(\lambda - S)^{-1}B$ compact for pure imaginary λ,

$\tilde{S} = S + B$ with $\mathcal{D}(\tilde{S}) = \mathcal{D}(S)$.

Suppose that $i\mathbb{R} \subset \rho(\tilde{S})$.

Then $\tilde{S}$ is exponentially dichotomous.

Sketch of proof

If $\tilde{S}$ generates a bisemigroup then the resolvent identity gives that it satisfies

$$E(t;\tilde{S})x - \int_{-\infty}^{\infty} E(t-\tau;S)BE(\tau;\tilde{S})x\,d\tau = E(t;S)x,$$

where $x \in \mathcal{H}$ and $0 \neq t \in \mathbb{R}$.

Conversely, consider the convolution equation

$$u(t,x) - \int_{-\infty}^{\infty} E(t-\tau;S)Bu(\tau,x),d\tau = E(t;S)x.$$

The symbol of the convolution integral equation

$$I + (\lambda - S)^{-1}B = (\lambda - S)^{-1}(\lambda - \tilde{S})$$

tends to I in the norm as $\lambda \to \infty$ in the strip $|\mathrm{Re}\,\lambda| \leq \varepsilon$.

Moreover, it is a compact perturbation of the identity which only takes invertible values on the imaginary axis. Thus there exists $\varepsilon_0 \in (0,\varepsilon]$ such that the symbol only takes invertible values on the strip $|\mathrm{Re}\,\lambda| \leq \varepsilon_0$.

By the Bochner-Phillips theorem the convolution equation has a unique solution

$$u(\cdot;x) = E(\cdot;\tilde{S})x.$$

Using results from Bart-Gohberg-Kaashoek we can conclude that this is a bisemigroup.

Fourth perturbation result: MR

Our next result uses the concept of *immediately norm continuous* semigroups.

A semigroup $T(t)$ is called immediately norm continuous if $T(t)$ is norm continuous for $t > 0$.

Examples

- Analytic semigroups,
- immediately compact semigroups (i.e., $T(t)$ is a compact operator for $t > 0$)
- immediately differentiable semigroups

K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations* (Springer GTM **194**, Berlin, 2000) diagram (4.26) in Chapter II.

Theorem 4. *Let $\mathcal{H}$ be a complex Hilbert space and let $(T(t))_{t\geq 0}$ be an uniform exponentially stable C_0-semigroup on $\mathcal{H}$. Then $(T(t))_{t\geq 0}$ is immediately norm continuous if and only if the resolvent $(\lambda - A)^{-1}$ of its infinitesimal generator A vanishes in the norm as $\lambda \to \infty$ along the imaginary line.*

K.-J. Engel and R. Nagel ibid. Section II 4.20.

Theorem 5. (van der Mee, Ran)

S exponentially dichotomous with immediately norm continuous constituent semigroups

B bounded,

$\tilde{S} = S + B$ with $\mathcal{D}(\tilde{S}) = \mathcal{D}(S)$.

Suppose that

$$\{\lambda \in \mathbb{C} : |Re\,\lambda| < \varepsilon\} \subset \rho(\tilde{S})$$

Then $\tilde{S}$ is exponentially dichotomous with immediately norm continuous constituent semigroups.

First application to the ARE

If A generates an immediately norm continuous exponentially stable semigroup, and D and Q are bounded, such that the resolvent of $H = \begin{pmatrix} A & -D \\ -Q & -A^* \end{pmatrix}$ contains a strip around $i\mathbb{R}$, then H generates a bisemigroup.

Moreover, these assumptions imply the existence of a bounded solution X of the ARE

$$(XDX - XA - A^*X - Q)x = 0, \quad x \in \mathcal{D}(A)$$

such that $A - DX$ is exponentially stable and X maps $\mathcal{D}(A)$ into $\mathcal{D}(A^*)$.

Second application to the ARE

Assume: A generates an exponentially stable semigroup, D compact and positive semidefinite, Q positive semidefinite.

Consider $S = \begin{pmatrix} A & 0 \\ -Q & -A^* \end{pmatrix}$.

Claim: this generates a bisemigroup.

Idea: let $X = \int_0^\infty e^{\tau A^*} Q e^{\tau A}\, d\tau$. Then X solves the Lyapunov equation

$$A^*X + XA = -Q \qquad \text{on } \mathcal{D}(A)$$

Put $T = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$, then $T^{-1}ST = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$.

Now view H as a compact perturbation of S, prove that H has no spectrum on $i\mathbb{R}$, to conclude that H generates a bisemigroup.

Theorem 6. *There exist unique positive semidefinite selfadjoint $-\Pi_+$ and Π_-, such that*

- *Π_+ is compact and Π_- is bounded,*
- *Let P be the separating projection for H, then* $\operatorname{Im} P = \operatorname{Im} \begin{pmatrix} I \\ \Pi_- \end{pmatrix}$, $\operatorname{Ker} P = \operatorname{Im} \begin{pmatrix} \Pi_+ \\ I \end{pmatrix}$,
- *Π_- maps $\mathcal{D}(A)$ into $\mathcal{D}(A^*)$ and Π_+ maps $\mathcal{D}(A^*)$ into $\mathcal{D}(A)$,*
- *Π_- is a solution of*

  $$(\Pi A + A^*\Pi + Q - \Pi D \Pi)x = 0, \qquad x \in \mathcal{D}(A),$$

  Π_+ is a solution of

  $$(A\Pi + \Pi A^* + \Pi Q \Pi - D)x = 0, \qquad x \in \mathcal{D}(A^*),$$

- *$A - D\Pi_-$ and $A + \Pi_+ Q$ generate exponentially decaying semigroups.*

Approximation

Let $\mathcal{H}_n$ be finite dimensional subspaces of the state space $\mathcal{H}$,
$\pi_n : \mathcal{H} \to \mathcal{H}_n$ the projection onto $\mathcal{H}_n$
$\imath_n : \mathcal{H}_n \to \mathcal{H}$ the injection of $\mathcal{H}_n$ into $\mathcal{H}$.

We assume $\imath_n \pi_n \to I$ strongly, i.e., for all $x \in \mathcal{H}$ we have $\imath_n \pi_n x \to x$.

As before, A generates a strongly continuous exponentially stable semigroup, $S = A \oplus -A^*$ on $\mathcal{H} \oplus \mathcal{H}$, Q is positive semidefinite, and D is compact and positive semidefinite.

Put $Q_n = \pi_n Q \imath_n, D_n = \pi_n D \imath_n$.

Let A_n generate a strongly continuous exponentially stable semigroup on $\mathcal{H}_n$, and let $S_n = A_n \oplus -A_n^*$ on $\mathcal{H}_n \oplus \mathcal{H}_n$.

The sequence of triples (A_n, Q_n, D_n) is called an *approximant* of the triple (A, Q, D) if for some $\varepsilon > 0$ we have

$$\lim_{n\to\infty} e^{\varepsilon|t|} \|\hat{\imath}_n E(t; -S_n) \hat{\pi}_n x - E(t; -S)x\| = 0$$

for every $x \in \mathcal{H} \oplus \mathcal{H}$, uniformly in $t \in \mathbb{R} \setminus \{0\}$.

Here $\hat{\imath}_n = \imath_n \oplus \imath_n$ and $\hat{\pi}_n = \pi_n \oplus \pi_n$.

Theorem 7. *Let $\Pi_{-,n}$ be the solution of the algebraic Riccati equation corresponding to (A_n, Q_n, D_n). If this sequence of triples is an approximant of the triple (A, Q, D) then*

- $\lim_{n\to\infty} \|\imath_n \Pi_{-,n} \pi_n x - \Pi_- x\| = 0$ *for every $x \in \mathcal{H}$,*
- $\lim_{n\to\infty} \|\imath_n e^{t(A_n - D_n \Pi_{-,n})} \pi_n - e^{t(A - D\Pi_-)}\| = 0$ *uniformly in t on compact intervals of* $[0, \infty)$.

References

H. Bart, I. Gohberg and M.A. Kaashoek: Wiener-Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators,
J. Funct. Anal. **68** 1986, 1–42,

P. Bubak, C.V.M. van der Mee and A.C.M. Ran, Approximation of solutions of Riccati equations, *SIAM J. Control and Opt.* to appear.

H. Langer, A.C.M. Ran, and B.A. van de Rotten, Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations, in: I. Gohberg and H. Langer, eds., *Linear Operators and Matrices* (Birkhäuser OT **130**, Basel and Boston, 2001) 235–254.

C.V.M. van der Mee and A.C.M. Ran, Perturbation results for exponentially dichotomous operators on general Banach spaces, *J. Funct. Anal.* **210** 2004, 193–213.

C.V.M. van der Mee and A.C.M. Ran, Additive and multiplicative perturbations of exponentially dichotomous operators on general Banach spaces, in: Proceedings IWOTA 2003, (Birkhäuser OT **160** Basel and Boston, 2005) 411–422.