Polar decomposition in Pontryagin space

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Infinite dimensional version of finite dimensional work by the same authors and Y. Bolshakov and B. Reichstein.

Pontryagin space

A Kreĭn space is a direct sum of a Hilbert space and an anti-Hilbert space, in other words:

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$$

equipped with the indefinite inner product

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In case $\dim \mathcal{G}_2 < \infty$ the space is called a Pontryagin space.

Example

$\mathcal{G}$ a Hilbert space, and H an invertible selfadjoint operator, with $\dim E_{(-\infty,0)}(H) < \infty$. Then

$$[x, y] = \langle Hx, y \rangle$$

defines an indefinite inner product on $\mathcal{G}$ and with this indefinite inner product it becomes a Pontryagin space.

Henceforth our Pontryagin space will be of this type.

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H-unitary operators: invertible isometries.

H-adjoint of an operator X: $X^{[*]}$ defined by

$$[Xx, y] = [x, X^{[*]}y]$$

for all x, y.

H-polar decomposition

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Finite dimensional case

Lemma 1. *X admits H-polar decomposition if and only if $X^{[*]}X = A^2$ for some H-selfadjoint A with* Ker $X =$ Ker A. *In this case we can take U to be H-unitary.*

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Idea of proof: one way is evident. The other way, define an H-isometry U on the range of A to the range of X by $UAx = Xx$. Then use Witt's theorem to extend U to an H-unitary matrix.

Witt's Theorem

Theorem 1. *Let $[\cdot,\cdot]$ be the scalar product in $\mathbb{C}^n$ defined by the invertible hermitian matrix H. Let $U_0 : V_1 \to V_2$, where V_1 and V_2 are subspaces in $\mathbb{C}^n$, be a nonsingular linear transformation that preserves the scalar product*

$$[U_0x, U_0y] = [x, y] \quad \textit{for every} \quad x, y \in V_1.$$

Then there exists a linear transformation U on $\mathbb{C}^n$ such that $[Ux, Uy] = [x, y]$ for every vectors x and y, and $Ux = U_0x$ for every $x \in V_1$.

Plan of attack

To characterize those matrices that allow H-polar decomposition

- Study matrices of the form $X^{[*]}X$.
- Determine which of these have an H-selfadjoint square root.
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The second item puts restrictions on the eigenvalues of $X^{[*]}X$ in $(\infty, 0]$. The third item puts additional constraints on $\operatorname{Ker} X$.

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Infinite dimensional analogue

Recall: an H-polar decomposition $X = UA$ is H-unitary if the operator U can be chosen to be H-unitary. Remember: it need only be an H-isometry defined on a linear set containing the range of A.

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But what if we want something extra??

After all, partial H-isometries are rather weak.

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Proof is based on a Pontryagin space version of Witt's theorem.

Witt's theorem for Pontryagin space

Theorem 3. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be closed subspaces of $\mathcal{G}$ and let U_0 be a continuous H-isometric operator mapping $\mathcal{V}_1$ onto $\mathcal{V}_2$. Assume that $\operatorname{codim} \mathcal{V}_1 \geq \operatorname{codim} \mathcal{V}_2$. Then there exists a linear operator $U : \mathcal{G} \to \mathcal{G}$ such that*

- $[Ux, Uy] = [x, y]$ *for all* $x, y \in \mathcal{G}$
- $Ux = U_0 x$ *for all* $x \in \mathcal{V}_1$.

Moreover, in case $\operatorname{codim} \mathcal{V}_1 = \operatorname{codim} \mathcal{V}_2$, U_0 can be extended to an H-unitary operator on $\mathcal{G}$.

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Idea of proof: 1. $X^{[*]}X$ is a finite rank perturbation of X^*X. Hence $\sigma(X^{[*]}X)\setminus[0,\infty)$ is a discrete set in $\mathbb{C}\setminus[0,\infty)$ with possible accumulation points in $[0,\infty)$. Each of the points in this discrete set is an eigenvalue of finite algebraic multiplicity.

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2. $X^{[*]}X$ is H-selfadjoint in a Pontryagin space, and hence the number of non-real points in the spectrum is finite.
3. Only to prove that the negative spectrum is a finite set.

If $\lambda < 0$ is an eigenvalue of $X^{[*]}X$ with eigenvector x then

$$\lambda[x, x] = [Xx, Xx].$$

As $\lambda < 0$ either $[x, x] = [Xx, Xx] = 0$, or $[x, x] < 0$ or $[Xx, Xx] < 0$. Now it is a matter of dimension count, and the fact that we are working in a Pontryagin space (finite dimensional negative part).

Intermezzo: Critical points

If A is H-selfadjoint in a Pontryagin space $\mathcal{G}$ then there is polynomial $p(\lambda)$ such that $p(A)$ is H-nonnegative, that is

$$\langle Hp(A)x, x\rangle = [p(A)x, x] \geq 0$$

for all $x \in \mathcal{G}$. (In Kreĭn space terminology: every H-selfadjoint operator is definitizable.)

The set of *critical points* is defined as

$$c(A) = \sigma(A) \cap \{\lambda \in \mathbb{R} \mid p(\lambda) = 0\}$$

Spectral function

H. Langer showed that there is a "spectral function" for A.

Define R_A to be the semiring of all bounded intervals with end-points not in $c(A)$.

Then there is a map $\Delta \to E(\Delta)$ from R_A into the set of H-selfadjoint projections in $\mathcal{G}$ satisfying the usual properties for a spectral function.

Two types of critical points

Let α be in $c(A)$, let $\lambda_0 < \alpha, \lambda_1 > \alpha$.

If for each such λ_0, λ_1 the limits

$$\lim_{\lambda \uparrow \alpha} E([\lambda_0, \lambda]), \lim_{\lambda \downarrow \alpha} E([\lambda, \lambda_1])$$

exist in the strong operator topology, then α is called a *regular* critical point.

Otherwise it is called a *singular* critical point.

Regular critical points are indeed "regular": one can define $E(\Delta)$ as long as Δ contains no singular critical points (or has them as its endpoints).

Back to the main issue
A decomposition for $X^{[*]}X$

Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Then $\mathcal{G}$ decomposes *H-orthogonally* as follows:

$$\mathcal{G} = \mathcal{M}_{00} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$$

where each of these subspaces is invariant under $X^{[*]}X$ and has the following properties

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- $\sigma(X^{[*]}X|_{\mathcal{M}_2}) \subset [0, \infty)$; $\operatorname{Ker} X^{[*]}X|_{\mathcal{M}_2} = (0)$,

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- $\sigma(X^{[*]}X|_{\mathcal{M}_2}) \subset [0, \infty)$; $\operatorname{Ker} X^{[*]}X|_{\mathcal{M}_2} = (0)$,

- $\mathcal{M}_3$ is finite dimensional and $X^{[*]}X$ restricted to $\mathcal{M}_3$ has no spectrum in $[0, \infty)$.

H-selfadjoint square roots of $X^{[*]}X$

Suppose that $X^{[*]}X$ does not have a singular critical point at zero.

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and let $H_i = P_i H|_{\mathcal{M}_i}$.

Then there exists an H-selfadjoint square root A of $X^{[*]}X$, that is $X^{[*]}X = A^2$, if and only if Z_{00} has an H_{00}-selfadjoint square root and Z_3 has an H_3-selfadjoint square root.

Note: this reduces the problem to the finite dimensional case.

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Taking the H_2-selfadjoint square root of Z_2 involves techniques from operator theory (functional calculus). This makes heavy use of the fact that zero is not a singular critical point.

Example

$\mathcal{G} = L_2([0,1]) \oplus \mathbb{C}^2$, $H = I \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Let M_t be the operator of multiplication by t on $L_2([0,1])$, take z a non-zero complex number, and define

$$Z = \begin{bmatrix} M_t & 0 & z\sqrt{t} \\ \bar{z}\langle \cdot, \sqrt{t} \rangle & 0 & |z|^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then Z is bounded H-selfadjoint, $\sigma(Z) = [0,1]$ and zero is a critical point of Z.

Moreover, $Z = A^2$ where the H-selfadjoint A is given by

$$A = \begin{bmatrix} M_{\sqrt{t}} & 0 & z \cdot 1 \\ \bar{z}\langle \cdot, 1 \rangle & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

Here $a \in \mathbb{R}$ is arbitrary.

H-polar decomposition

Theorem 4. *Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Assume in addition that the range of X is closed. Then X admits H-polar decomposition if and only if Z_{00} has an H_{00}-polar decomposition and Z_3 has an H_3-selfadjoint square root (and in that case automatically also an H_3-polar decomposition), and $\mathcal{M}_1 \subset \operatorname{Ker} X$.*

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Again, this works via reduction to the finite dimensional case. The part on $\mathcal{M}_2$ is now much more difficult and requires a further decomposition.

H-unitary H-polar decomposition

Theorem 5. *Suppose that $X^{[*]}X$ does not have a singular critical point at zero. Assume in addition that the range of X is closed and* $\dim \operatorname{Ker} X = \operatorname{codim} \operatorname{Im} X$. *Then X admits H-unitary H-polar decomposition if and only if it admits H-polar decomposition.*