Communications in Mathematical Physics manuscript No.
(will be inserted by the editor)

Critical thresholds and the limit distribution in the Bak-Sneppen model

Ronald Meester[1] **and Dmitri Znamenski**[1]

Divisie Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit Amsterdam, de Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands; e-mail: `rmeester@cs.vu.nl`, `dznamen@cs.vu.nl`

Received: date / Accepted: date

Abstract: One of the key problems related to the Bak-Sneppen evolution model is to compute the limit distribution of the fitnesses in the stationary regime, as the size of the system tends to infinity. Simulations in [3], [1] and [4] suggest that the one-dimensional limit marginal distribution is uniform on $(p_c, 1)$, for some $p_c \sim 0.667$.

In this paper we define three critical thresholds related to avalanche characteristics. We prove that if these critical thresholds are the same and equal to some p_c (we can only prove that two of them are the same) then the limit distribution is the product of uniform distributions on $(p_c, 1)$, and moreover $p_c < 0.75$. Our proofs are based on a self-similar graphical representation of the avalanches.

MSC 2000 subject classifications. 60K35, 82B26, 92D15.

Key words. Bak-Sneppen, evolution, interacting particle system, self-organised criticality, coupling, stationary distribution, phase transition.

1. Introduction

The Bak-Sneppen model was introduced in [2], as a simple model of evolution, and has received a lot of attention in the literature: recently an Internet search engine found about 900 links by the keyword *Bak-Sneppen*.

The model is defined as follows. Consider a system with N species. These species are represented by N vertices on a circle, evenly spaced, say. Now each of these species is assigned a so called *fitness*, a number between 0 and 1. The higher the fitness, the better chance of surviving the species has. The dynamics of evolution is modelled as follows. Every discrete time step, we choose the vertex with minimal fitness, and we think of the corresponding species as disappearing completely. This species is then replaced by a new one, with a fresh and independent fitness, uniformly distributed on $[0, 1]$.

So far, the dynamics does not have any interaction between the species, and does not result in an interesting process. Indeed, if we only replace the species with the lowest fitness, then it is easy to see that the system converges to a situation with all fitnesses equal to 1. Interaction is introduced by also replacing the two *neighbours* of the vertex with lowest fitness by new species with independent fitnesses. This interaction represents co-evolution of related species. The neighbour interaction makes the model very attractive and highly non-trivial from a mathematical point of view.

The Bak-Sneppen model turns out to be also of some practical interest. For instance, in [16], [17] and [18], variations of the Bak-Sneppen model related to real evolution of bacteria populations are considered and in [19] and [20], variations related to macro-economical processes are studied. To avoid the mathematical difficulties, some simplified versions of the model have also been proposed, for example, the *mean field* version of the model ([9], [10], [11]) and discrete versions of the model ([12], [13] and [14]).

It is simple to run the model on a computer. Simulations then suggest the following behaviour, for large N ([3], [1] and [4]). It appears that the one-dimensional marginals are uniform (in the limit for $N \to \infty$) on $(p_c, 1)$ for some p_c whose numerical value is close to 0.667, see [4]. In this article, we make significant progress towards this conjecture.

The fitnesses of the vertices are random variables with values in $[0, 1]$ and we update them according to the uniform distribution on $[0, 1]$. For computational reasons however, it is convenient for the fitnesses to have values in $[0, \infty]$ and to update them according to the exponential distribution with parameter 1, say (see for example [5]). In this new *exponential setup* (denoted in the sequal by the *BS-process*) a threshold b corresponds to the threshold $q(b) = 1 - e^{-b}$ in the original *uniform setup*. The conjecture in the exponential setup says that the one-dimensional stationary marginals (in the limit for $N \to \infty$) are exponentially distributed above b_c, for some b_c whose numerical value is close to $1.0996 \simeq q^{-1}(0.667)$.

To prepare for the main results of this article, in Section 2 we define the so called *locking thresholds representation* of the process, where we ignore all information irrelevant for the distribution of the process. This representation captures the essential characteristics of the BS-process, but at the same time, it has a much simpler spatial structure. Thus the representation has a potential to improve, for example, the simulation results.

It is natural to define *avalanches* of fitnesses from threshold $b \in [0, \infty]$: start counting at the moment that all fitnesses are above b and finish the counting at the first next moment that all fitnesses are above b again. In Section 3 we will give a more precise definition of an avalanche, and relate to it three characteristics: the *mean range* $R_N(b)$, *mean duration* $D_N(b)$ and the probability $P_N(b)$ that an avalanche is of range N. In Section 4 we prove that the above avalanche characteristics have limits, as $N \to \infty$, to be denoted by $R_\infty(b)$, $D_\infty(b)$ and $P_\infty(b)$ respectively, and that these limits are non-decreasing in b. We associate to them three *critical thresholds* b_c^r, b_c^d and b_c^p respectively, non-trivial in the sense that the values 0 and ∞ are excluded.

The central results of this article are stated in Section 3 and consist of three parts. First we relate mean duration $D_N(b)$ and mean range $R_N(b)$ via a differential equation, and prove that their critical thresholds are the same, i.e. $b_c^r = b_c^d$. Secondly, we prove that if all critical thresholds are the same and equal to b_c, say, then the limit distribution is the product of exponential distributions above b_c (or in terms of the uniform setup, the limit distribution is product of the uniform distributions on $[q_c, 1]$, where $q_c = q(b_c)$). Finally we prove that $b_c^r < 2 \log 2$ (in the uniform setup the upper bound is 0.75) with the help of a differential inequality for $R_\infty(b)$.

Most of our proofs are based on the so called *graphical representation* of an avalanche which is defined in Section 4. The graphical representation and some of the monotonicity properties of avalanches are borrowed from our previous work [15]. We include them here for convenience.

2. The locking thresholds representation

Let $N \geq 3$, and let $\Lambda(N) = \{-N+1, \dots, -1, 0\}$ index the set of N vertices on the circle, so that 0 and $-N+1$ are neighbours. We use negative indices to have a notation compatible with [15]. Later we will add and subtract the indices $\Lambda(N)$ as integers, identifying them modulo N. For example, sometimes we denote vertex $-N+1$ by vertex 1. To simplify expressions, we will omit $(\mathrm{mod}\ N)$ in the algebraic operations on $\Lambda(N)$ as long as it does not lead to a confusion.

For any $N \geq 3$ consider a BS-process on $\Lambda(N)$, and for any $n \in \mathbb{N}$, let $\mathcal{X}_N(n) = \{X_{N,x}(n)\}_{x \in \Lambda(N)}$ be the collection of fitnesses at time $n \geq 0$. We assume that at the initial time $n = 0$ all the fitnesses are i.i.d. and exponentially distributed. For any $x_1 < \cdots < x_k \in \Lambda(N)$, let $F_{N,x_1,\dots,x_k}(n,\cdot)$ denote the joint distribution function of $X_{N,x_1}(n), \dots, X_{N,x_k}(n)$ at time $n \geq 0$, i.e. for any $b_1, \dots, b_k \in \mathbb{R}_+$, we define

$$F_{N,x_1,\dots,x_k}(n, b_1, \dots, b_k) = P\Big(X_{N,x_1}(n) \leq b_1, \dots, X_{N,x_k}(n) \leq b_k\Big).$$

We will now inductively define the so called *locking thresholds* $\mathcal{Y}_N(n) = \{Y_{N,x}(n)\}_{x \in \Lambda(N)}$ and prove that, given $\mathcal{Y}_N(n)$, the fitnesses $\mathcal{X}_N(n)$ are independent and exponentially distributed above their locking thresholds. More precisely, we will show that for any $x_1 < \cdots < x_k \in \Lambda(N)$, and $b_1, \dots, b_k \in \mathbb{R}_+$ we have the representation

$$F_{N,x_1,\dots,x_k}(n, b_1, \dots, b_k) = \int\limits_0^\infty \cdots \int\limits_0^\infty \left(\prod_{i=1}^{k} \Theta_{s_1}(b_i) \right) \mathbb{G}_{N,x_1,\dots,x_k}(n, ds_1, \dots, ds_k), \tag{2.1}$$

where, $\Theta_s(\cdot)$ is the exponential distribution function above s, i.e.

$$\Theta_s(b) = \begin{cases} 0, & \text{if } b < s, \\ 1 - \exp(-b+s), & \text{if } b \geq s, \end{cases}$$

and $\mathbb{G}_{N,x_1,\ldots,x_k}(n,\cdot)$ is the joint distribution of $\mathcal{Y}_{N,x_1}(n),\ldots,\mathcal{Y}_{N,x_k}(n)$.

Let us define $Y_{N,x}(0) = 0$, for all $x \in \Lambda(N)$. Since the collection $\mathcal{X}_N(0)$ is independent and exponentially distributed, we have the basis of the induction. Suppose that, for some $n \geq 0$, the locking thresholds $\mathcal{Y}_N(n)$ are defined, and that $\mathcal{X}_N(n)$, given $\mathcal{Y}_N(n)$, is an independent and exponentially distributed collection above their locking thresholds. Let $x^* = x^*_N(n)$ be the vertex with minimal fitness at time n. Then, given $\mathcal{Y}_N(n)$, and given $X_{N,x^*}(n)$, the fitnesses $(X_{N,y}(n))_{y\in\Lambda(N)\setminus\{x^*\}}$ are independent and exponentially distributed above thresholds $(\max\{Y_{N,y}(n), X_{N,x^*}(n)\})_{y\in\Lambda(N)\setminus\{x^*\}}$. According to the update rules, we update at time n the neighbourhood $\mathcal{N}(x^*_N(n)) = \{x^*_N(n)-1, x^*_N(n), x^*_N(n)+1\}$ of $x^*_N(n)$ and replace their fitnesses by three new independent and exponentially distributed random variables. Hence, we define

$$Y_{N,y}(n+1) = \begin{cases} \max\{Y_{N,y}(n), X_{N,x^*}(n)\}, & \text{for } y \in \Lambda(N)\setminus\mathcal{N}(x^*_N(n)), \\ 0, & \text{for } y \in \mathcal{N}(x^*_N(n)), \end{cases}$$

and observe that $\mathcal{X}_N(n+1)$, given $\mathcal{Y}_N(n+1)$, is an independent and exponentially distributed collection above $\mathcal{Y}_N(n+1)$, i.e. we have the step of induction and (2.1) is proved.

The locking thresholds are useful for the study of the limit behaviour of the process as follows. For any $x_1 < \cdots < x_k \in \mathbb{Z}$, $b_1,\ldots,b_k \in \mathbb{R}_+$, $n \geq 1$ and N sufficiently large, we have $x_1 < \cdots < x_k \in \Lambda(N) \pmod N$, and thus

$F_{N,x_1,\ldots,x_k}(n, b_1, \ldots, b_k)$ is well-defined. For any $N \geq 3$ and $n \in \mathbb{N}$, let $G_N(n, \cdot)$ denote the distribution function of $Y_{N,0}(n)$.

Lemma 2.1 Suppose there exists $0 < b_c < \infty$, such that for any $b < b_c$,

$$\limsup_{N\to\infty} \limsup_{n\to\infty} G_N(n, b) = 0,$$

and for any $b' > b_c$,

$$\liminf_{N\to\infty} \liminf_{n\to\infty} G_N(n, b') = 1.$$

Then the limit distribution in the BS-process exists and is equal to the product of $\Theta_{b_c}(\cdot)$, i.e. for any $x_1 < \cdots < x_k \in \mathbb{Z}$ and $b_1, \ldots, b_k \in \mathbb{R}_+$,

$$\lim_{N\to\infty} \lim_{n\to\infty} F_{N,x_1,\ldots,x_k}(n, b_1, \ldots, b_k) = \prod_{i=1}^{k} \Theta_{b_c}(b_i).$$

PROOF: The proof is a simple consequence of the representation (2.1), and for simplicity we give it for the one-dimensional marginals $F_{N,0}(n, b)$ only. Let $b \geq 0$, and observe that for any $y \geq 0$, we have

$$|\Theta_y(b) - \Theta_{b_c}(b)| \leq 2. \tag{2.2}$$

Since $\Theta_y(b)$ is a continuous function in y, for any $0 < \varepsilon < b_c$ there exists $0 < \delta < \varepsilon$ such that

$$|\Theta_y(b) - \Theta_{b_c}(b)| \leq \varepsilon, \text{ for any } y \in [b_c - \delta, b_c + \delta].$$

Hence, according to (2.1) and (2.2), we have

$$\begin{aligned}
|F_{N,0}(n, b) - \Theta_{b_c}(b)| &\leq 2G_N(n, b_c - \delta) + \varepsilon\Big(G_N(n, b_c + \delta) - G_N(n, b_c - \delta)\Big) \\
&\quad +2(1 - G_N(n, b_c + \delta)) \\
&\leq 2G_N(n, b_c - \delta) + \varepsilon + 2(1 - G_N(n, b_c + \delta)),
\end{aligned}$$

and thus, due to the conditions of the lemma,

$$\lim_{N\to\infty}\lim_{n\to\infty}|F_{N,0}(n,b)-\Theta_{b_c}(b)|<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this proves the lemma.

□

3. Critical thresholds and main results

In this section we define *avalanches* on $\Lambda(N)$ and, in the second half of the section, on $\mathbb{Z}$. After that we relate them to a number of critical thresholds and show that the distribution of a locking threshold, in the limit (first the time $n\to\infty$, and then the size of the system $N\to\infty$) is concentrated between the smallest and the largest critical threshold.

We say that in the time interval $[n,n+d)$, an *avalanche from threshold* $b\in[0,\infty)$ (also referred to as a b-*avalanche*) with *origin* at $x\in\Lambda(N)$ and *duration* $d\geq 1$ occurs, if at time n, x is the vertex with minimal fitness, above threshold b, and $n+d$ is the first moment after n with all fitnesses again above b. The *range set* of the b-avalanche is the collection of vertices updated during the avalanche, and the *range* of the b-avalanche is the number of different vertices in the range set. Note that, according to this definition, if at times n and $n+1$ all the fitnesses are above b, then in the time interval $[n,n+1]$ an avalanche of range 3 and duration 1 occurs, even though there were no fitnesses below b.

For any $b>0$, the BS-process can be considered as a sequence of b-avalanches. Since during a b-avalanche the dynamics is completely determined by the position of the origin and the vertices with fitness below b, the range set of the b-avalanche $[n,n+d]$ depends only on the position of the origin x, but not on the details of the initial configuration at time n. If we shift the whole range set by $-x$ vertices, we obtain a set whose distribution is independent of x and of the initial configuration of fitnesses. The origin x itself naturally does depend on the configuration of fitnesses at time n, because x is the vertex with the minimal fitness at time n. The duration of the avalanche is clearly independent of the initial configuration of fitnesses. Thus, for any $b > 0$, the BS-process is a sequence of b-avalanches with i.i.d. durations and i.i.d. (shifted) range sets. These facts suggest that we should study a single b-avalanche in some detail.

Consider a b-avalanche on $\Lambda(N)$, with the origin at x. Its principal characteristics are the *range set* $\xi_N(x, b)$ and the *duration* $\eta_N(x, b)$. Two useful functions of the principal characteristics are the *range* of the avalanche $r_N(x, b) = |\xi_N(x, b)|$, and the indicator function to have a *spanning avalanche*, $\mathbf{1}\{r_N(x, b) = N\}$.

We define the corresponding mean values, independent of x, as follows

$$R_N(b) = E(r_N(x, b)),$$

$$D_N(b) = E(\eta_N(x, b)),$$

$$P_N(b) = P(r_N(x, b) = N).$$

From now on, we omit x in the notation of the avalanche characteristics if $x = 0$, i.e., for any $b \geq 0$, $\xi_N(b) = \xi_N(0, b)$, $\eta_N(b) = \eta_N(0, b)$, $r_N(b) = r_N(0, b)$. The following lemma is intuitively obvious. Its proof will be given in Section 4.

Lemma 3.1 $R_N(b)$, $D_N(b)$ and $P_N(b)$ are non-decreasing in b.

We now look at the limit behavior of the above functions, as N tends to infinity.

The notion of an avalanche on $\Lambda(N)$ can be naturally extended to an avalanche on $\mathbb{Z}$, as follows. Assume that every vertex $x \in \mathbb{Z}$ accommodates a *fitness*, a random variable with value in $[0, \infty]$. Choose an arbitrary threshold $b > 0$. If the number of vertices with fitness below b is finite and positive, then the update rules of BS-model are still well-defined. Suppose, therefore, that at time 0 we have a configuration with all fitnesses above b, and that we choose an arbitrary $x \in \mathbb{Z}$, regardless the values of the fitnesses, as the origin. We start by updating x and its two neighbours, $x-1$ and $x+1$. If, after that, among $x-1, x, x+1$ there are vertices with fitnesses below b, we choose the one with minimal fitness, and update it together with its two neighbours, and so on. As soon as we get a configuration with all fitnesses above b, we stop the procedure. Then we define the *duration* $\eta(x, b)$, and the *range set* $\xi(x, b)$ of the avalanche in the obvious way. Note, however, that it is perhaps possible that we never stop and that there always is at least one fitness below the threshold. In this case, we say that an *infinite avalanche* occurs. For an infinite avalanche, we set $\eta(x, b) = \infty$, and define $\xi(x, b)$ as the union of the vertices updated during $[0, m]$, $m \in \mathbb{N}$. It is easy to see that for an infinite avalanche, $|\xi(x, b)| = \infty$ a.s.

The pair $(\xi(x, b) - x, \eta(x, b))$ is clearly independent of the configuration at time 0, for the same reasons as before. Similar to the avalanches on $\Lambda(N)$, we consider the *range* $r(x, b) = |\xi(x, b)|$, and the indicator function to have an *infinite avalanche* $\mathbf{1}\{r(x, b) = \infty\}$.

Define the corresponding mean values, independent of x:

$$R_\infty(b) = E(r(x, b)),$$

$$D_\infty(b) = E(\eta(x, b)),$$

$$P_\infty(b) = P(r(x, b) = \infty).$$

As before, we omit x in the notation of the avalanche characteristics if $x = 0$, i.e., for any $b \geq 0$, $\xi(b) = \xi(0, b)$, $\eta(b) = \eta(0, b)$, $r(b) = r(0, b)$. The following theorem relates the avalanches on $\Lambda(N)$ to these on $\mathbb{Z}$.

Theorem 3.2 For any $b > 0$, and any $N \geq 3$,

$$R_N(b) \leq R_\infty(b), \quad \lim_{N\to\infty} R_N(b) = R_\infty(b),$$

$$P_N(b) \geq P_\infty(b), \quad \lim_{N\to\infty} P_N(b) = P_\infty(b),$$

$$D_N(b) \leq D_\infty(b), \quad \lim_{N\to\infty} D_N(b) = D_\infty(b).$$

The proof of this theorem will be given in Section 5, with a coupling argument.

The above theorem, together with Lemma 3.1, shows that $R_\infty(b)$, $D_\infty(b)$ and $P_\infty(b)$ are non-decreasing in b. Now we are ready for the definitions of the critical thresholds.

According to Lemma 3.1 in [15], $P_\infty(68) > 0$. Hence there exists a *finite* critical threshold

$$b_c^p = \inf\{b > 0 \,:\, P_\infty(b) > 0\}.$$

Since for any $b > 0$ such that $P_\infty(b) > 0$, we also have $R_\infty(b) = \infty$, there exists a finite critical threshold

$$b_c^r = \inf\{b > 0 \,:\, R_\infty(b) = \infty\},$$

and $b_c^r \leq b_c^p$. Since, for any $b > 0$, $D_\infty(b) \geq R_\infty(b) - 2$, infinite range implies infinite duration, and hence there exists a finite critical threshold

$$b_c^d = \inf\{b > 0 \,:\, D_\infty(b) = \infty\},$$

with $b_c^d \leq b_c^r$. Thus the three critical thresholds are ordered by definition as

$$b_c^d \leq b_c^r \leq b_c^p.$$

It is not hard to see that $b_c^d > 0$, so all critical values are non-trivial.

Theorem 3.3 We have

$$b_c^d = b_c^r, \tag{3.1}$$

and as long as $R_\infty(b)$ is finite, $D_\infty(b)$ is differentiable, and

$$\tfrac{d}{db} D_\infty(b) = D_\infty(b) R_\infty(b). \tag{3.2}$$

The proof of this theorem is given in Section 6. The following two results are proved in Section 7 and Section 8.

Theorem 3.4 If $b_c^d = b_c^r = b_c^p = b_c$ say, then the limit distribution in the BS-process is the product of exponential distributions above b_c.

Theorem 3.5 It is the case that

$$b_c^r \leq 2 \log 2.$$

4. The self-similar graphical representation

Let $N \geq 3$ and consider a b-avalanche on $\Lambda(N)$, starting at time 0 at the vertex 0. So at time 0, the vertex 0 and its two neighbours are updated. Since this corresponds to a minimal avalanche from threshold 0, we can write this in terms of the range set and the duration as $\xi_N(0) = \{-1, 0, 1\}$, $\eta_N(0) = 1$. We can now graphically illustrate the continuation of this b-avalanche on $\Lambda(N) \times \mathbb{R}^+$ (space $\times$ fitness) as follows. Look for the vertex with minimal fitness, and call this vertex x. Suppose that the fitness of x is equal to $s < b$. (Note that x must be the vertex 0 or one of its two neighbours.) Due to the lack of memory property of the exponential distribution, the fitnesses of the other two vertices in $\xi_N(0)$ are independent and exponentially distributed on $[s, \infty)$. We continue updating according to the appropriate rules, and wait until all fitnesses are above the threshold s. This in itself constitutes an s-avalanche, starting at x. We denote by $x + \hat{\xi}_N(s)$ the range set of this s-avalanche. In the graphical representation, we draw an arrow from the space-fitness point (x, s) to the space-fitness points (y, s), for all $y \in x + \hat{\xi}_N(s)$. In terms of the range set we write this as $\xi_N(s) = \xi_N(0) \cup \{x + \hat{\xi}_N(s)\}$, where $\xi_N(s)$ is the set of vertices updated to the end of the s-avalanche.

After the s-avalanche has ended, the fitnesses of all vertices in $\xi_N(s)$ are independent and exponentially distributed on $[s, \infty)$, due to the lack of memory property of the exponential distribution. We now look for the minimal fitness among all vertices in $\xi_N(s)$. If this minimal fitness is above b, then the b-avalanche has stopped. If this minimal fitness is equal to t, where $s < t < b$, and is associated with the vertex y, say, then we start, as before, a t-avalanche with origin y. We continue updating until all fitnesses are above t. If $y+\hat{\xi}_N(t)$ denotes the range set of this t-avalanche, then we draw an arrow in the graphical representation from the space-fitness point (y, t) to all space-fitness points (z, t), for $z \in y + \hat{\xi}_N(t)$. In terms of the range set, we write this as $\xi_N(t) = \xi_N(s) \cup \{y + \hat{\xi}_N(t)\}$, where $\xi_N(t)$ is the set of vertices updated at the end of the t-avalanche.

We continue in the obvious way: this process will stop a.s. as soon as all fitnesses are above b. The idea of avalanches forming a hierarchical structure of subavalanches is also mentioned in [5], in a slightly different context.

So, we have defined a random graph on $\Lambda(N) \times \mathbb{R}^+$. This random graph is a subgraph of a graphical representation GR_N defined formally as follows.

Let $\Pi(N) = \{\Pi_{N,k}\}_{k \in \Lambda(N)}$ be a collection of independent homogeneous Poisson processes. For each process $\Pi_{N,k}$ we perform the following procedure. At the j-th arrival $\tau_{k,j}$ of $\Pi_{N,k}$, we draw independently the pair $\Big(\hat{\xi}_N(\tau_{k,j}), \hat{\eta}_N(\tau_{k,j})\Big)$, the (range set, duration) of a typical $\tau_{k,j}$-avalanche with origin at 0. Then the pair $\left(k+\hat{\xi}_N(\tau_{k,j}), \hat{\eta}_N(\tau_{k,j})\right)$ is distributed as the (range set, duration) of a typical $\tau_{k,j}$-avalanche, with origin at k. We draw arrows in $\Lambda(N)\times\mathbb{R}^+$ from $(k,\tau_{k,j})$ to $(y,\tau_{k,j})$, for all $y\in k+\hat{\xi}_N(\tau_{k,j})$.

For any $b_1<b_2$ we say that (x,b_1) is *connected* to (x,b_2) by a *time segment*. A *path in* GR_N is a sequence $(x_0,s_0),\dots,(x_n,s_n)$ of points in $\Lambda(N)\times\mathbb{R}^+$, such that every pair (x_j,s_j), (x_{j+1},s_{j+1}) is connected by either by a time segment or an arrow. For any $A,B\subseteq\Lambda(N)$, and $b_1\le b_2\in\mathbb{R}$, we write $(A,b_1)\rightsquigarrow(B,b_2)$ in GR_N, if there exists at least one path in GR_N from (x,b_1) to (y,b_2), for some $x\in A$ and $y\in B$. See Figure 1 for an illustration.

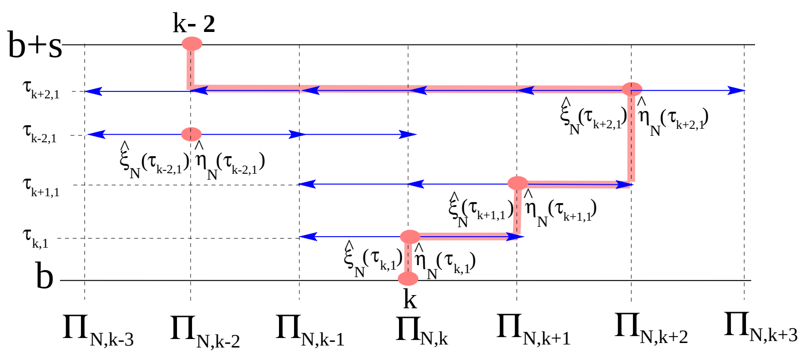

Fig. 1. The graphical representation GR_N, where, for instance, $(k,b)\rightsquigarrow(k-2,b+s)$ in GR_N.

Then, for any $b>0$, the range set $\xi_N(b)$ of a b-avalanche with origin at 0 is the collection of all $x\in\Lambda(N)$ such that $(\{-1,0,1\},0)\rightsquigarrow(x,b)$ in GR_N. For any $b\ge0$ and $(k,j)\in\Lambda(N)\times\mathbb{N}$, we call the pair $\left(\hat{\xi}_N(\tau_{k,j}),\hat{\eta}_N(\tau_{k,j})\right)$, a *subavalanche in* GR_N, if $(\{-1,0,1\},0)\rightsquigarrow(k,\tau_{k,j})$ in GR_N, and a b-*subavalanche in* GR_N, if, in addition, $\tau_{k,j}\le b$. Then, for any $b>0$, the duration $\eta_N(b)$ of a b-avalanche with origin at 0, is one plus the total duration of all b-subavalanches in GR_N.

The graphical representation provides us with the following monotonicity properties. For any $A \subset \Lambda(N)$ and $b, s \geq 0$ we denote by $\xi_N^{(A,b)}(s)$ the collection of all $x \in \Lambda(N)$ such that $(A, b) \rightsquigarrow (x, b+s)$ in GR_N, and we denote by $\eta_N^{(A,b)}(s)$ the sum of $\hat{\eta}_N(\tau_{k,j})$ over all $b < \tau_{k,j} \leq b+s$, such that $(A, b) \rightsquigarrow (k, \tau_{k,j})$ in GR_N. Then for any $A \subseteq B \subseteq \Lambda(N)$, $0 \leq s_1 \leq s_2$, and $b \geq 0$

$$\begin{aligned} &\xi_N^{(A,b)}(s_1) \subseteq \xi_N^{(B,b)}(s_2), \\ &\eta_N^{(A,b)}(s_1) \leq \eta_N^{(B,b)}(s_2). \end{aligned} \tag{4.1}$$

In the particular cases of the range set $\xi_N(x, b) = \xi^{(\{x-1,x,x+1\},0)}(b)$ and the duration $\eta_N(x, b) = \eta^{(\{x-1,x,x+1\},0)}(b)$ of an avalanche from threshold b and with origin at x, (4.1) gives us

$$\begin{aligned} &\xi_N(x, b_1) \subseteq \xi_N(x, b_2), \\ &\eta_N(x, b_1) \leq \eta_N(x, b_2), \qquad \text{if } b_1 \leq b_2. \end{aligned} \tag{4.2}$$

Proof of Lemma 3.1: It follows from (4.2) that for any $b_1 \leq b_2$

$$E(|\xi_N(b_1)|) \leq E(|\xi_N(b_2)|),$$

$$E(\eta_N(b_1)) \leq E(\eta_N(b_2)),$$

$$E(\mathbf{1}\{|\xi_N(b_1)| = N\}) \leq E(\mathbf{1}\{|\xi_N(b_2)| = N\}).$$

$\square$

We define GR, a graphical representation for the process on $\mathbb{Z}$, in almost the same way as GR_N; the only thing we need to take care of are the infinite avalanches. We do this by restricting GR to the fitnesses less then b_c^p, i.e. GR is a random graph on the space-fitness diagram $\mathbb{Z} \times [0, b_c^p)$. We do the formal definition of GR and of the related things just without subscript N. So, let $\Pi(\mathbb{Z}) = \{\Pi_k\}_{k \in \mathbb{Z}}$ be the collection of independent homogeneous Poisson processes restricted to the interval $[0, b_c^p)$. For any $\tau_{k,j} \in \Pi_k$, we denote by $\left(\hat{\xi}(\tau_{k,j}), \hat{\eta}(\tau_{k,j})\right)$, the (range set, duration) of the $\tau_{k,j}$-avalanche. The notions of $(A, b_1) \rightsquigarrow (B, b_2)$ in GR, *subavalanche in* GR and b-*subavalanche in* GR are defined the same way. Again, for any $b < b_c^p$, the range set $\xi(b)$ of a b-avalanche with origin at 0 is the collection of all $x \in \mathbb{Z}$ such that $(\{-1, 0, 1\}, 0) \rightsquigarrow (x, b)$ in GR, and the duration $\eta(b)$ of a b-avalanche with origin at 0, is one plus the total duration of all b-subavalanches in GR.

The graphical representation GR provides us the same monotonicity properties as (4.1) for GR_N. Thus, in particular, the range set $\xi(b)$ and the duration $\eta(b)$ are monotone in b.

5. Proof of Theorem 3.2

In this section we define a slightly different graphical representation, $\widehat{GR}_N$, for avalanches on $\Lambda(N)$. The reason to do this is that we want to couple avalanches on $\Lambda(N)$ with avalanches on $\mathbb{Z}$, and our current graphical representation GR_N is not so suitable for this purpose. We will also need a graphical representation $\widehat{GR}$ for avalanches on $\mathbb{Z}$, see below.

The main difference between GR_N and $\widehat{GR}_N$ is the specification of the Poisson processes $\Pi(N)$. In $\widehat{GR}_N$ we specify these *during* the construction of s-avalanches, by selecting the required number of Poisson processes from an infinite sequence of Poisson processes $\hat{\Pi}(\infty)$. An avalanche of range k uses the first k processes of $\hat{\Pi}(\infty)$. We use the same sequence $\hat{\Pi}(\infty)$ for the construction of avalanches on $\Lambda(N)$, for *any* $N \geq 3$, and for avalanches on $\mathbb{Z}$. This gives a transparent coupling of avalanches on $\Lambda(N)$ and $\mathbb{Z}$.

We define the new graphical representation as follows. Let $\hat{\Pi}(\infty) = \Pi^1, \Pi^2, \dots$ be a sequence of independent homogeneous Poisson processes. We will use the first N processes of $\hat{\Pi}(\infty)$ for the new graphical representation of s-avalanches on $\Lambda(N)$, with origin at 0.

We define $\xi_N(0) = \{-1, 0, 1\}$, $\eta_N(0) = 1$, and associate with $\xi_N(0)$ the first three Poisson processes. We write this as $\Pi_{N,-1} = \Pi^1$, $\Pi_{N,0} = \Pi^2$ and $\Pi_{N,1} = \Pi^3$, where $\Pi_{N,k}$ denotes the process associated with vertex $k \in \Lambda(N)$. Let τ_1^N be the first arrival in the superposition of $\Pi_{N,k}$, $k \in \{-1, 0, 1\}$, and let κ_1^N be the position of the corresponding process: $\tau_1^N \in \Pi_{\kappa_1^N}$. Then we have $\xi_N(s) = \{-1, 0, 1\}$, $\eta_N(s) = 1$, for $0 < s < \tau_1^N$. At $s = \tau_1^N$ we draw independently a pair $\left(\hat{\xi}_N(\tau_1^N), \hat{\eta}_N(\tau_1^N)\right)$, distributed as the (range set, duration) of a typical τ_1^N-avalanche with origin at 0. Then the pair $\left(\kappa_1^N + \hat{\xi}_N(\tau_1^N), \hat{\eta}_N(\tau_1^N)\right)$ is distributed as the (range set, duration) of a typical τ_1^N-avalanche, with origin at κ_1^N. As in GR_N, we draw arrows in $\Lambda(N) \times \mathbb{R}^+$ from (κ_1^N, τ_1^N) to (y, τ_1^N), for all $y \in \kappa_1^N + \hat{\xi}_N(\tau_1^N)$. We define $\xi_N(\tau_1^N) = \{-1, 0, 1\} \cup \left\{\kappa_1^N + \hat{\xi}_N(\tau_1^N)\right\}$ and $\eta_N(\tau_1^N) = 1 + \hat{\eta}_N(\tau_1^N)$. If $\xi_N(\tau_1^N)$ is larger then $\{-1, 0, 1\}$ we associate with $\xi_N(\tau_1^N) \setminus \{-1, 0, 1\}$ the next $r_N(\tau_1^N) - r_N(0)$ Poisson processes from $\pi(\infty)$, and assign them to $(\Pi_{N,k})_{k \in \xi_N(\tau_1^N) \setminus \{-1,0,1\}}$. In this case we first add the required number of processes to the left of $\{-1, 0, 1\}$ and then to the right of $\{-1, 0, 1\}$.

We define τ_2^N as the first arrival after τ_1^N in the superposition of $\Pi_{N,k}$, $k \in \xi_N(\tau_1^N)$, and continue in the obvious way, each time adding a certain (random) number of vertices and corresponding Poisson process to our collection, until we have a collection of size N. The first time that we have N vertices, these N vertices form $\Lambda(N)$ and the corresponding Poisson processes define $\Pi(N)$. It is clear that this happens (with probability one) after a finite number of steps, and this means that we have defined $(\xi_N(s), \eta_N(s))$ for all $s \geq 0$. We denote the collection of arrivals below threshold b in the newly formed $\Lambda(N)$ by $\mathcal{T}(N)$.

We now define the graphical representation $\widehat{GR}$, for an avalanche on $\mathbb{Z}$. We proceed as for $\widehat{GR}_N$; the only thing we need to take care of are the infinite avalanches. We do this by restricting $\widehat{GR}$ to the fitnesses less then b_c^p, i.e. $\widehat{GR}$ is a random graph on the space-fitness diagram $\mathbb{Z} \times [0, b_c^p)$. We again use the sequence $\hat{\Pi}(\infty)$ to specify the required number of the Poisson processes. We denote by $\mathcal{T}$ the collection of arrivals of all specified Poisson processes in $\widehat{GR}$.

The coupling just described will be used in the second part of the proof of Theorem 3.2 below.

Proof of Theorem 3.2: Fix $b > 0$. We distinguish between two cases, $P_\infty(b) > 0$ and $P_\infty(b) = 0$.

First, suppose that $P_\infty(b) > 0$. In this case we have $R_\infty(b) = \infty$ and $D_\infty(b) = \infty$. We will couple a b-avalanche on $\mathbb{Z}$, with origin at 0, with a sequence of b-avalanches on $\Lambda(N), N \geq 3$, with origin at 0, in such a way that

$$\begin{aligned} &|\xi(b)| = \infty \;\text{ implies } r_N(b) = N, \text{ for all } N \geq 3. \\ &|\xi(b)| < \infty \;\text{ implies } r_N(b) = r(b), \text{ for } N \text{ large enough.} \end{aligned} \tag{5.1}$$

Then the theorem follows from (5.1), Fatou's lemma and the relation $\eta_N(b) \geq r_N(b) - 2$.

Let $\mathcal{G} = \left(G_j^{(1)}, G_j^{(2)}, G_j^{(3)}\right)_{i\in\mathbb{N}}$ be a sequence of triples, where $\left\{G_j^{(i)}\right\}$ are independent and exponentially distributed with parameter 1. We can use the sequence $\mathcal{G}$ as a sequence of updates to define a b-avalanche on $\mathbb{Z}$. Indeed, consider at the initial moment an arbitrary configuration of fitnesses above threshold b, and replace the fitnesses of $\{-1, 0, 1\}$ by $\left(G_1^{(1)}, G_1^{(2)}, G_1^{(3)}\right)$. If within

$\{-1, 0, 1\}$ there are vertices with fitnesses below the threshold b, we choose the one with minimal fitness, y say, and replace the fitnesses in $\{y-1, y, y+1\}$ by $\left(G_2^{(1)}, G_2^{(2)}, G_2^{(3)}\right)$, and so on. We can also, at the same time, use the sequence $\mathcal{G}$ to define a b-avalanche on $\Lambda(N)$. Since the avalanche on $\Lambda(N)$ is identical to the avalanche on $\mathbb{Z}$ until it spans the whole system, we have (5.1).

Next, suppose that $P_\infty(b) = 0$. The above coupling via the sequence $\mathcal{G}$ still gives us the first two lines of the theorem, but not the third line, because we might have $\eta_N(b) > \eta(b)$, if $r(b) \geq N$. We use the more complicated coupling described before this proof.

Since $P_\infty(b) = 0$, we can construct $(\xi(s), \eta(s))$, $s \leq b$ with the graphical representation $\widehat{GR}$ restricted to the thresholds $[0, b]$. Then we will couple this $\widehat{GR}$ with a sequence of $\widehat{GR}_N$, $N \geq 3$, restricted to the thresholds $[0, b]$, in such a way that $(\xi_N(s), \eta_N(s))$, $s \leq b$, corresponding to $\widehat{GR}_N$ satisfies, for any $s \leq b$,

$$\begin{aligned} \xi_N(s) = \xi(s) \,(\bmod N\,),\ \eta_N(s) = \eta(s), \quad &\text{on } |\xi(s)| < N, \\ r_N(s) = N, \eta_N(s) \leq \eta(s), \quad &\text{on } |\xi(s)| \geq N. \end{aligned} \tag{5.2}$$

The theorem then follows from Fatou's lemma, because for all N big enough, we have $|\xi(b)| < N$.

A construction of the above $\widehat{GR}_N$ requires the specification of $\hat{\Pi}(\infty)$ and the b-subavalanches. We will borrow $\hat{\Pi}(\infty)$ from $\widehat{GR}$ and then couple the b-subavalanches with these on $\mathbb{Z}$ in a proper way.

Suppose first that $\eta(b) = 1$. This means that in $\widehat{GR}$ the Poisson processes Π_{-1}, Π_0 and Π_1 have no arrivals at the time interval $[0, b]$, and hence $\xi(b) = \{-1, 0, 1\}$. Since we use the same $\hat{\Pi}(\infty)$ for the construction of $\widehat{GR}_N$ we also have that in $\widehat{GR}_N$ the Poisson processes $\Pi_{N,-1}$, $\Pi_{N,0}$ and $\Pi_{N,1}$ have no arrivals at the time interval $[0,b]$, and hence $\xi_N(b) = \{-1,0,1\}$, $\eta_N(b) = 1$ and we have constructed the coupling satisfying (5.2), on the event $\eta(b) = 1$.

Suppose now in an inductive fashion that for some $n \geq 1$ we can construct the coupling satisfying (5.2), for all $N \geq 3$, on $\eta(b) \leq n$. We now construct $\widehat{GR}_N$, for all $N \geq 3$, on $\eta(b) \leq n+1$.

If a b-avalanche on $\widehat{GR}$ has duration at most $n+1$ then all b-subavalanches on $\widehat{GR}$ have duration at most n, and we can couple them by induction, i.e. for any $\tau_i \in \mathcal{T}$ and any $N \geq 3$ we can define $(\hat{\xi}_N(\tau_i), \eta_N(\tau_i))$ satisfying:

$$\begin{aligned} \hat{\xi}_N(\tau_i) = \hat{\xi}(\tau_i)\,(\bmod N\,),\, \hat{\eta}_N(\tau_i) = \hat{\eta}(\tau_i), &\quad \text{on } |\hat{\xi}(\tau_i)| < N, \\ |\hat{\xi}(\tau_i)| = N, \hat{\eta}_N(\tau_i) \leq \hat{\eta}(\tau_i), &\quad \text{on } |\hat{\xi}(\tau_i)| \geq N. \end{aligned} \tag{5.3}$$

If $|\xi(b)| < N$ then, then $|\hat{\xi}(\tau_i)| < N$, for every $\tau_i \in \mathcal{T}$ and due to the first line of (5.3) the b-avalanches on $\widehat{GR}_N$ and $\widehat{GR}$ are identical and we have the first line in (5.2).

If $|\xi(b)| \geq N$, we define $\tau(N,b)$ as the first arrival in $\mathcal{T}$ such that $|\xi(\tau(N,b))| \geq N$ in $\widehat{GR}$. For any $s \in [0, \tau(N,b))$, we have $|\xi(s)| < N$, in $\widehat{GR}$ and, by the same reasoning as above, the first line in (5.2). Since $\tau(N,b) \in \mathcal{T}(N)$ we have, by the second line of (5.3), that $|\xi_N(\tau(N,b)| = N$. Hence, for any $t \in (s,b]$, $\widehat{GR}_N$ uses the first N Poisson processes of $\pi(\infty)$, while $\widehat{GR}$ uses at least these first N processes. Thus any arrival in $\mathcal{T}(N)$ is in $\mathcal{T}$, and we have the second line of (5.2).

□

6. Proof of Theorem 3.3 via differential equations for $D_N(b)$ and $R_N(b)$

Theorem 6.1 For any $N \geq 3$ $D_N(b)$ is differentiable with respect to b, and

$$\frac{d}{db}D_N(b) = D_N(b)R_N(b).$$

The proof of the theorem is a simple corollary of the following two lemmas.

Lemma 6.2 For any $b > 0$ we have

$$\tfrac{D_N(b+\varepsilon)-D_N(b)}{\varepsilon} \geq D_N(b)R_N(b)\Big(1+o(1)\Big), \text{ as } \varepsilon \to 0. \tag{6.1}$$

Lemma 6.3 For any $b > 0$ we have

$$\tfrac{D_N(b+\varepsilon)-D_N(b)}{\varepsilon} \leq D_N(b)\tfrac{R_N(b)}{1-\varepsilon R_N(b)}\Big(1+o(1)\Big), \text{ as } \varepsilon \to 0. \tag{6.2}$$

Proof of Theorem 3.3: Suppose that $R_\infty(b) < \infty$. Since $R_N(b) \leq R_\infty(b)$, and $D_N(0) = 1$, for any $N \geq 3$, Theorem 6.1 gives us

$$D_N(b) \leq \exp(R_N(b)b) \leq \exp(R_\infty(b)b),$$

and hence $D_\infty(b) < \infty$. Thus $R_\infty(b) < \infty$ implies $D_\infty(b) < \infty$, that is, $b_c^d = b_c^r$.

To obtain (3.2), one can simply take the limit $N \to \infty$ in (6.2) and (6.1), and then the limit $\varepsilon \to \infty$.

$\square$

It remains to prove Lemma 6.2 and Lemma 6.3.

Proof of Lemma 6.2: It follows from the properties of GR_N that

$$D_N(b) = E\Big(\eta_N(b)\Big) = E\Big(\sum_{\substack{x\in\Lambda(N),\tau_{x,i}\in\Pi_{N,x}\cap(0,b] \\ \text{s.t. } (\{-1,0,1\},0)\rightsquigarrow(x,\tau_{x,i}) \text{ in } GR_N}} \hat{\eta}_N(\tau_{x,i})\Big).$$

Hence

$$\begin{aligned} D_N(b+\varepsilon) - D_N(b) &= E\Bigg(\sum_{\substack{x \in \Lambda(N), \tau_{x,i} \in \Pi_{N,x} \cap [b,b+\varepsilon) \\ \text{s.t. } (\{-1,0,1\},0) \rightsquigarrow (x,\tau_{x,i}) \text{ in } GR_N}} \hat{\eta}_N(\tau_{x,i}) \Bigg) \\ &= E\Bigg(\sum_{\substack{x \in \Lambda(N), \tau_{x,i} \in \Pi_{N,x} \cap [b,b+\varepsilon) \\ \text{s.t. } (\xi_N(b),b) \rightsquigarrow (x,\tau_{x,i}) \text{ in } GR_N}} \hat{\eta}_N(\tau_{x,i}) \Bigg). \end{aligned} \tag{6.3}$$

For every $y \in \Lambda(N)$, define $\tau(y)$ as the first arrival in $\Pi_{N,y}$ after time b. Since, for any $y \in \xi_N(b)$, we have always $(\xi_N(b), b) \rightsquigarrow (y, \tau(y))$in GR_N, (6.3) is at least

$$E\Bigg(\sum_{y \in \xi_N(b) \text{ s.t. } \tau(y) < b+\varepsilon} \hat{\eta}_N(\tau(y)) \Bigg) = E\Bigg(\sum_{y \in \xi_N(b)} s_N(y,b) \Bigg), \tag{6.4}$$

where

$$s_N(y,b) = \hat{\eta}_N(\tau(y)) \mathbf{1}\{\tau(y) < b+\varepsilon\}.$$

We now couple $\hat{\eta}_N(\tau(y))$ with a smaller random variable, independent of $\tau(y)$. Consider GR_N. At time $\tau(y)$, we have in GR_N an independent subavalanche from threshold $\tau(y)$. Let GR'_N be a graphical representation for this avalanche. Then $\hat{\eta}_N(\tau(y))$ is one plus the total duration of all $\tau(y)$-subavalanches in GR'_N. Define $\hat{\eta}'_N(y,b)$ to be one plus the total duration of all b-subavalanches in GR'_N. Then $\hat{\eta}'_N(y,b)\mathbf{1}\{\tau(y) < b+\varepsilon\} \le s_N(y,b)$, and clearly $\hat{\eta}'_N(y,b)$ is independent of $\tau(y)$. It is easy to see that $\hat{\eta}'_N(y,b)$ is distributed as the duration of a b-avalanche, and therefore $E(\hat{\eta}'_N(y,b)) = D_N(b)$. Hence

$$\begin{aligned} E\Big(\hat{\eta}'_N(y,b) \mathbf{1}\{\tau(y) < b+\varepsilon\} \Big) &\ge D_N(b) P(\tau(y) < b+\varepsilon) \\ &= D_N(b) \Big(\varepsilon + o(\varepsilon) \Big). \end{aligned}$$

Taking into account that $(s_N(y,b))_{y\in\Lambda(N)}$, and thus $(\hat\eta'_N(y,b))_{y\in\Lambda(N)}$, are independent of $\xi_N(b)$, we finish the estimate (6.4) as

$$E\Big(\sum_{y\in\xi_N(b)} s_N(y,b)\Big) \ge E\Big(\sum_{y\in\xi_N(b)} \hat\eta'_N(y,b)\Big) \tag{6.5}$$

$$= E(\xi_N(b))E(\hat\eta'_N(0,b)) = R_N(b)D_N(b)\Big(\varepsilon + o(\varepsilon)\Big).$$

□

Proof of Lemma 6.3: First of all, we rewrite (6.2) in a more convenient form. For any $b,\varepsilon>0$ such that $0<\varepsilon<(R_N(b))^{-1}$, the above inequality is equivalent to

$$D_N(b+\varepsilon) - D_N(b) \le R_N(b)D_N(b+\varepsilon)\Big(\varepsilon + o(\varepsilon)\Big). \tag{6.6}$$

To prove (6.6) we again use the decomposition (6.3),

$$D_N(b+\varepsilon) - D_N(b) = E\Big(\sum_{\substack{x\in\Lambda(N),\tau_{x,i}\in\Pi_{N,x}\cap[b,b+\varepsilon)\\ \text{s.t. } (\xi_N(b),b)\rightsquigarrow(x,\tau_{x,i}) \text{ in } GR_N}} \hat\eta_N(\tau_{x,i})\Big).$$

For every $y\in\Lambda(N)$, define $\tau(y)$ as the first arrival in $\Pi_{N,y}$ after time b. Observe that if $(\xi_N(b),b)\rightsquigarrow(x,\tau_{x,i})$ in GR_N, for some $x\in\Lambda(N),\tau_{x,i}\in\Pi_{N,x}\cap[b,b+\varepsilon)$, then there exists at least one $y\in\xi_N(b)$ such that $\tau(y)<b+\varepsilon$, and $(\xi_N(b),b)\rightsquigarrow(y,\tau(y))\rightsquigarrow(x,\tau_{x,j})$ in GR_N. See Figure 2 for an illustration.

Hence we can continue the estimate as

$$\le E\Big(\sum_{y\in\xi_N(b),\tau(y)<b+\varepsilon}\Big(\hat\eta_N(\tau(y))$$

$$+ \sum_{\substack{x\in\Lambda(N),\,\tau_{x,j}\in\Pi_{N,x}\cap[\tau(y),b+\varepsilon)\\ \text{s.t. } (\xi_N(b),b)\rightsquigarrow(y,\tau(y))\rightsquigarrow(x,\tau_{x,j}) \text{ in } GR_N}} \hat\eta_N(\tau_{x,j})\Big)\Big) \tag{6.7}$$

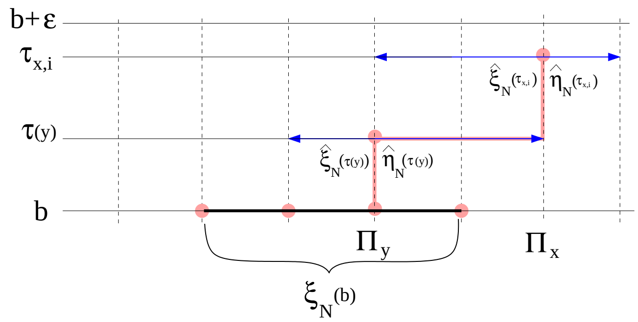

Fig. 2. Any path out of $\xi_N(b)$ should use at least one arrow inside $\xi_N(b)$.

and since, for any $y \in \xi_N(b)$, we have always $(\xi_N(b), b) \rightsquigarrow (y, \tau(y))$in GR_N,

$$= E\Bigg(\sum_{y\in\xi_N(b), \tau(y)<b+\varepsilon} \Bigg(\hat{\eta}_N(\tau(y)) + \sum_{\substack{x\in\Lambda(N), \tau_{x,j}\in\Pi_{N,x}\cap[\tau(y),b+\varepsilon), \\ \text{s.t. } (y,\tau(y))\rightsquigarrow(x,\tau_{x,j}) \text{ in } GR_N}} \hat{\eta}_N(\tau_{x,j}) \Bigg)\Bigg) \tag{6.8}$$

$$= E\Bigg(\sum_{y\in\xi_N(b)} S_N(y,b) \Bigg),$$

where

$$S_N(y,b) = \Bigg(\hat{\eta}_N(\tau(y)) + \sum_{\substack{x\in\Lambda(N), \tau_{x,j}\in\Pi_{N,x}\cap[\tau(y),b+\varepsilon) \\ \text{s.t. } (y,\tau(y))\rightsquigarrow(x,\tau_{x,j}) \text{ in } GR_N}} \hat{\eta}_N(\tau_{x,j}) \Bigg) \mathbf{1}\{\tau(y) < b+\varepsilon\}.$$

We will now prove that the conditional expectation of $S_N(y,b)$ given $\tau(y) < b+\varepsilon$, is the same as the unconditional expectation of $\eta_N(b+\varepsilon)$. Indeed, at arrival $\tau(y)$ we have in GR_N an independent subavalanche from threshold $\tau(y)$. We construct this subavalanche via an independent graphical representation GR'_N. If $\tau(y) \geq b+\varepsilon$ we do nothing. If $\tau(y) < b+\varepsilon$, we glue the part of GR'_N before threshold $\tau(y)$, with the part of GR_N after threshold $\tau(y)$, shifted by $-y$ vertices, see Figure 3.

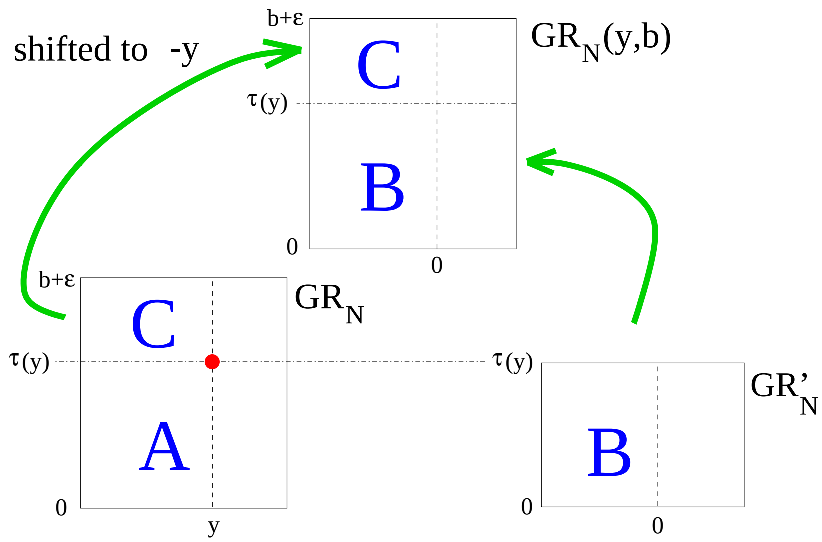

Fig. 3. We glue the part of GR'_N before threshold $\tau(y)$ with the part of GR_N after threshold $\tau(y)$ to construct $GR_N(y, b)$.

We denote the result by $GR_N(y, b)$. Since GR_N and GR'_N are independent and $\tau(y)$ is a stopping time with respect to $\Pi(N)$, $GR_N(y, b)$ is again a graphical representation. It is easy to see that $S_N(y, b)$, given $\tau(y) < b + \varepsilon$, is the total duration of all $(b + \varepsilon)$-subavalanches in $GR_N(y, b)$. Hence the conditional expectation of $S_N(y, b)$, given $\tau(b) < b + \varepsilon$, is the same as the expectation of $\eta_N(y, b + \varepsilon)$, i.e. the expectation of $\eta_N(b + \varepsilon)$. Therefore,

$$E(S_N(y, b) \,|\, \tau(b) < b + \varepsilon) = D_N(b + \varepsilon),$$

and hence,

$$\begin{aligned} E(S_N(y, b)) &= D_N(b + \varepsilon) P(\tau(y) < b + \varepsilon) \\ &= D_N(b + \varepsilon)\Big(\varepsilon + o(\varepsilon)\Big). \end{aligned}$$

Hence, taking into account that for any $y \in \Lambda(N)$, $S_N(y,b)$ and $\xi_N(b)$ are independent, we obtain that the r.h.s. of (6.7) is equal to

$$R_N(b)D_N(b+\varepsilon)\Big(\varepsilon + o(\varepsilon)\Big). \tag{6.9}$$

So we have (6.6) and the lemma.

$\square$

7. Proof of Theorem 3.4

Lemma 7.1 For any $b < b_c^d$ we have

$$\limsup_{N\to\infty} \limsup_{n\to\infty} G_N(n,b) = 0.$$

Lemma 7.2 Suppose for some $0 < b < b' < \infty$, we have

$$\lim_{N\to\infty} NP_N(b)D_N(b')/D_N(b) = \infty.$$

Then for any $b'' > b'$ we have

$$\liminf_{N\to\infty} \liminf_{n\to\infty} G_N(n,b'') = 1.$$

If $b' > b > b_c^p$ then $D_N(b') \geq D_N(b)$ and $P_N(b) \geq P_\infty(b) > 0$, uniformly on $N \geq 3$. This gives the following corollary of Lemma 7.2.

Corollary 1. *For any $b'' > b_c^p$ we have*

$$\liminf_{N\to\infty} \liminf_{n\to\infty} G_N(n,b'') = 1.$$

Proof of Theorem 3.4: Suppose $b_c^d = b_c^r = b_c^p$ and equal to b_c. Then, by Lemma 7.1 and Corollary 1, for any $b' > b_c$,

$$\liminf_{N\to\infty} \liminf_{n\to\infty} G_N(n,b') = 1,$$

and for any $b'' < b_c$,

$$\limsup_{N\to\infty} \limsup_{n\to\infty} G_N(n, b'') = 0.$$

Hence, by Lemma 2.1, the limit distribution in the BS-process exists and is equal to the product of exponential distributions above b_c.

$\square$

Note that the condition of Lemma 7.2 is weaker then that of Corollary 1, and to have the statement of Theorem 3.4 it would be sufficient to prove that for any $b' > b > b_c^r$

$$\lim_{N\to\infty} N P_N(b) D_N(b')/D_N(b) = \infty.$$

It remains to prove Lemma 7.1 and Lemma 7.2.

Proof of Lemma 7.1: Let $P_N(b,k)$ denote the probability that a b-avalanche has range k. Let $D_N(b|k)$ denote the mean duration of a b-avalanche, given the avalanche has range k. Decompose the Bak-Sneppen process into the sequence of b-avalanches. Since the b-avalanches have i.i.d. (range, duration), the probability that at time n we are in a b-avalanche of range k converges, as $n \to \infty$, to

$$\frac{D_N(b|k) P_N(b,k)}{\sum_{l=3}^{N} D_N(b|l) P_N(b,l)} = \frac{D_N(b|k) P_N(b,k)}{D_N(b)}.$$

For any $k \geq 3$, given that we are in a b-avalanche of range k, we have at most k locking thresholds below b. Thus for any $N \geq 3$ and $n \geq 3$, we have

$$\lim_{n\to\infty} G_{N,0}(n,b) = \lim_{n\to\infty} P(Y_{N,0}(n) \leq b) \leq \frac{\sum_{k=3}^{N} \frac{k}{N} D_N(b|k) P_N(b,k)}{D_N(b)}. \tag{7.1}$$

It follows from the coupling of the avalanches on $\Lambda(N)$ and $\mathbb{Z}$, introduced in the proof of Theorem 3.2, that for $k < N$, the values of $D_N(b|k)$ and $P_N(b,k)$ are independent of N,

$$D_N(b|k) = D(b|k) = E\left(\eta(b)\,\Big|\,|\xi(b)| = k\right),$$

$$P_N(b,k) = P(b,k) = P(|\xi(b)| = k).$$

Hence

$$D_N(b) = D_N(b|N)P_N(b,N) + \sum_{k=3}^{N} D(b|k)P(b,k).$$

Since, for any $b < b_c^d$, $D_\infty(b) < \infty$, we have, according to Lemma 3.2,

$$\lim_{N\to\infty} D_N(b) = E(\eta(b)) = \sum_{k=3}^{\infty} D(b|k)P(b,k) < \infty.$$

Hence for any $\varepsilon > 0$, there exists $K(\varepsilon) \geq 3$ such that for any $N > K(\varepsilon)$,

$$\begin{aligned}
&\sum_{k=K(\varepsilon)+1}^{N} D_N(b|k)P_N(b,k) \\
&\leq \sum_{k=K(\varepsilon)+1}^{N-1} D(b|k)P(b,k) + D_N(b|N)P_N(b,N) < \varepsilon.
\end{aligned}$$

Hence the r.h.s. of (7.1) is at most

$$\begin{aligned}
&\frac{\sum_{k=3}^{K(\varepsilon)} \frac{K(\varepsilon)}{N} D_N(b|k)P_N(b,k) + \sum_{k=K(\varepsilon)+1}^{N} D_N(b|k)P_N(b,k)}{\sum_{k=3}^{N} D_N(b|k)P_N(b,k)} \\
&\leq \frac{\frac{K(\varepsilon)}{N} D_N(b) + \varepsilon}{D_N(b)} \\
&\leq 2\varepsilon, \qquad \text{for } N > K(\varepsilon)/\varepsilon.
\end{aligned} \tag{7.2}$$

Combining (7.1) and (7.2) we have

$$\limsup_{N\to\infty} \lim_{n\to\infty} G_{N,0}(n,b) \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have the lemma.

$\square$

Proof of Lemma 7.2: We will modify the proof of Theorem 1.1 in [15]. Let $0 < b < b' < \infty$ be fixed and satisfy the condition of the lemma. Fix some arbitrary $b'' > b'$. For any $N \geq 3$ consider a BS-process on $\Lambda(N)$, such that at the initial time all the fitnesses are i.i.d. and exponentially distributed. Define a sequence $(\tau_{j,A}, \tau_{j,R})_{j\in\mathbb{N}}$ of stopping times, with respect to the natural filtration, as follows: $\tau_{0,A} = \tau_{0,R}$ and they are equal to the first moment that all the locking thresholds are above b. For any $j \in \mathbb{N}$, $\tau_{j+1,A}$ is the end of the first b-avalanche of range N after $\tau_{j,R}$, and $\tau_{j+1,R}$ is the first moment after time $\tau_{j+1,A}$ such that all the fitnesses are above threshold b''. See Figure 4 for an illustration. For any $j \in \mathbb{N}$, we call the time interval $I_j^N(b, b'') = [\tau_{j,R}, \tau_{j+1,R})$ the j-th

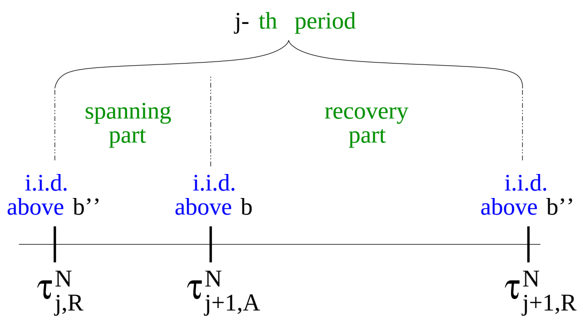

Fig. 4. $I_j^N(b, b'') = [\tau_{j,R}, \tau_{j+1,R})$ is the j-th *period*. It consists of the *avalanche part* $I_{j,A}^N(b) = [\tau_{j,R}, \tau_{j+1,A})$, and the *recovery part* $I_{j,R}^N(b, b'') = [\tau_{j+1,A}, \tau_{j+1,R})$.

period, and within the period $I_j^N(b, b'')$ we distinguish between the *avalanche part* $I_{j,A}^N(b) = [\tau_{j,R}, \tau_{j+1,A})$, and the *recovery part* $I_{j,R}^N(b, b'') = [\tau_{j+1,A}, \tau_{j+1,R})$. Observe that the recovery part can be empty, if at time $\tau_{j+1,A}$ the minimal fitness is larger than b''. For any time n, we denote by $j(n)$ the number of the period containing n, i.e. by definition $n \in I_{j(n)}^N(b, b'')$. Suppose n is in the recovery part of its period, i.e. $n \in I_{j(n),R}^N(b, b'')$. Then, we claim that at time n, all the locking thresholds are below b''. Indeed, during the b-avalanche of range N at the end of $I^N_{j(n)+1,A}(b)$ every locking threshold has been updated by a new value, below b. Hence at time $\tau_{j(n)+1,A} - 1$, the end of this b-avalanche, all the locking thresholds are below $b < b''$, and at time interval $[\tau_{j(n)+1,A}, n]$, we assign to the locking thresholds only values below b''. Hence at any $n \in I^N_{j(n),R}(b, b'')$, all the locking thresholds are below b''. Therefore we have

$$\begin{aligned} G_N(n, b'') &= P\left(\mathcal{Y}_{N,0}(n) < b''\right) \\ &\geq P\left(\text{ for all } x \in \Lambda(N),\, \mathcal{Y}_{N,x}(n) < b''\right) \\ &\geq P\left(n \in I^N_{j(n),R}(b, b'')\right). \end{aligned}$$

Thus to prove the lemma it suffices to show that

$$\liminf_{N\to\infty} \liminf_{n\to\infty} P\Big(n \in I^N_{j(n),R}(b, b'')\Big) = 1.$$

It is clear that at every $\tau_{j,R}$ the fitnesses are i.i.d. and exponentially distributed above the threshold b'', and at every $\tau_{j+1,A}$ the fitnesses are i.i.d. and exponentially distributed above the threshold b. Thus the sequences of lengths $\Big(|I^N_{j,A}(b)|\Big)_{j\in\mathbb{N}}$ and $\Big(|I^N_{j,R}(b, b'')|\Big)_{j\in\mathbb{N}}$ are independent, and each consists of i.i.d. random variables. Since both $|I^N_{j,A}(b)|$ and $|I^N_{j,R}(b, b'')|$ have non-lattice distributions, in the stationary regime with N vertices we can write (using standard alternating renewal process theory),

$$\lim_{n\to\infty} P\Big(\, n \in I^N_{j(n),R}(b, b'')\Big) = \frac{E\Big(|I^N_{0,R}(b,b'')|\Big)}{E\Big(|I^N_{0,A}(b)|\Big)+E\Big(|I^N_{0,R}(b,b'')|\Big)}. \tag{7.3}$$

The duration of the avalanche part $|I^N_{0,A}(b)|$ can be decomposed into two parts: the duration of the b-avalanche of range N, and the waiting time until this avalanche. We denote by W_N a typical waiting time before the b-avalanche of range N, and by A_N the duration of this avalanche. We already found in [15] that

$$E(W_N) = \left(\frac{1}{P_N(b)} - 1\right) E\left(\eta_N(b) \,\middle|\, \xi_N(b) < N\right).$$

Thus

$$\begin{aligned} E\Big(|I_{0,A}^N(b)|\Big) &= E(W_N) + E(A_N) \\ &= \left(\tfrac{1}{P_N(b)} - 1\right) E\Big(\eta_N(b) \,\Big|\, \xi_N(b) < N\Big) \\ &\qquad\qquad + E\Big(\eta_N(b) \,\Big|\, \xi_N(b) = N\Big) \\ &= D_N(b)/P_N(b). \end{aligned} \tag{7.4}$$

Furthermore

$$\begin{aligned} E\Big(|I_{0,R}^N(b, b'')|\Big) &= E\Big(\eta_N^{(\Lambda(N),b)}(b'' - b)\Big) \\ &\geq E\Big(\eta_N^{(\Lambda(N),b')}(b'' - b')\Big) \\ &\geq (b'' - b' + o(b'' - b'))ND(b'), \end{aligned} \tag{7.5}$$

where the last inequality is obtained in the same way as (6.5) in the proof of Lemma 6.2. Equation (7.4) together with inequality (7.5) shows that the r.h.s. of (7.3) is at least

$$\frac{(b'' - b' + o(b'' - b'))ND(b')}{D_N(b)/P_N(b) + (b'' - b' + o(b'' - b'))ND(b')},$$

and the last expression tends to 1, as N tends to infinity, according to the condition of our lemma.

$\square$

8. The uppper bound for b_c^r

For any $b \geq 0$, let $\ell(b)$ denote the leftmost vertex of $\xi(b)$, i.e.

$$\ell(b) = \min\Big\{k \in \mathbb{Z} \,:\, k \in \xi(b)\Big\}.$$

Let $L_\infty(b)$ denote the expectation of $\ell(b)$. It is clear that $L_\infty(b)$ is decreasing in b, and it becomes $-\infty$ at the same point as the function $R_\infty(b)$.

Lemma 8.1 If $R_\infty(b) < \infty$, we have

$$\limsup_{\varepsilon\to 0} \tfrac{L_\infty(b+\varepsilon)-L_\infty(b)}{\varepsilon} \leq -\tfrac{1}{2}L_\infty^2(b) + \tfrac{1}{2}L_\infty(b). \tag{8.1}$$

Proof of Theorem 3.5: By definition we have, $\ell(0) = -1$, and hence $L_\infty(0) = -1$. Thus $L_\infty(b)$ decays at least as fast as the solution of

$$\begin{cases} y'(b) = -\frac{y^2(b)}{2} + \frac{y(b)}{2} \\ y(0) = -1, \end{cases}$$

The above differential equation can be solved analytically:

$$y(b) = \frac{-1}{2e^{-b/2} - 1},$$

and the solution blows up at $b = 2\log 2$. Thus $b_c^r < 2\log 2$.

□

Proof of Lemma 8.1: Fix any $b > 0$, such that $R_\infty(b) < \infty$. Since $L_\infty(\cdot)$ and $R_\infty(\cdot)$ have the same critical thresholds, we automatically have $L_\infty(b) > -\infty$. Consider the range set as a function of threshold $b' \leq b$ on GR. Fix $\varepsilon > 0$. We will estimate the mean of the following increment

$$\mathit{\Delta}(b,\varepsilon) = \ell(b+\varepsilon) - \ell(b).$$

Let $Big(x(b), \tau(b)\Big)$ be the position and the moment of the first arrival in the superposition of the Poisson processes

$$\bigcup_{x \in [\ell(b), 0]} \Pi_x,$$

of GR after time (threshold) b. It follows from the definition of $x(b)$ and $\tau(b)$ that

$$\begin{aligned} P(\tau(b) < b + \varepsilon \,|\, \ell(b) = l) &= -(l+1)(\varepsilon + o(\varepsilon)), \text{ as } \varepsilon \to 0, \\ P(x(b) = x \,|\, \ell(b) = l) &= \tfrac{-1}{l+1},\ x \in [l, 0], \end{aligned} \tag{8.2}$$

Moreover, $x(b)$ and $\tau(b)$ are conditionally independent given $\ell(b)$. In the graphical representation GR, at the moment $\tau(b)$ we have the subavalanche $\Big(\hat{\xi}(\tau(b)), \eta(\tau(b))\Big)$. If $\tau(b) < b + \varepsilon$ we have

$$\Delta(b, \varepsilon) \leq \min\Big(m(b) + x(b), \ell(b)\Big) - \ell(b), \tag{8.3}$$

where $m(b)$ is the leftmost point of $\hat{\xi}(\tau(b))$, i.e.

$$m(b) = \min\Big\{k \,:\, k \in \hat{\xi}(\tau(b))\Big\}.$$

Observe that the conditional distribution of $m(b)$ given $\ell(t)$ depends on $\ell(b)$, but only through the value of $\tau(b)$. Since any $\tau(b)$-avalanche contains a b-avalanche, we can couple $m(b)$ with $\hat{\ell}(b)$, a random variable independent of $\tau(b)$ and $\ell(b)$, and distributed as the leftmost point of a b-avalanche with origin at 0, and therefore $E(\hat{\ell}(b)) = L_\infty(b)$. So the r.h.s. of (8.3) is at most

$$\min\Big(\hat{\ell}(b) + x(b), \ell(b)\Big) - \ell(b). \tag{8.4}$$

The random variables $\hat{\ell}(b)$ and $\ell(b)$ have finite expectations, but we have no information about their second moments. To deal with this, we use the truncations

$$\begin{aligned} a_M(b) &= \max\Big(\ell(b), -M\Big), \\ \hat{a}_M(b) &= \max\Big(\hat{\ell}(b), -M\Big), \qquad M \geq 1. \end{aligned}$$

Since,

$$\lim_{M\to\infty} a_M(b) = \ell(b),$$

$$\lim_{M\to\infty} \hat{a}_M(b) = \hat{\ell}(b),$$

we also have by Fatou's lemma

$$\begin{aligned}
&\lim_{M\to\infty} E(a_M(b)) = E(\ell(b)) = L_\infty(b), \\
&\lim_{M\to\infty} E(\hat{a}_M(b)) = E(\hat{\ell}(b)) = L_\infty(b).
\end{aligned} \tag{8.5}$$

The expression in (8.4) is at most

$$\begin{aligned}
&\leq \Big(\min\Big(\hat{a}_M(b) + x(b), \ell(b)\Big) - \ell(b)\Big)\mathbf{1}\Big\{x(b) \leq \ell(b) - a_M(b)\Big\} \qquad (8.6) \\
&= \sum_{x=\ell(b)}^{\ell(b)-\max(a_M(b),\hat{a}_M(b))} \Big(\hat{a}_M(b) + x - \ell(b)\Big)\mathbf{1}\Big\{x(b) = x\Big\} \\
&= \sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} \Big(j + \hat{a}_M(b)\Big)\mathbf{1}\Big\{x(b) = \ell(b) + j\Big\}.
\end{aligned}$$

Combining the above estimates with (8.3) and taking expectations on both sides, we get

$$\begin{aligned}
E(\Delta(b,\varepsilon)) &\leq E\Bigg(\mathbf{1}\{\tau(b) < b + \varepsilon\} \sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} (j + \hat{a}_M(b))\,\mathbf{1}\{x(b) = \ell(b) + j\}\Bigg) \\
&= \sum_{l=-\infty}^{-1} E\Bigg(\mathbf{1}\{\tau(b) < b + \varepsilon\} \sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} (j + \hat{a}_M(b)) \\
&\qquad \times \mathbf{1}\{x(b) = \ell(b) + j\} \,\Bigg|\, \ell(b) = l\Bigg) P\left(\ell(b) = l\right).
\end{aligned} \tag{8.7}$$

Since $\tau(b)$ and $x(b)$ are conditionally independent given $\ell(b)$, and since $\hat{a}_M(b)$ and $\ell(b)$ are independent, the r.h.s. is equal to

$$\begin{aligned}
&\sum_{l=-\infty}^{-1} P\left(\tau(b) < b + \varepsilon \,\middle|\, \ell(b) = l\right) E\Bigg(\sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} (j + \hat{a}_M(b)) \,\Bigg|\, \ell(b) = l\Bigg) \\
&\qquad \times P\left(x(b) = \ell(b) + j \,\middle|\, \ell(b) = l)\right) P\left(\ell(b) = l\right).
\end{aligned} \tag{8.8}$$

After substituting (8.2) in the above expression, we get

$$\sum_{l=-\infty}^{-1} E\Big((\varepsilon + o(\varepsilon)) \sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} (\hat{a}_M(b) + j) \;\Big|\; \ell(b) = l\Big) P\left(\ell(b) = l\right)$$

$$= E\Big(\sum_{j=0}^{-\max(a_M(b),\hat{a}_M(b))} (\hat{a}_M(b) + j) \Big)(\varepsilon + o(\varepsilon))$$

$$= E\left(-\tfrac{\hat{a}_M(b)(\hat{a}_M(b)-1)}{2}\mathbf{1}\{\hat{a}_M(b) > a_M(b)\}\right)(\varepsilon + o(\varepsilon))$$

$$+E\left(\left(-(a_M(b)-1)\hat{a}_M(b) + \tfrac{a_M(b)(a_M(b)-1)}{2}\right)\mathbf{1}\{\hat{a}_M(b) \le a_M(b)\}\right)(\varepsilon + o(\varepsilon))$$

$$= E\Big(\left(-\tfrac{\hat{a}_M(b)(\hat{a}_M(b)-1)}{2} + (a_M(b)-1)\hat{a}_M(b) - \tfrac{a_M(b)(a_M(b)-1)}{2}\right)$$

$$\times \mathbf{1}\{\hat{a}_M(b) > a_M(b)\}\Big)(\varepsilon + o(\varepsilon))$$

$$+E\left(-(a_M(b)-1)\hat{a}_M(b) + \tfrac{a_M(b)(a_M(b)-1)}{2}\right)(\varepsilon + o(\varepsilon))$$

$$= E\left(\left(-\tfrac{(\hat{a}_M(b)-a_M(b))^2}{2} - \tfrac{\hat{a}_M(b)-a_M(b))}{2}\right)\mathbf{1}\{\hat{a}_M(b) > a_M(b)\}\right)(\varepsilon + o(\varepsilon))$$

$$+\left(-E(a_M(b))E(\hat{a}_M(b)) + E(\hat{a}_M(b)) + \tfrac{(E(a_M(b)))^2}{2} - \tfrac{E(a_M(b))}{2}\right)(\varepsilon + o(\varepsilon))$$

$$\le E\left(\left(-\tfrac{(\hat{a}_M(b)-a_M(b))^2}{2}\right)\mathbf{1}\{\hat{a}_M(b) > a_M(b)\}\right)(\varepsilon + o(\varepsilon))$$

$$+\left(-E(a_M(b))E(\hat{a}_M(b)) + E(\hat{a}_M(b)) + \tfrac{(E(a_M(b)))^2}{2} - \tfrac{E(a_M(b))}{2}\right)(\varepsilon + o(\varepsilon)). \tag{8.9}$$

Since $a_M(b)$ and $\hat{a}_M(b)$ are i.i.d., the difference $a_M(b) - \hat{a}_M(b)$ has a symmetric distribution. Hence

$$E\left(\Big(a_M(b) - \hat{a}_M(b)\Big)^2 \mathbf{1}\{a_M(b) > \hat{a}_M(b)\}\right) = \frac{1}{2}E\Big(a_M(b) - \hat{a}_M(b)\Big)^2,$$

and we can continue (8.9) as

$$= \Big(-E\tfrac{(a_M(b)-\hat{a}_M(b))^2}{4} - (Ea_M(b))^2 + E\tfrac{a_M(b)}{2} + E\tfrac{(a_M(b))^2}{2}\Big)$$

$$= \Big(-\tfrac{(Ea_M(b))^2}{2} + \tfrac{Ea_M(b)}{2}\Big)(\varepsilon + o(\varepsilon)).$$

Substituting the above estimates in (8.7) results in

$$E\Big(\Delta_N(b,\varepsilon)\Big) \le \Big(\tfrac{-(Ea_M(b))^2}{2} + \tfrac{Ea_M(b)}{2}\Big)(\varepsilon + o(\varepsilon)).$$

The lemma is now straightforward by (8.5).

□

Acknowledgement: We would like to thank Natalia Davydova for helpful discussions.

References

1. Bak P. (1996) *How Nature Works*, New-York: Springer-Verlag.
2. Bak P., Sneppen K. (1993) Punctuated equilibrium and criticality in a simple model of evolution, *Phys. Rev. Let.* **74**, 4083-4086.
3. Jensen H.J. (1998) *Self-Organized Criticality*, Cambridge Lecture Notes in Physics.
4. Grassberger P. (1995) The Bak-Sneppen model for punctuated evolution, *Physics Letters A* **200**, No. 3, 277-282.
5. Maslov S. (1996) Infinite hierarchy of exact equations in the Bak-Sneppen model, *Phys. Rev. Let.* **77**, 1182-1186.
6. Moreno Y., Vazquez (2002) The Bak-Sneppen Model on Scale-Free Networks, *Europhys. Lett.* **57**, 765-771.
7. Cafiero R., De los Rios P., Valleriani A., Vega J.L. (1999) Levy-Nearest-Neighbors Bak-Sneppen Model, *Physical Review* E.**60** (2 Part A), R1111-R1114.
8. De Los Rios P., Marsili M., Vendruscolo M. (2001) High Dimentional Bak-Sneppen model, *Phys. Rev. Let.* **80**, No. 26, 5746-5749.

9. Pis'mak Yu.M. (2002) Self-Organized Criticality in simple model of evolution: exact description of scaling laws, *Acta Physica Slovatica* **52**, No. 6, 525-532.

10. de Boer J., Derrida B., Flyvbjerg H., Jackson A.D., Wettig T. (1994) Simple Model of Self-Organized Biological Evolution, *Phys. Rev. Let.* **73**, 906-909.

11. Marsili M., De Los Rios P., Maslov S. (1998) Expansion around the mean-field solution of the Bak-Sneppen model, *Physical Review Letters* **80**, No. 7, 1457-1460.

12. Barbay J., Kenyon C. (2001) On the discrete Bak-Sneppen model of self-organized criticality, *Proceedings Of The Twelth Annual ACM-SIAM Symposium On Discrete Algorithms (SODA)* Washington DC, January 2001.

13. Meester R., Znamenski D. (2002) Non-triviality of a discrete Bak-Sneppen evolution model, *Journal of Statistical Physics* **109**, No. 516, 987-1004.

14. Jovanovic B., Buldyrev S.V., Havlin S., Stanley H.E. (1994) Punctuated Equilibrium and 'History Dependent' Percolation, *Phys. Rev. E* **50**, R2403-R2406.

15. Meester R., Znamenski D. (2002) On the limit behaviour of the Bak-Sneppen evolution model, to appear in *Annals of Probability.*

16. Bose I., Chaudhuri I. (2001) Bacterial evolution and the Bak-Sneppen model, *International Journal of Modern Physics C* **12**, No. 5, 675-683.

17. Kovalev O.V., Pis'mak Yu. M., Vechernin V.V. (1997) Self-Organized Criticality in the Model of Biological Evolution Describing Interaction of "Coenophilous" and " Coenophobous" Species, *Europhys. Lett.* **40**, 471-476.

18. Donangelo R., Fort H. (2002) A model for mutation in bacterial populations, *to appear in Phys. Rev. Let.*

19. Cuniberti G., Valleriani A., Vega J. L. (2001) Effects of regulation on a self-organized market, *Quantitative Finance* **1** 332-338.

20. Yamano T. (2001) Regulation effects on market with Bak-Sneppen model in high dimensions, *International Journal of Modern Physics C* **12**, No. 9, 1329-1333.

Communicated by name