Perturbations of bisemigroups

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Motivated in part by:

H. Bart, I. Gohberg and M.A. Kaashoek: Wiener-Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators, *J. Funct. Anal.* **68** 1986, 1-42,

and in part by:

H. Langer, A.C.M. Ran, and B.A. van de Rotten, Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations, in: I. Gohberg and H. Langer, eds., *Linear Operators and Matrices* (Birkhäuser OT **130**, Basel and Boston, 2001) 235–254.

Motivation

LQ-optimal control.

System $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$.

A generator of an exponentially decaying C_0-semigroup acting on a separable Hilbert space $\mathcal{H}$

$B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, where also $\mathcal{U}$ is a separable Hilbert space.

Cost function

$J(u, x_0) = \int_0^\infty \langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle \, dt$

$Q = Q^* \geq 0, Q \in \mathcal{L}(\mathcal{H}),\ R = R^* > 0, R \in \mathcal{L}(\mathcal{U})$.

Known result:

$min_{u \text{ admissible}} J(u, x_0) = \langle X_+ x_0, x_0\rangle$

where X_+ satisfies

1. for $x \in \mathcal{D}(A)$ we have

$$(X_+ BR^{-1}B^*X_+ - A^*X_+ - X_+A - Q)x = 0,$$

2. $A - BR^{-1}B^*X_+$ generates an exponentially decaying C_0-semigroup.

See, e.g.,

R.F. Curtain and H. Zwart: *An Introduction to Infinite-Dimensional Linear Systems Theory,* Springer Verlag, 1995.

Finite dimensional approach

$$H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}$$

Let $\mathcal{M}_+$ be the invariant subspace of H corresponding to the open left half plane

$\mathcal{M}_+ = \text{Im} \begin{pmatrix} I \\ X \end{pmatrix}$ for some $X = X^*$

and X satisfies the algebraic Riccati equation

$$XBR^{-1}B^*X - A^*X - XA - Q = 0$$

and

$$\sigma(A - BR^{-1}B^*X) = \sigma(H|_{\mathcal{M}_+})$$

Infinite dimensional analogue

Problem We would like to see the infinite dimensional result in the same way

Idea: set $H_0 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$

This generates an exponentially decaying *bisemigroup*

View H as a bounded perturbation of this.

Under what conditions does H generate an exponentially decaying bisemigroup

If so, what can we say about $\mathcal{M}_+$?

Bisemigroups

A closed and densely defined linear operator S on a Banach space $\mathcal{X}$ is called *exponentially dichotomous* if there is a projection P commuting with S such that

$-S|_{\text{Im}\,P}$ and $S|_{\text{Ker}\,P}$

generate exponentially decaying C_0-semigroups.

The *bisemigroup* generated by S

$$E(t;S) = \begin{cases} e^{tS}(I-P), & t>0 \\ -e^{tS}P, & t<0. \end{cases}$$

Its *separating projection* P is given by

$P = -E(0^-;S) = I_{\mathcal{X}} - E(0^+;S)$.

There is a $\varepsilon > 0$ such that

$\{\lambda \in \mathbb{C} : |\text{Re}\,\lambda| \le \varepsilon\} \subset \rho(S)$

and for every $x \in \mathcal{X}$

$(\lambda - S)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t;S)x\,dt, \qquad |\text{Re}\,\lambda| \le \varepsilon.$

Hence for every $x \in \mathcal{X}$ we have

$\|(\lambda - S)^{-1}x\| \to 0$

as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : |\text{Re}\,\lambda| \le \varepsilon'\}$ for some $\varepsilon' \in (0, \varepsilon]$.

We call the restrictions of e^{tS} to Ker P and of e^{-tS} to Im P the *constituent semigroups* of the exponentially dichotomous operator S.

First perturbation result: BGK

Theorem 1. (Bart, Gohberg, Kaashoek)

S exponentially dichotomous.

$\tilde{S}$ closed, densely defined with

$$\mathcal{D}(\tilde{S}^2) \subset \mathcal{D}(S^2),$$

and such that there is $\varepsilon' > 0$ with

$$\{\lambda \in \mathbb{C} : |Re\,\lambda| \le \varepsilon'\} \subset \rho(S) \cap \rho(\tilde{S})$$

and

$$\sup_{|Re\,\lambda| \le \varepsilon'} |\lambda^2| \|(\lambda - \tilde{S})^{-1} - (\lambda - S)^{-1}\| < \infty.$$

Then $\tilde{S}$ is exponentially dichotomous.

Formula for separating projection: for $x \in \mathcal{D}(\tilde{S}^2)$

$$\tilde{P}x = -\frac{1}{2\pi i} \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{1}{\lambda^2} (\lambda - \tilde{S})^{-1} \tilde{S}^2 x\,d\lambda$$

Try to apply to Hamiltonian operator

$$S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$$

$$\tilde{S} = H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}$$

The domain condition

$$\mathcal{D}(\tilde{S}^2) \subset \mathcal{D}(S^2)$$

is complicated.

Can we do better?

Two possibilities: strenghten the assumptions on S, or strenghten the assumptions on the perturbation.

Second perturbation result: LRR

Assume S generates an analytic bisemigroup (i.e., the constituent semigroups are analytic).

Then the separating projection

$$Px = -\frac{1}{2\pi i}\int_{\varepsilon'-i\infty}^{\varepsilon'+i\infty} \frac{1}{\lambda^2} S(\lambda - S)^{-1} Sx \, d\lambda$$

for $x \in \mathcal{D}(S)$.

Theorem 2. (Langer, Ran, van de Rotten) *S dichotomous, and for some ε*

$$\{\lambda \in \mathbb{C} : |Re\,\lambda| \leq \varepsilon\} \subset \rho(S)$$

and for some $\gamma > 0$ and $\beta > \frac{1}{2}$

$$\|(\lambda - S)^{-1}\| < \frac{\gamma}{1 + |\lambda|^\beta}, \quad |Re\,\lambda| \leq \varepsilon$$

For B bounded let $\tilde{S} = S + B$. Assume that

$$\{\lambda \in \mathbb{C} : |Re\,\lambda| \leq \varepsilon\} \subset \rho(\tilde{S})$$

Then $\tilde{S}$ is dichotomous as well.

Third perturbation result: MR

Theorem 3. (van der Mee, Ran)

S exponentially dichotomous

B compact,

$\tilde{S} = S + B$ with $\mathcal{D}(\tilde{S}) = \mathcal{D}(S)$.

Suppose that $i\mathbb{R} \subset \rho(\tilde{S})$.

Then $\tilde{S}$ is exponentially dichotomous.

Fourth perturbation result: MR

Our next result uses the concept of *immediately norm continuous* semigroups.

A semigroup $T(t)$ is called immediately norm continuous if $T(t)$ is norm continuous for $t > 0$.

Examples

- Analytic semigroups,
- immediately compact semigroups (i.e., $T(t)$ is a compact operator for $t > 0$)
- immediately differentiable semigroups

K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations* (Springer GTM **194**, Berlin, 2000). (4.26) in Chapter II.

Theorem 4. *Let $\mathcal{H}$ be a complex Hilbert space and let $(T(t))_{t\geq 0}$ be an uniform exponentially stable C_0-semigroup on $\mathcal{H}$. Then $(T(t))_{t\geq 0}$ is immediately norm continuous if and only if the resolvent $(\lambda - A)^{-1}$ of its infinitesimal generator A vanishes in the norm as $\lambda \to \infty$ along the imaginary line.*

K.-J. Engel and R. Nagel ibid. Section II 4.20.

Theorem 5. (van der Mee, Ran)

S exponentially dichotomous with immediately norm continuous constituent semigroups

B bounded,

$\tilde{S} = S + B$ with $\mathcal{D}(\tilde{S}) = \mathcal{D}(S)$.

Suppose that

$$\{\lambda \in \mathbb{C} : |Re\,\lambda| < \varepsilon\} \subset \rho(\tilde{S})$$

Then $\tilde{S}$ is exponentially dichotomous with immediately norm continuous constituent semigroups.

Sketch of proof

If $\tilde{S}$ generates a bisemigroup then the resolvent identity gives that it satisfies

$$E(t;\tilde{S})x - \int_{-\infty}^{\infty} E(t-\tau;S)BE(\tau;\tilde{S})x\,d\tau = E(t;S)x,$$

where $x \in \mathcal{H}$ and $0 \neq t \in \mathbb{R}$.

Conversely, consider the convolution equation

$$u(t,x) - \int_{-\infty}^{\infty} E(t-\tau;S)Bu(\tau,x), d\tau = E(t;S)x.$$

The symbol of the convolution integral equation

$$I + (\lambda - S)^{-1}B = (\lambda - S)^{-1}(\lambda - \tilde{S})$$

tends to I in the norm as $\lambda \to \infty$ in the strip $|\mathrm{Re}\,\lambda| \leq \varepsilon$.

Moreover, it is a compact perturbation of the identity which only takes invertible values on the imaginary axis. Thus there exists $\varepsilon_0 \in (0, \varepsilon]$ such that the symbol only takes invertible values on the strip $|\mathrm{Re}\,\lambda| \leq \varepsilon_0$.

By the Bochner-Phillips theorem the convolution equation has a unique solution

$$u(\cdot;x) = E(\cdot;\tilde{S})x.$$

Using results from Bart-Gohberg-Kaashoek we can conclude that this is a bisemigroup.

Application to the ARE

If A generates an immediately norm continuous exponentially stable semigroup, and D and Q are bounded, such that the resolvent of $H = \begin{pmatrix} A & -D \\ -Q & -A^* \end{pmatrix}$ contains a strip around $i\mathbb{R}$, then H generates a bisemigroup.

Moreover, these assumptions imply the existence of a bounded solution X of the ARE

$$(XDX - XA - A^*X - Q)x = 0, \quad x \in \mathcal{D}(A)$$

such that $A - DX$ is exponentially stable and X maps $\mathcal{D}(A)$ into $\mathcal{D}(A^*)$.

As another application: if only one of D or Q is compact and A generates an exponentially stable semigroup then H generates a bisemigroup and we again get the existence of a bounded solution of the ARE.