

Sharp phase transition and critical behaviour in 2D divide and colour models

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Abstract

Consider subcritical Bernoulli bond percolation with fixed parameter $p < p_c$. We define a dependent site percolation model by the following procedure: for each bond cluster, we colour all vertices in the cluster black with probability r and white with probability $1 - r$, independently of each other. On the square lattice, defining the critical probabilities for the site model and its dual, $r_c(p)$ and $r_c^*(p)$ respectively, as usual, we prove that $r_c(p) + r_c^*(p) = 1$ for all subcritical p . On the triangular lattice, where our method also works, this leads to $r_c(p) = 1/2$, for all subcritical p . On both lattices, we obtain exponential decay of cluster sizes below $r_c(p)$, divergence of the mean cluster size at $r_c(p)$, and continuity of the percolation function in r on $[0, 1]$. We also discuss possible extensions of our results, and formulate some natural conjectures. Our methods rely on duality considerations and on recent extensions of the classical RSW theorem.

Keywords: dependent percolation, sharp phase transition, critical behaviour, duality, DaC model, RSW theorem, $p_c = 1/2$.

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1 Introduction

1.1 Definition of the model and main results

Despite the vast literature on two-dimensional percolation and the tremendous progress made in its analysis since its introduction as a mathematical theory in [5], the exact value of the critical density is known only for a handful of models. The latter cases are typically Bernoulli (*independent*) percolation models endowed with certain *duality* properties, which play a crucial role in the determination of the critical point. In this paper, we will be concerned with the study of (the value of) the critical point and the “phase diagram” of certain two-dimensional *dependent* percolation models.

Our main object of interest is the two-dimensional *Divide and Color (DaC) model* introduced by Häggström [16]. For our purposes, it will be sufficient to consider the simplest version of the model, which can be described as follows. Given a graph $G = (\mathcal{V}, \mathcal{E})$ with vertex set \mathcal{V} and edge set \mathcal{E} , assign to each edge $e \in \mathcal{E}$ value 1 (present/open) with probability p and value 0 (absent/closed) with probability $1 - p$, independently of all other edges. Denote the resulting $\{0, 1\}^{\mathcal{E}}$ -valued configuration by Y , and the corresponding distribution by ν_p . Call p -clusters (“protoclusters”) the maximal connected components of the graph obtained by removing from G all the closed edges. Next, colour the vertices of each p -cluster black with probability r and white with probability $1 - r$, independently of all other p -clusters. Denote by X the resulting configuration of black and white vertices, by $\mathbb{P}_{p,r}$ the corresponding colouring distribution, and call r -clusters the maximal connected (via the edge set \mathcal{E}) white and black subsets of the vertex set \mathcal{V} . These are for us the “real clusters”, whose percolation properties we are interested in. Indeed, we will be mainly interested in the percolation properties of X for fixed density p of open edges, and will consider the edge configuration Y as an auxiliary object, needed to define X . We will in fact argue later in the introduction that our results should still be valid if the product measure ν_p on configurations of edges – corresponding to Bernoulli bond percolation – is replaced by some other measure with analogous properties of translation invariance and ergodicity – e.g., the random cluster measures of which Bernoulli bond percolation is a special case. Nonetheless, our arguments make substantial use of properties of the product measure ν_p , and cannot be immediately applied to models defined using other measures.

We first restrict our attention to the square lattice, with vertex set \mathbb{Z}^2 and edge set \mathcal{E}^2 given by the edges between nearest neighbour elements of \mathbb{Z}^2 , and its matching graph, with vertex set \mathbb{Z}^2 and edge set given by the edges between nearest and next-nearest neighbour elements of \mathbb{Z}^2 (i.e., the

previous graph with two edges added to each face of the graph along the two diagonals – see Figure 1). We remind the reader that the measure ν_p

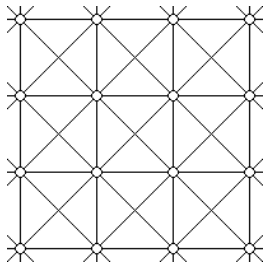


Figure 1: Part of the matching graph of the square lattice.

on the square lattice has a percolation phase transition at $p = 1/2$ [17]. We denote by $\Theta(p, r)$ the probability that in the DaC model with parameters p and r the origin $(0, 0)$ of the square lattice is contained in an infinite black r -cluster, and by $\Theta^*(p, 1 - r)$ the probability that it is contained in an infinite white $*$ -cluster, where a $*$ -cluster is a connected set of vertices of the matching graph of the square lattice (i.e., connections along the diagonals are allowed). For fixed p , we let $r_c(p) = \sup \{r : \Theta(p, r) = 0\}$ and $r_c^*(p) = \sup \{r : \Theta^*(p, r) = 0\}$. In [16], Theorem 2.6, it is shown that $r_c(p)$ and $r_c^*(p)$ are non-trivial. For fixed p and r , we call the model *critical* if (i) $\Theta(p, r) = 0$ and (ii) the mean size of the black r -cluster of the origin is divergent (we call the size of a cluster C its cardinality $|C|$, i.e. the number of vertices in the cluster). In this context, we have the following results.

Theorem 1.1. (*Duality*) For all $p < 1/2$, $r_c(p) + r_c^*(p) = 1$.

We remark that van den Berg has recently proved [2] that this relation holds for a large class of percolation models using methods different from those of this paper. However, the DaC model does not seem to fit in the framework treated in [2].

Theorem 1.2. (*Exponential decay*) For $p < 1/2$ and $r < r_c(p)$, the size of the black r -cluster C_0^r of the origin has an exponentially decaying tail, i.e. there exists a constant $c(p, r) > 0$ such that

$$\mathbb{P}_{p,r}(|C_0^r| \geq n) \leq e^{-c(p,r)n}$$

for all $n \in \mathbb{N}$.

The proof of this result is quite similar to the proof of Theorem 2 in [6] (see also the proof of Theorem 5 in [7]), and we do not give it here.

Theorem 1.3. (*Criticality*) *The DaC model is critical for $0 \leq p < 1/2$ and $r = r_c(p)$, and for $p = 1/2$ and $r \in (0, 1)$. It is not critical for $1/2 < p \leq 1$, where $\Theta(p, r) > 0$ for all $r > 0$.*

Theorem 1.1 amounts to a duality relation between black percolation and white $*$ -percolation. Together with the other two theorems, it provides a complete picture of the phase diagram of the DaC model, summarized below.

Corollary 1.4. (*Phase diagram*)

- For all $p < 1/2$, there exists $r_c(p) \in [1/2, 1)$ such that:
 1. If $r < r_c(p)$, there exists an infinite white $*$ -cluster a.s. and the size of the black r -cluster of the origin has an exponentially decaying tail.
 2. If $r = r_c(p)$, $\Theta(p, r_c(p)) = \Theta^*(p, 1 - r_c(p)) = 0$ and the mean size of the black r -cluster of the origin and of the white $*$ -cluster of the origin are infinite.
 3. If $r > r_c(p)$, there exists an infinite black r -cluster a.s. and the size of the white $*$ -cluster of the origin has an exponentially decaying tail.
- For $p = 1/2$, $\Theta(1/2, r) = 0$ for all $r \in (0, 1)$ and the mean size of the black r -cluster of the origin is infinite.
- For all $p > 1/2$, $\Theta(p, r) > 0$ for all $r \in (0, 1]$.

It is interesting to notice that the two regions of the phase diagram where $\Theta(p, r) > 0$, namely (1) $0 \leq p < 1/2$ and $r > r_c(p)$, and (2) $1/2 < p \leq 1$ and $r > 0$, have different properties. In the first one, there is an infinite black r -cluster (somewhere) with probability 1. This follows from the ergodicity of the measure $\mathbb{P}_{p,r}$ with respect to translations when $p < 1/2$ (see Section 2), and the fact that the event that $x \in \mathbb{Z}^2$ belongs to an infinite black r -cluster is translation invariant, and has strictly positive probability. In the second region, where $p > 1/2$, the probability that there is an infinite black r -cluster is bounded away from 1 for all $r < 1$. This also shows that when $p > 1/2, r \in (0, 1)$, the measure $\mathbb{P}_{p,r}$ is not ergodic with respect to translations (because of the presence of a unique infinite p -cluster).

Another result that follows easily from Theorem 1.3 is the continuity of the percolation function $\Theta(p, r)$ as a function of r for $p < 1/2$.

Corollary 1.5. *For all $p < 1/2$, $\Theta(p, r)$ is a continuous function of r .*

The methods used to prove our main results above are not restricted to the square lattice. In particular, they can be applied to the DaC model on the triangular lattice to obtain the following theorem, where $p_c(\mathbb{T}) := 2 \sin(\pi/18)$ is the critical density for Bernoulli bond percolation on the triangular lattice [25] (see also [23, 24]).

Theorem 1.6. *(Critical point) In the context of the DaC model on the triangular lattice, for all $p < p_c(\mathbb{T})$, $r_c(p) = 1/2$.*

We remark that Theorem 1.6 looks stronger than Theorem 1.1 because of the self-duality of site percolation on the triangular lattice, with $*$ -clusters being of the same nature as r -clusters, which immediately implies $r_c(p) = r_c^*(p)$. The proofs of Theorems 1.2 and 1.3 and Corollaries 1.4 and 1.5 can also be easily adapted to the triangular lattice.

The DaC model discussed in this section is one of the simplest dependent percolation models that one can think of. Indeed, it is defined using only product measures. Nonetheless, despite its simplicity, even results that are by now considered standard for Bernoulli percolation, such as those discussed in this paper, appear to be much harder to prove for the DaC model than in the independent case. The proofs of such results, although based on ideas developed for Bernoulli percolation, require various original arguments that could potentially be helpful in analysing other dependent percolation models.

The structure of the DaC model is very similar in spirit to that of the random cluster model. Following this analogy, the DaC model can be generalised and seen as a particular member of a larger family of models, as explained in the next section.

1.2 Other models

As mentioned just before Theorem 1.6, our methods are robust in the sense that they work, with obvious modifications, on different lattices. Another natural extension of our results would be to replace the product measure ν_p with other measures. In particular, we have in mind the class of random cluster measures (of which ν_p is a special case – see, e.g., [14]). These are dependent percolation models that unify in a single two-parameter family a variety of stochastic processes of significant importance for probability and statistical physics, including Bernoulli percolation, Ising and Potts models. They are characterized by two parameters, $0 \leq p \leq 1$ and $q > 0$, with $q = 1$ corresponding to the Bernoulli percolation measure ν_p . For $q \geq 1$, they have positive correlation and are believed to have a phase transition in p with exponential decay of correlations for $p < p_c$. For a detailed account on the random cluster model, the reader is referred to [14].

Conjecture 1.7. *Let $\tilde{X} = \tilde{X}(p, q, r)$ be a configuration generated as in the DaC model but with ν_p replaced by a random cluster measure with parameters $p < p_c$ and $q > 0$. For this model, we conjecture that $r_c(p) + r_c^*(p) = 1$ on the square lattice and $r_c(p) = 1/2$ on the triangular lattice. Furthermore, we conjecture that results analogous to Theorems 1.2 and 1.3 and Corollaries 1.4 and 1.5 also hold.*

One major obstacle in proving the conjecture using the methods of this paper is the lack of a proof of exponential decay of correlations for random cluster measures with $q \neq 1, 2$. The case $q = 1$ is treated in this paper. The case $q = 2$ is particularly interesting since for $r = 1/2$ it corresponds to the Ising model; we address this case in a forthcoming paper. On the triangular lattice, where self-duality holds, it is easy to see that, at least for $q \geq 2$, the model defined by \tilde{X} is critical, in our definition of the term, at the purported critical point.

Proposition 1.8. *Let $\tilde{X}(p, q, r)$ be the model defined above on the triangular lattice with $p < p_c$, $q \geq 2$, and $r = 1/2$. Then there is no infinite r -cluster a.s. but the mean size of the black r -cluster of the origin is infinite.*

Proposition 1.8 implies that $r_c(p) \geq 1/2$ and suggests that indeed $r_c(p) = 1/2$. It has a “universality” flavour since it suggests that self-duality alone (almost) determines the critical value $r_c(p) = 1/2$. One should however compare this result to Theorem 1.3, which shows that when $p = p_c$ (i.e., either $1/2$ on the square lattice or $2 \sin(\pi/18)$ on the triangular lattice) the DaC model has a critical segment, $r \in (0, 1)$, rather than a single critical point. The difference between the two cases is that when $p < p_c$ the size of p -clusters has an exponentially decaying tail, while at p_c the mean p -cluster size diverges and p -clusters form circuits around the origin at all scales, turning the percolation of r -clusters effectively into a one-dimensional problem.

1.3 Scaling limits

It is natural to ask what the continuum scaling limit (when the lattice spacing is sent to zero) of the DaC model is on the two critical “curves” (1) $p \in [0, p_c)$, $r = r_c(p)$ and (2) $p = p_c$, $r \in (0, 1)$.

For the first critical curve, based on universality considerations, we expect the scaling limit to be the same as for critical Bernoulli site percolation, corresponding to $p = 0$ and $r = r_c(0)$. In particular, we expect crossing probabilities to converge to Cardy’s formula [10] (as proved by Smirnov [22] for critical Bernoulli site percolation on the triangular lattice) and the set of all interfaces between black r -clusters and white $*$ -clusters to converge to

the Continuum Nonsimple Loop process described in [8, 9]. This is in line with the general principle that short range correlations, as produced by the p -clusters below p_c , do not affect the critical behaviour and the scaling limit. The DaC model in this regime can be seen as Bernoulli site percolation on a random graph whose vertices are the p -clusters, and as long as $p < p_c$, the random graph will be, in some sense, “close” to the underlying regular lattice. In other words, under the action of the renormalisation group, the critical curve (1) should have a unique fixed point, namely, $p = 0$, $r = r_c(0)$.

We expect similar considerations to hold when the product measure ν_p is replaced by a different random cluster measure (or even in greater generality), and make the following natural conjecture, stated for simplicity for the triangular lattice.

Conjecture 1.9. *Let $\tilde{X}(q, p, r)$ be the model of Proposition 1.8. For all $q > 0$ and all $p < p_c$, the (site percolation) scaling limit of $\tilde{X}(q, p, 1/2)$ is the same as the scaling limit of critical Bernoulli (site) percolation.*

In the case of the second critical curve, we expect a different situation, with different scaling limits for different values of r . We expect, for instance, that the scaling limit of crossing probabilities will depend on r and will not in general be given by Cardy’s formula.

1.4 Strategy of the proof of Theorem 1.1

The proof of our main result, Theorem 1.1, follows a “modern” version (using Russo’s formula) of the celebrated proof [17] by Kesten that the critical probability for Bernoulli (independent) bond percolation on the square lattice is $1/2$ (see also [21]). However, since we are dealing with a *dependent* percolation model, the proof requires various modifications, needed for instance to avoid gathering “too much information.” Also, in this modified version of Kesten’s strategy, and due to the dependence structure of the DaC model, we cannot apply the “traditional” RSW theorem. We will instead use a recent version of it taken from [3], which is a strengthened form of the RSW type theorem in [6].

We now describe briefly (and somewhat imprecisely) what one would do in the case of Bernoulli percolation, corresponding to $p = 0$. Some notational remarks first: we shall omit the subscript r from the notation of the DaC measure $\mathbb{P}_{p,r}$ and also often write r_c and r_c^* for $r_c(p)$ and $r_c^*(p)$ respectively if no confusion is possible.

It follows from standard arguments that $1 - r_c^* \leq r_c$. Therefore, the main task is to prove that this inequality is not strict. We proceed by contradiction, assuming that the open interval $(1 - r_c^*, r_c)$ is non-empty. Let $V_{n,3n}^b$ denote

the presence of a vertical black crossing of an $n \times 3n$ rectangle and $H_{n,3n}^{w*}$ the presence of a horizontal white $*$ -crossing of the same rectangle (precise definitions will be given in Section 2.1 below). Accordingly, let $H_{n,3n}^b$ denote the event that there is a horizontal black crossing of an $n \times 3n$ rectangle, $V_{n,3n}^{w*}$ that there is a vertical white $*$ -crossing of the same rectangle. Since there is a.s. no percolation of black vertices for $r \in (1 - r_c^*, r_c)$, it is easy to prove that $\limsup_{n \rightarrow \infty} \mathbb{P}_r(V_{n,3n}^b) < 1$. But then, by the simple but crucial observation that, no matter how one chooses to colour the vertices inside the $n \times 3n$ rectangle, there is always either a vertical black r -crossing or a horizontal white $*$ -crossing, $\liminf_{n \rightarrow \infty} \mathbb{P}_r(H_{n,3n}^{w*}) > 0$ for all $r \in (1 - r_c^*, r_c)$. This implies that there is a uniform positive probability to have a horizontal white $*$ -crossing in the lowest half of an $n \times 8n$ rectangle for all n large enough. (We note that Kesten used squares in his proof. In our case, due to the dependence structure, it will be more convenient to use rectangles. To avoid confusion and prepare the reader for what will come, we employ the same rectangles here.)

Consider the lowest such crossing π_1 and look for a black vertical r -crossing in the left half of the same $n \times 8n$ rectangle from the top of the rectangle to π_1 . Since there is no white $*$ -percolation for $r \in (1 - r_c^*, r_c)$, with the help of the RSW theorem, one can show that a black vertical crossing from the top of the rectangle to π_1 exists with probability bounded away from zero. Consider the leftmost such crossing π_2 . Due to the properties of lowest crossings, the presence of such a black crossing implies the presence of a white vertex x on π_1 which is pivotal for the event $H_{n,8n}^{w*}$.

Next, center at x a sequence of nested annuli intersecting the $n \times 8n$ rectangle. Inside the portion of each annulus intersecting the rectangle and lying above π_1 and to the right of π_2 , look for a black crossing joining π_2 with π_1 . Once again, the existence of such crossings with uniform positive probability is assured by the RSW theorem. Every such crossing gives another white vertex on π_1 which is pivotal for $H_{n,8n}^{w*}$. In this way, choosing the annuli appropriately, one can find many pivotal vertices with high probability. Using Russo's formula it is then possible to conclude that $\mathbb{P}_r(H_{n,8n}^{w*})$ has a very large (negative) derivative for all $r \in (1 - r_c^*, r_c)$, obtaining a contradiction.

The argument above relies on properties of lowest and leftmost crossings, and in particular uses the fact that a lowest (respectively, leftmost) crossing can be found without exploring the area above (resp., to the right of) the crossing itself. In the case of Bernoulli percolation this implies that the configuration above the lowest crossing can be coupled to an independent configuration, and the probability to find a black r -crossing in the left half of the rectangle can be bounded below using the RSW theorem. The same type of argument applies to the portions of annuli to the right of π_2 and above π_1 ,

where the configurations can again be coupled to independent configurations, and the probabilities of finding the appropriate crossings bounded below once again using the RSW theorem.

In our case, similar arguments can be used, but the dependence in the model makes them significantly more complex. Moreover, as remarked above, the “traditional” RSW theorem cannot be used, and we have to resort to a more recent version [3] which is weaker but more general and, as it turns out, still sufficiently strong for our purposes.

To deal with the dependence structure of the model, in some situations we will “fatten” certain collections of vertices (e.g., vertices forming a crossing) by adding to them their p -clusters. This procedure identifies closed “barriers” of edges with the property that colour configurations on different sides of a barrier are conditionally independent (conditioned on the barrier).

We will also use algorithmic constructions carefully designed to explore certain domains looking for monochromatic crossings without obtaining too much information. This will allow us to couple in a useful way the DaC measure $\mathbb{P}_{p,r}$ conditioned on some specific σ -algebras corresponding to the information obtained while looking for crossings with an unconditional version.

1.5 Outline of the paper

In Section 2, we present the definitions and introduce notation. Then, we collect the tools which are needed to prove the main results. These include known results such as the exponential decay property of subcritical Bernoulli bond percolation, the FKG inequality for the measure $\mathbb{P}_{p,r}$, and the modern RSW theorem from [3]. Then we give the natural analogue of Russo’s formula for the DaC model (Theorem 2.8), and finally state that percolation occurs with positive probability if and only if certain rectangles can be crossed with high probability (Lemma 2.11).

Section 3 contains the proof of Theorem 1.1. In Section 4, we prove Theorem 1.3 and Corollaries 1.4 and 1.5. Finally, in Section 5, we prove our results on the triangular lattice.

2 Preliminaries

2.1 Basic definitions and notation

We consider the square lattice, with vertices the points of \mathbb{Z}^2 , and edges between adjacent vertices (that is, between vertices at Euclidean distance 1).

With the usual abuse of notation, we denote both the graph and its vertex set by \mathbb{Z}^2 , and we write \mathcal{E}^2 for the edge set of this graph.

The state space of our configurations is defined as $\Omega := \Omega_D \times \Omega_C$, where $\Omega_D := \{0, 1\}^{\mathcal{E}^2}$ corresponds to Bernoulli bond percolation, and $\Omega_C := \{0, 1\}^{\mathbb{Z}^2}$ corresponds to colouring. We identify 0 with the colour white, and 1 with black. The probability measure $\mathbb{P}_{p,r}$ is the measure (on the usual σ -algebra on Ω) obtained by the procedure described in the introduction.

We introduce the set $\tilde{\Omega} \subset \Omega$ as the set of configurations such that vertices in the same p -cluster have the same colour, and we equip $\tilde{\Omega}$ with a partial ordering as follows. For $\omega_1 = (\eta_1, \xi_1), \omega_2 = (\eta_2, \xi_2) \in \tilde{\Omega}$ we say that $\omega_1 \geq \omega_2$ if, for all $x \in \mathbb{Z}^2$, we have $\xi_1(x) \geq \xi_2(x)$. Note that the ordering depends on the colours of the vertices only, not on the bond configurations. All the configurations in this paper are silently assumed to be in $\tilde{\Omega}$. We call an event $A \subset \tilde{\Omega}$ *increasing* if $\omega \in A$ and $\omega' \geq \omega$ implies $\omega' \in A$. A is a *decreasing event* if A^c is increasing.

We call a sequence of vertices (x_0, x_1, \dots, x_n) in \mathbb{Z}^2 a (*self-avoiding*) *path* if for all $i = 0, \dots, n-1$, x_i and x_{i+1} are adjacent, and for any $0 \leq i < j \leq n$, $x_i \neq x_j$. The definition of a **-path* is similar, but for $i = 0, \dots, n-1$, the vertices x_i and x_{i+1} need to be just **-adjacent* instead of adjacent, which means that their Euclidean distance is 1 or $\sqrt{2}$. A (**-circuit*) is defined in the same way as a (**-path*) except that $x_n = x_0$. A *horizontal crossing* of a rectangle $R = [a, b] \times [c, d]$, with $a, b, c, d \in \mathbb{Z}$, is a path x_0, x_1, \dots, x_n such that $x_0 \in \{a\} \times [c, d]$, $x_n \in \{b\} \times [c, d]$ and for all i , $x_i \in R$. A *vertical crossing* of the same rectangle is a path x_0, x_1, \dots, x_n such that $x_0 \in [a, b] \times \{d\}$, $x_n \in [a, b] \times \{c\}$ and for all i , $x_i \in R$. **-circuits*, *horizontal *-crossings*, and *vertical *-crossings* are defined by replacing paths by **-paths* in the above definitions.

A *black path* is a path $\pi = x_0, x_1, \dots, x_n$ such that for all $i = 0, \dots, n$, x_i is black (i.e. $X(x_i) = 1$). *Black circuits*, *black horizontal crossings*, *black vertical crossings* are defined analogously. A *black cluster* is a maximal subset K of \mathbb{Z}^2 such that between any two vertices of K there exists a black path. The definitions of *white path*, *white circuit*, *white horizontal crossing*, *white vertical crossing*, *white cluster* are obtained by replacing black with white. Black and white (**-paths*), (**-circuits*), (**-crossings*), and (**-clusters*) are defined analogously.

Let $S_{n,m}$ denote the rectangle $[0, n] \times [0, m]$, with $n, m \in \mathbb{N}$. Denote by $V_{n,m}^b$ the event that there is a vertical black crossing in the rectangle $S_{n,m}$; let $H_{n,m}^b$ be the corresponding event with a horizontal crossing. Furthermore, let B_n^b denote the event that there is a black circuit surrounding the midpoint in the annulus $A_n := S_{3n,3n} \setminus (S_{n,n} + (n, n))$. Here and later, for a set S and a vector v , we use the notation $S + v := \{x : x - v \in S\}$. The analogous events

with white crossings are denoted by $V_{n,m}^w$, $H_{n,m}^w$, and B_n^w , respectively. A $*$ in the notation will indicate that we are referring to $*$ -crossings and $*$ -circuits – for example, $V_{n,m}^{w*}$ denotes the event that there is a vertical white $*$ -crossing in $S_{n,m}$.

Let d denote the L_1 distance. The distance between two sets of vertices V_1 and V_2 is defined by $d(V_1, V_2) := \min\{d(x, y) : x \in V_1, y \in V_2\}$. Let $\partial B(v, n)$ denote the circle of radius n with center at vertex v in the metric d , i.e., $\partial B(v, n) := \{w : d(v, w) = n\}$. For a vertex $v \in \mathbb{Z}^2$, let C_v^p be the open p -cluster of v , i.e., the set of vertices that can be reached from v through edges that are open in the underlying Bernoulli bond percolation with parameter p . Let us define the *dependence range* of a vertex v by $\mathcal{D}(v) := \max\{n \in \mathbb{N} : C_v^p \cap \partial B(v, n) \neq \emptyset\}$.

We call an edge set $E = \{e_1, e_2, \dots, e_k\}$ a *barrier* if removing e_1, e_2, \dots, e_k (but not their end-vertices) separates the graph \mathbb{Z}^2 into two or more disjoint connected subgraphs, of which exactly one is infinite. We call the infinite component of $(\mathbb{Z}^2, \mathcal{E}^2) \setminus E$ the *exterior* of E , and denote it by $ext(E)$. We call the union of the finite components the *interior* of E , and denote it by $int(E)$. (Note that a barrier as defined above corresponds to a dual circuit in bond percolation. However, since we work with a different sort of duality throughout this paper, we adopt a different term to avoid confusion.) $E = \{e_1, e_2, \dots, e_k\}$ is a *closed barrier* if E is a barrier and e_i is closed in the Bernoulli bond percolation (i.e. $Y(e_i) = 0$). For a vertex set $A \subset \mathbb{Z}^2$, let ΔA denote the *edge boundary* of A , that is, $\Delta A := \{(x, y) \in \mathcal{E}^2 : x \in A, y \in \mathbb{Z}^2 \setminus A\}$. Note that for $p < p_c$, the edge boundary of any p -cluster is a closed barrier.

2.2 Preliminary results

In this subsection we collect several results that are mostly known, follow directly from known results, or can be proved using variations of classical arguments. The exception is Lemma 2.4, which is new and very important in the forthcoming construction.

Theorem 2.1. ([19],[1]) *If $p < 1/2$, there exists $\psi(p) > 0$ such that for all n we have*

$$\nu_p(\mathcal{D}(0) \geq n) < e^{-n\psi(p)}.$$

A short and simple proof of the following classical result is given in Section 4.2 of [7].

Lemma 2.2. *The restriction to a rectangle R of any colour configuration $\xi \in \Omega_C$ contains either a black vertical crossing or a white horizontal $*$ -crossing of R , but never both. In particular, for any n, m we have*

$$V_{n,m}^b = (H_{n,m}^{w*})^c.$$

The proof of the next result, which shows positive correlation for the DaC model, was obtained by Häggström and Schramm and included in [16].

Theorem 2.3. ([16]) *Let A, B be increasing events. Then, for any $p, r \in [0, 1]$,*

$$\mathbb{P}_{p,r}(A \cap B) \geq \mathbb{P}_{p,r}(A)\mathbb{P}_{p,r}(B).$$

We shall also need a result which makes precise (and generalizes) the observation that an edge between two vertices of the same colour is more likely to be open than an edge between vertices whose colours are unknown. Let us consider the following scenario for Lemma 2.4 below: let B_1, B_2, \dots, B_k be barriers, e_1, \dots, e_l edges in $U := \bigcup_{i=1}^k \text{int}(B_i)$, u_1, \dots, u_m vertices in U (where $k, l, m \in \{0, 1, 2, \dots\}$), and denote $\bigcap_{i=1}^k \text{ext}(B_i)$ by V . Fix states $s_i \in \{0, 1\}$, $i = 1, \dots, l$ and colours $c_j \in \{0, 1\}$, $j = 1, \dots, m$. Let v_1, \dots, v_n be vertices in V (where $n \in \{0, 1, 2, \dots\}$), and let $\kappa \in \{0, 1\}$ be a colour. Let I denote the event that B_1, B_2, \dots, B_k are closed, $Y(e_i) = s_i$, $X(u_j) = c_j$, and v_1, \dots, v_n all have colour κ .

Lemma 2.4. *The conditional distribution of the edges in V , conditioned on the event I described above, stochastically dominates the measure ν_p .*

Proof. We shall prove the lemma with an iteration, determining the states of edges in V one after another. Let f_1, f_2, \dots, f_g be edges in V , and $t_1, t_2, \dots, t_g \in \{0, 1\}$ states, where $g \in \{0, 1, 2, \dots\}$. Let us consider the events $A := \bigcap_{i=1}^g \{Y(f_i) = t_i\}$, and $J := I \cap A$. Take an edge e in V whose state is not determined by A . Note that there is no further restriction on the location of e : it may be incident on 0, 1 or 2 vertices from $\{v_1, \dots, v_n\}$. We shall first show that

$$\mathbb{P}_{p,r}(e \text{ open} | J) \geq p. \tag{1}$$

It is easy to see that (1) is equivalent to

$$(1 - p)\mathbb{P}_{p,r}(e \text{ open}, J) \geq p\mathbb{P}_{p,r}(e \text{ closed}, J),$$

which is also equivalent to

$$(1 - p)\mathbb{P}_{p,r}(J | e \text{ open})\nu_p(e \text{ open}) \geq p\mathbb{P}_{p,r}(J | e \text{ closed})\nu_p(e \text{ closed}).$$

Since $\nu_p(e \text{ open}) = p$ and $\nu_p(e \text{ closed}) = 1 - p$, it remains to show that

$$\mathbb{P}_{p,r}(J|e \text{ open}) \geq \mathbb{P}_{p,r}(J|e \text{ closed}). \quad (2)$$

This may be seen as follows. Since the first step in constructing a configuration corresponds to Bernoulli bond percolation, the states of edges other than e are independent of the state of e . For J to occur, the edges in $\bigcup_{i=1}^k B_i$ need to be closed. In that case, the colouring of u_1, \dots, u_m is not influenced by the state of e , since every vertex in question is in the interior of one of the closed barriers. Therefore, the only thing left to prove is that the probability that the vertices v_1, \dots, v_m all have colour κ is greater given that e is open than given that e is closed. This follows immediately from a very simple coupling between $\mathbb{P}_{p,r}(J|e \text{ open})$ and $\mathbb{P}_{p,r}(J|e \text{ closed})$ in which all the edges except e are in the same state, since when e is open the number of p -clusters that need to be assigned colour κ is smaller than or equal to the number of p -clusters that need to be assigned colour κ when e is closed. This observation proves (2), finishing the proof of (1).

The full stochastic domination can be shown using (1) iteratively as follows. We shall condition on I . Fix a deterministic ordering of the edges in V and use the following iteration:

1. Start with $A = \{0, 1\}^{\mathcal{E}(V)}$ where $\mathcal{E}(V)$ is the set of edges contained in V .
2. Determine the state s_e of the first edge $e \in \mathcal{E}(V)$ in the ordering whose state is not yet determined by A , according to the conditional distribution $\mathbb{P}_{p,r}(\cdot | I \cap A)$.
3. $A := A \cap \{Y(e) = s_e\}$.
4. Go back to step 2.

It is clear that every edge in V gets a state drawn from the correct distribution after finitely many steps. On the other hand, we know from (1) that for all A , the marginal of $\mathbb{P}_{p,r}(\cdot | I \cap A)$ on Ω_D dominates $\nu_p(\cdot)$. This proves the desired stochastic domination. \square

In the proof of Theorem 1.1, we will use an RSW type theorem that was recently obtained by van den Berg, Brouwer and Vágvölgyi [3]. This is a stronger version of the RSW type theorem used by Bollobás and Riordan in [6]. Such results are weaker than the classical RSW theorem but more general, and can be applied to models for which the classical RSW theorem has not been proved. We remark that in our proof of $r_c(p) + r_c^*(p) \leq 1$ in

Section 3 we seem to need the full strength of the result of van den Berg, Brouwer and Vágvölgyi, as the weaker form proved by Bollobás and Riordan does not seem to suffice for our purposes. Stated for the DaC model, the result reads as follows.

Lemma 2.5. *For any $p < 1/2, r \in [0, 1]$, we have*

- (a) *If $\limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,\rho n}^b) > 0$ for some $\rho > 0$,
then $\limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,\rho n}^b) > 0$ for all $\rho > 0$;*
- (b) *If $\limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,\rho n}^{w*}) > 0$ for some $\rho > 0$,
then $\limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,\rho n}^{w*}) > 0$ for all $\rho > 0$.*

Proof. First we prove (a). Following [3], Section 4.3, and [7], Section 5.1, it suffices to check conditions (1)–(5) below. For a set R and $\lambda \neq 0$, we write λR for $\{x : x/\lambda \in R\}$. We consider the following five conditions.

- (1) For any rectangle R , if h and v are a horizontal and a vertical crossing of R , respectively, then $d(h, v) \leq 1$.
- (2) Increasing events are positively correlated.
- (3) The model has the symmetries of \mathbb{Z}^2 , i.e., is invariant under translations by the vectors $(1, 0)$ and $(0, 1)$, reflections through the coordinate axes of \mathbb{Z}^2 , and rotations of 90 degrees.
- (4) Disjoint regions are asymptotically independent as we “zoom out” (the precise formulation of this condition will be given in Lemma 2.6).
- (5) For any fixed rectangle R there is a constant C such that the length of a horizontal crossing of λR is bounded from above by λ^C if λ is large enough.

Condition (1) clearly holds here, since horizontal and vertical black crossings of the same rectangle have at least one vertex in common. Condition (2) is given by Theorem 2.3. Condition (3) can be checked easily. For condition (4), see Lemma 2.6 below. Condition (5) obviously holds, since the model is discrete.

For the proof of (b), the same conditions need to be checked with $*$ -crossings instead of crossings in (1) and (5), and increasing events replaced by decreasing events in (2). The new first condition still holds, since even

though a horizontal *-crossing and a vertical one of the same rectangle do not necessarily have a vertex in common, they are at distance at most 1 from each other. The new second condition, namely that decreasing events are positively correlated, is an easy consequence of Theorem 2.3, since the complement of a decreasing event is an increasing event. \square

The next lemma immediately implies weak mixing and ergodicity for the DaC model when $p < p_c$.

Lemma 2.6. *Let $p < 1/2, r \in [0, 1]$. Then for disjoint rectangles R_1 and R_2 , for any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, for any events A_1, A_2 defined in terms of the colouring of vertices in λR_1 and λR_2 respectively, we have*

$$|\mathbb{P}_{p,r}(A \cap B) - \mathbb{P}_{p,r}(A)\mathbb{P}_{p,r}(B)| \leq \varepsilon.$$

Proof. Fix $p < p_c$. Let R_1 and R_2 be rectangles at distance $k > 0$. Take arbitrary events A_1, A_2 defined in terms of the colours of vertices in R_1 and R_2 respectively. Let $K = K_{R_1, R_2}$ be the event that R_1 and R_2 are separated by a closed barrier in the bond configuration. If K occurs, then the colours of the vertices in R_1 and R_2 are conditionally independent. Therefore, A_1 and A_2 are conditionally independent, conditioned on K . The law of total probability, together with the previous observation, gives

$$\mathbb{P}_{p,r}(A_1 \cap A_2) = \mathbb{P}_{p,r}(A_1|K)\mathbb{P}_{p,r}(A_2|K)\nu_p(K) + \mathbb{P}_{p,r}(A_1 \cap A_2|K^c)\nu_p(K^c) \quad (3)$$

Since

$$\mathbb{P}_{p,r}(A_1) = \mathbb{P}_{p,r}(A_1|K)\nu_p(K) + \mathbb{P}_{p,r}(A_1|K^c)\nu_p(K^c) \quad (4)$$

and

$$\mathbb{P}_{p,r}(A_2) = \mathbb{P}_{p,r}(A_2|K)\nu_p(K) + \mathbb{P}_{p,r}(A_2|K^c)\nu_p(K^c), \quad (5)$$

by substituting the right hand sides of equations (3), (4) and (5), using the triangle inequality, we obtain

$$\begin{aligned} & |\mathbb{P}_{p,r}(A_1 \cap A_2) - \mathbb{P}_{p,r}(A_1)\mathbb{P}_{p,r}(A_2)| \\ & \leq |\mathbb{P}_{p,r}(A_1|K)\mathbb{P}_{p,r}(A_2|K)| |\nu_p(K) - \nu_p(K)^2| + |\nu_p(K^c)| |Q|, \end{aligned}$$

where $|Q| \leq 4$ since Q is the sum of four products of probabilities.

We also need to notice that if none of the vertices in $R_1 \cup R_2$ has a dependence range of at least (say) $\frac{k}{3}$ in the initial random bond configuration Y , then K occurs. Therefore,

$$\begin{aligned} \nu_p(K^c) &\leq \nu_p\left(\bigcup_{v \in R_1 \cup R_2} \{\mathcal{D}(v) \geq \frac{k}{3}\}\right) \\ &\leq (|R_1| + |R_2|)\nu_p(\mathcal{D}(0) \geq \frac{k}{3}) \\ &\leq (|R_1| + |R_2|)e^{-\psi(p)\frac{k}{3}}, \end{aligned}$$

according to Theorem 2.1.

It immediately follows that for the probability of the event $L := K_{\lambda R_1, \lambda R_2}$, we have $\nu_p(L^c) \leq (|R_1| + |R_2|)\lambda^2 e^{-\psi(p)\frac{\lambda k}{3}} \rightarrow 0$ as $\lambda \rightarrow \infty$. Fix $\varepsilon > 0$, and choose λ_0 so large that for $\lambda > \lambda_0$, we have $\nu_p(L^c) < \frac{\varepsilon}{8}$. Take arbitrary events A_1 and A_2 , defined in terms of the colours of vertices in λR_1 and λR_2 , respectively. Since $\nu_p(L) \geq 1 - \frac{\varepsilon}{8}$ implies $|\nu_p(L) - \nu_p(L)^2| \leq \frac{\varepsilon}{4}$, we obtain

$$\begin{aligned} &|\mathbb{P}_{p,r}(A_1 \cap A_2) - \mathbb{P}_{p,r}(A_1)\mathbb{P}_{p,r}(A_2)| \\ &\leq |\mathbb{P}_{p,r}(A_1|L)\mathbb{P}_{p,r}(A_2|L)| \cdot |\nu_p(L) - \nu_p(L)^2| + |\nu_p(L^c)| \cdot 4 \\ &\leq 1 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{8} \cdot 4 \\ &< \varepsilon, \end{aligned}$$

proving the lemma. □

Corollary 2.7. *For $p < 1/2$, the measure $\mathbb{P}_{p,r}$ is weakly mixing and therefore ergodic with respect to translations.*

We also need a version of Russo's formula [21] (see also [13]). Let A be an event, and let $\omega = (\eta, \xi)$ be a configuration from $\tilde{\Omega}$. Let C be an open p -cluster from η . We call C *pivotal* for the pair (A, ω) if $I_A(\omega) \neq I_A(\omega')$ where I_A is the indicator function of A , $\omega' = (\eta, \xi')$, and ξ' agrees with ξ everywhere except that the colour of the vertices in C is different.

Theorem 2.8. *Let W be a set of vertices with $|W| < \infty$, and let A be an increasing event that depends only on the colours of vertices in W . Then we have, for any $p \in [0, 1]$,*

$$\frac{d}{dr}\mathbb{P}_{p,r}(A) = \mathbb{E}_{p,r}(n(A)),$$

where $n(A)$ is the number of p -clusters which are pivotal for A .

Proof sketch. Let us denote the (finite) set of partitions of the vertices in W which are compatible with a bond configuration by \mathcal{P}_W , and the (random) partitioning determined by the initial bond percolation by \mathcal{G}_W . One can follow the proof of Russo's formula in e.g. [11] to obtain for any $g_W \in \mathcal{P}_W$ that

$$\frac{d}{dr} \mathbb{P}_{p,r}(A | \mathcal{G}_W = g_W) = \mathbb{E}_{p,r}(n(A) | \mathcal{G}_W = g_W).$$

Since

$$\frac{d}{dr} \mathbb{P}_{p,r}(A) = \frac{d}{dr} \sum_{g_W \in \mathcal{P}_W} \mathbb{P}_{p,r}(A | \mathcal{G}_W = g_W) \nu_p(\mathcal{G}_W = g_W).$$

and the sum is finite, the sum and the derivative can be interchanged, giving

$$\begin{aligned} \frac{d}{dr} \mathbb{P}_{p,r}(A) &= \sum_{g_W \in \mathcal{P}_W} \nu_p(\mathcal{G}_W = g_W) \frac{d}{dr} \mathbb{P}_{p,r}(A | \mathcal{G}_W = g_W) \\ &= \sum_{g_W \in \mathcal{P}_W} \nu_p(\mathcal{G}_W = g_W) \mathbb{E}_{p,r}(n(A) | \mathcal{G}_W = g_W) \\ &= \mathbb{E}_{p,r}(n(A)) \end{aligned}$$

□

Corollary 2.9. *If A is a decreasing event depending on colours of vertices in a finite set $W \subset \mathbb{Z}^2$, then we have, for all $p \in [0, 1]$,*

$$\frac{d}{dr} \mathbb{P}_{p,r}(A) = -\mathbb{E}_{p,r}(n(A)).$$

The following lemma gives a finite size criterion for percolation (see [21], Lemma 2).

Lemma 2.10. *There exists a constant $\varepsilon > 0$ which satisfies the following property. If there exists $N \in \mathbb{N}$ such that*

$$(N + 1)(3N + 1) \nu_p(\mathcal{D}(0) \geq \frac{N}{3}) \leq \varepsilon \tag{6}$$

and

$$\mathbb{P}_{p,r}(V_{N,3N}^b) > 1 - \varepsilon,$$

then $\Theta(p, r) > 0$. If there exists $N \in \mathbb{N}$ such that (6) holds and

$$\mathbb{P}_{p,r}(V_{N,3N}^{w*}) > 1 - \varepsilon,$$

then $\Theta^*(p, 1 - r) > 0$.

We do not give the proof of this lemma here as it uses a well-known coupling argument with a 1-dependent bond percolation model on \mathbb{Z}^2 (see, e.g., the proof of Theorem 2.6 in [16] or the proof of Theorem 1 in [6]). Using Theorem 2.1, Lemma 2.10, and standard arguments, we obtain the following lemma, which relates the occurrence of percolation to the probability of crossing large rectangles.

Lemma 2.11. *For $p < 1/2$, we have*

$$(a) \limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,3n}^b) = 1 \text{ if and only if } \Theta(p, r) > 0.$$

$$(b) \limsup_{n \rightarrow \infty} \mathbb{P}_{p,r}(V_{n,3n}^{w*}) = 1 \text{ if and only if } \Theta^*(p, 1 - r) > 0.$$

In order to state the final result in this section, taken from [12], we need the following notation. Let μ be a probability measure on colour configurations where the vertices of \mathbb{Z}^2 are each declared black or white. Let us denote the event that the origin is in an infinite black cluster by $0 \leftrightarrow \infty$, and the number of infinite black clusters by N .

Theorem 2.12. ([12]) *Assume that*

- (1) μ is invariant under horizontal and vertical translations and axis reflections.
- (2) μ is ergodic (separately) under horizontal and vertical translations.
- (3) For any increasing events E and F ,

$$\mu(E \cap F) \geq \mu(E)\mu(F).$$

- (4) $0 < \mu(0 \leftrightarrow \infty) < 1$.

If assumptions (1)-(4) hold, then

$$\mu(N = 1) = 1.$$

Moreover, any finite set of vertices is surrounded by a black circuit with probability 1 and, equivalently, all white $$ -clusters are finite with probability 1.*

3 Proof of Theorem 1.1

In this section, we shall prove that for any $p < p_c$ we have $r_c(p) + r_c^*(p) = 1$. This can be split in two parts. The first one is an easy consequence of Theorem 2.12, stated in the previous section.

Theorem 3.1. *For $p < p_c$, we have $r_c(p) + r_c^*(p) \geq 1$.*

Proof. We apply Theorem 2.12. Let us fix $p < p_c$, and assume that $r_c(p) + r_c^*(p) < 1$. Then, we may choose some $r \in (r_c(p), 1 - r_c^*(p))$. Since $r > r_c(p)$, we have $\Theta(p, r) > 0$. On the other hand, it is clear that $\Theta(p, r) \leq r$. This gives $0 < \Theta(p, r) < 1$, i.e. (4) for the measure $\mathbb{P}_{p,r}$. Condition (3) is provided by Theorem 2.3, (2) by Corollary 2.7, and (1) clearly holds for $\mathbb{P}_{p,r}$. Therefore, all white $*$ -clusters are finite with probability 1. However, this cannot be the case since $r < 1 - r_c^*(p)$. \square

To prove the difficult direction, $r_c(p) + r_c^*(p) \leq 1$, we shall use ideas described in [21], some of which are based on Kesten's proof of $p_c = 1/2$ for Bernoulli bond percolation on \mathbb{Z}^2 (see [17]). However, here the proof is considerably more difficult due to the dependence structure of the DaC model. Some difficulties are of a geometrical nature, others arise from the fact that we have to use an RSW type theorem which is weaker than the RSW theorem available for Bernoulli (independent) percolation, and used by Kesten [17] in his celebrated proof.

Theorem 3.2. *For any $p < p_c$, $r_c(p) + r_c^*(p) \leq 1$.*

Proof. We shall prove this theorem by contradiction. Assume that for some $p < p_c$, $r_c(p) + r_c^*(p) > 1$, and fix such a p . Most of the time in the rest of the proof, this p will not appear in our notation. By the assumption above, we can choose $1 - r_c^* < r_1 < r_2 < r_c$. Since for all $r \in (1 - r_c^*, r_c)$, $\Theta(p, r) = 0$ and $\Theta^*(p, 1 - r) = 0$, we have by Lemma 2.11 that

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{r_2}(V_{n,3n}^b) < 1,$$

and

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{r_1}(V_{n,3n}^{w*}) < 1.$$

Applying Lemma 2.2, we obtain from these inequalities that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_{r_2}(H_{n,3n}^{w*}) > 0, \tag{7}$$

and

$$\liminf_{n \rightarrow \infty} \mathbb{P}_{r_1}(H_{n,3n}^b) > 0. \tag{8}$$

Inequality (7) implies that there exists $\gamma > 0$ and an integer N_0 such that for all $n > N_0$, we have $\mathbb{P}_{r_2}(H_{n,3n}^{w*}) \geq \gamma$. Since $H_{n,3n}^{w*}$ is a decreasing event, by monotonicity this inequality holds in the whole interval: for all $r \in [r_1, r_2]$ and all $n > N_0$,

$$\mathbb{P}_r(H_{n,3n}^{w*}) \geq \gamma. \quad (9)$$

Since the measure \mathbb{P}_{r_1} is invariant under 90 degree rotations, Theorem 2.5 and inequality (8) imply that

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{r_1}(V_{n,3n}^b) > 0, \quad (10)$$

which implies that there exists $\alpha > 0$ and a sequence of side lengths $n_k \rightarrow \infty$ as $k \rightarrow \infty$ such that for every k ,

$$\mathbb{P}_{r_1}(V_{n_k,3n_k}^b) \geq \alpha. \quad (11)$$

For later purposes we remark that the FKG inequality (Theorem 2.3) and a standard pasting argument imply that for each k we have, for all $i \in \mathbb{N}, i \geq 1$,

$$\mathbb{P}_{r_1}(V_{n_k, (2i+1)n_k}^b) \geq \alpha^{2i-1}. \quad (12)$$

Indeed, consider rectangles $[0, n_k] \times [2jn_k, (2j+3)n_k]$ for $j = 0, 1, \dots, i-1$ and squares $[0, n_k] \times [2ln_k, (2l+1)n_k]$ for $l = 1, 2, \dots, i-1$. If there are black vertical crossings in these i rectangles of size $n_k \times 3n_k$ and black horizontal crossings in the $i-1$ squares of size $n_k \times n_k$, then there is a black vertical crossing in the $n_k \times (2i+1)n_k$ rectangle since horizontal and vertical crossings of the same square meet. Using Theorem 2.3 and the fact that the probability of a horizontal black crossing in a square is bounded below by the probability of a vertical crossing in an $n_k \times 3n_k$ rectangle, we obtain (12).

Let us now fix an integer L with the property that if we consider L Bernoulli (i.e. independent) trials, each with success probability $\alpha^4/2$, then the probability that there are at least $\frac{32}{(r_2-r_1)\gamma\alpha^{63}} + 1$ successes is at least $1/2$.

Next, we choose an element m_1 of the sequence $\{n_k\}$ (for which (11) holds) so large that it satisfies

$$(8m_1^2 + 8m_1)e^{-\psi(p)\frac{m_1}{100}} \leq \min\left(\frac{\alpha^4}{2L}, \frac{\alpha^{63}}{4(\alpha^{63} + 1)}\right) \quad (13)$$

and

$$m_1 > \frac{600}{\psi(p)}, \quad (14)$$

where $\psi(p)$ is the constant corresponding to our fixed p in Theorem 2.1. Then take other elements m_2, m_3, \dots, m_L in the sequence $\{n_k\}$ satisfying

$$m_{i+1} > \frac{151}{49}m_i \quad (15)$$

for $i = 1, 2, \dots, L-1$. Finally, using the constant N_0 from (9), we set $N = 8n_j$ for some j such that

$$N > \max(N_0, \frac{9}{2}m_L). \quad (16)$$

As $N > N_0$, for all $r \in [r_1, r_2]$, we have

$$\mathbb{P}_r(H_{N,3N}^{w*}) \geq \gamma. \quad (17)$$

Since the annulus $A_n = S_{3n,3n} \setminus (S_{n,n} + (n, n))$ can be split into four overlapping rectangles, each with sides of length n and $3n$, a standard argument, based on pasting crossings and the FKG inequality (Theorem 2.3) implies that, for $i = 1, 2, \dots, L$,

$$\mathbb{P}_{r_1}(B_{m_i}^b) \geq (\mathbb{P}_{r_1}(V_{m_i,3m_i}^b))^4.$$

Since m_1, \dots, m_L , and $\frac{N}{8}$ are elements of the sequence $\{n_k\}$, we get by (11) and (12) that $\mathbb{P}_{r_1}(V_{m_i,3m_i}^b) \geq \alpha$ and $\mathbb{P}_{r_1}(V_{\frac{N}{8},8N}^b) \geq \mathbb{P}_{r_1}(V_{\frac{N}{8},65\frac{N}{8}}^b) \geq \alpha^{63}$. By monotonicity these inequalities hold in the whole interval $[r_1, r_2]$. Hence, for all $r \in [r_1, r_2]$ and for $i = 1, \dots, L$, we obtain

$$\mathbb{P}_r(B_{m_i}^b) \geq \alpha^4, \quad (18)$$

and

$$\mathbb{P}_r(V_{\frac{N}{8},8N}^b) \geq \alpha^{63}. \quad (19)$$

We have now made all the preparation needed for the essential part of the proof. In the second part, we shall show that there are uniformly many pivotal clusters for the event $H_{N,8N}^{w*}$, in expectation, in the interval $r \in [r_1, r_2]$. More precisely, we will show that for all $r \in [r_1, r_2]$, we have

$$\mathbb{E}_r(n(H_{N,8N}^{w*})) > \frac{1}{r_2 - r_1}, \quad (20)$$

where $n(H_{N,8N}^{w*})$ denotes the number of p -clusters that are pivotal for the event $H_{N,8N}^{w*}$.

Before giving the proof, let us explain how this statement leads to a contradiction. By putting Corollary 2.9 and (20) together, we obtain

$$\max_{r \in [r_1, r_2]} \frac{d}{dr} \mathbb{P}_r(H_{N,8N}^{w*}) < -\frac{1}{r_2 - r_1}.$$

However, this cannot be the case since it would imply

$$\begin{aligned} \mathbb{P}_{r_2}(H_{N,8N}^{w*}) &\leq \mathbb{P}_{r_1}(H_{N,8N}^{w*}) + (r_2 - r_1) \max_{r \in [r_1, r_2]} \frac{d}{dr} \mathbb{P}_r(H_{N,8N}^{w*}) \\ &< \mathbb{P}_{r_1}(H_{N,8N}^{w*}) - 1, \end{aligned}$$

which is clearly impossible.

Note that it was the assumption that the interval $(1 - r_c^*, r_c)$ is non-empty that enabled us to choose a sub-interval $[r_1, r_2]$ of positive length where the derivative of $\mathbb{P}_r(H_{N,8N}^{w*})$ is uniformly bounded away from 0 by $-\frac{1}{r_2 - r_1}$. Since this leads to a contradiction, we conclude that $r_c \leq 1 - r_c^*$, as stated in Theorem 3.2. It remains to prove (20).

Proof of inequality (20). It will be convenient to introduce the following notation to denote certain parts of $S := S_{N,8N}$. We will denote by $t(S) := [0, N] \times \{8N\}$ the *top of S*, by $b(S) := [0, N] \times \{0\}$ its *bottom*, by $l(S) := \{0\} \times [0, 8N]$ its *left side*, by $r(S) := \{N\} \times [0, 8N]$ its *right side*, by $UH(S) := [0, N] \times [4N + 1, 8N]$ its *upper half*, and by $LH(S) := [0, \frac{N}{2}] \times [0, 8N]$ its *left half*.

We shall now present a construction of black and white paths in S which guarantees the existence of many pivotal clusters, and which succeeds with a high enough probability to provide the desired lower bound for $\mathbb{E}_r(n(H_{N,8N}^w))$. The construction consists of three parts. In the first part, we show that with probability bounded away from 0, there is a horizontal white $*$ -crossing in the lowest part of S .

Part 1. We start looking for the lowest white horizontal $*$ -crossing of S . It is well-known that the lowest such $*$ -crossing can be found (when it exists) by checking only the colours of vertices (in S) below the $*$ -crossing and on it. (The meaning of expressions such as “below, above, to the right of” can be made precise via the Jordan Curve Theorem.)

Recall that by (17), the probability of the event $H_{N,3N}^{w*}$ is uniformly bounded below by γ . Suppose that $H_{N,3N}^{w*}$ occurs. Denote the lowest horizontal white $*$ -crossing in $S_{N,3N}$ by Π_h^{w*} . We shall later use the fact that Π_h^{w*} is also the lowest horizontal white $*$ -crossing in S . So far we have checked sites only below or on Π_h^{w*} , but not above it. However, since the model is dependent, we do have some information above Π_h^{w*} ; for example that the p -clusters of the vertices in Π_h^{w*} are white. Therefore, let us consider the thickened $*$ -crossing

$$\Gamma_h^{w*} := \bigcup_{x \in \Pi_h^{w*}} C_x^p.$$

We denote the portion of S above Γ_h^{w*} by $A(\Gamma_h^{w*})$. We also need to define

the following sets (see Figure 2):

$$\begin{aligned}
R_i &:= \left(\left[(i-1)\frac{N}{8}, i\frac{N}{8} \right] \times [0, 8N] \right) \cap A(\Gamma_h^{w*}) \text{ for } i = 1, 2, 3, 4, \\
R_5 &:= \left(\left[4\frac{N}{8}, 7\frac{N}{8} \right] \times [0, 8N] \right) \cap A(\Gamma_h^{w*}), \\
R_6 &:= \left(\left[7\frac{N}{8}, N \right] \times [0, 8N] \right) \cap A(\Gamma_h^{w*}), \\
C_{L_1} &:= \bigcup_{x \in S \setminus A(\Gamma_h^{w*})} C_x^p.
\end{aligned}$$

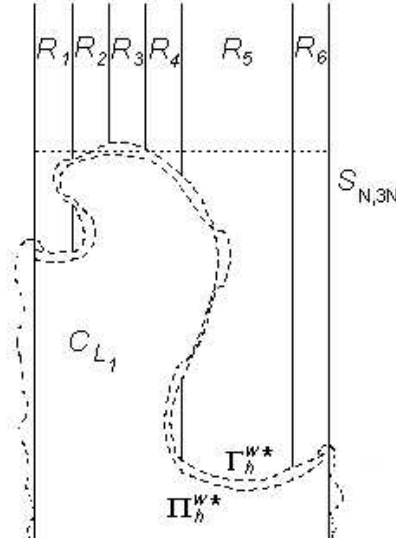


Figure 2: Middle part of the rectangle S . The top side of $S_{N,3N}$ is indicated by a dotted segment. The lower broken line represents the lowest horizontal white $*$ -crossing Π_h^{w*} of $S_{N,3N}$. The higher broken line represents the upper boundary of the thickened $*$ -crossing Γ_h^{w*} . The dotted lines at the left and right side of S form part of the boundary of C_{L_1} .

We know that the edges in the edge boundary $\Delta_1 := \Delta C_{L_1}$ are closed in the underlying Bernoulli percolation, hence Δ_1 forms a closed barrier. Note that this barrier is obtained without checking the states of edges or colours of vertices in $\text{ext}(\Delta_1)$.

We now claim that with high probability, $\{\Gamma_h^{w*} \cap UH(S) = \emptyset\}$, and $\{C_{L_1} \cap R_i = \emptyset\}$ for $i = 2, 3, 4, 5$. Indeed, if all the vertices in $S_{N,3N}$ have a dependence range smaller than N , then the first equality holds. The only way that any of the latter equalities could be false is that there is a vertex below Γ_h^{w*} whose

p -cluster extends above Γ_h^{w*} so much that it intersects one of the rectangles R_2, R_3, R_4, R_5 . For this to happen, there has to be a vertex in $S_{N,3N}$ with a dependence range of at least $\frac{N}{8}$. Hence, using crude estimations, we give an upper bound for the probability that at least one equation is false:

$$\begin{aligned}
\nu_p \left(\bigcup_{x \in S_{N,3N}} \{ \mathcal{D}(x) \geq \frac{N}{8} \} \right) &\leq \nu_p \left(\bigcup_{x \in S_{N,3N}} \{ \mathcal{D}(x) \geq \frac{N}{100} \} \right) \\
&\leq \sum_{x \in S_{N,3N}} \nu_p \left(\mathcal{D}(x) \geq \frac{N}{100} \right) \\
&\leq (N+1)(3N+1) \nu_p \left(\mathcal{D}(0) \geq \frac{N}{100} \right) \\
&\leq (8N^2 + 8N) e^{-\psi(p) \frac{N}{100}} \\
&\leq (8m_1^2 + 8m_1) e^{-\psi(p) \frac{m_1}{100}} \\
&\leq \frac{\alpha^4}{2L},
\end{aligned}$$

by the choice of m_1 (see (13)). We have also used Theorem 2.1 and the monotonicity of $f(x) = (8x^2 + 8x)e^{-\psi(p) \frac{x}{100}}$ for $x > \frac{600}{\psi(p)}$ (this is justified because of (14)). Therefore, with probability at least $1 - \frac{\alpha^4}{2L}$, we have no information about the bond configuration or the colour of the vertices in R_2, R_3, R_4, R_5 , so their union provides an unexplored region in $A(\Gamma_h^{w*})$, which contains $UH(S)$.

We now condition on the events $\{\Gamma_h^{w*} \cap UH(S) = \emptyset\}$, $\{C_{L_1} \cap R_i = \emptyset\}$ for $i = 2, 3, 4, 5$, and continue with the second part of our construction.

Part 2. In this part, our task is to find the leftmost vertical black path in $R_2 \cup R_3 \cup R_4$ from $t(S)$ to Γ_h^{w*} . Here and later, if W is a set of white vertices, then by ‘‘a black path to W ’’ we mean a black path to some vertex at distance 1 from W . Let us consider the following event: $E_2 := \{\text{there is a black path from } t(S) \text{ to } \Gamma_h^{w*} \text{ that does not leave } R_2 \cup R_3 \cup R_4\}$. Let us denote the σ -algebra generated by the information we have so far by \mathcal{F}_c , and let us denote the conditional measure $\mathbb{P}_r(\cdot | \mathcal{F}_c)$ by $\mathbb{P}_r^{(c)}$. We shall first show that for all $r \in [r_1, r_2]$, we a.s. have

$$\mathbb{P}_r^{(c)}(E_2) \geq \frac{\alpha^{63}}{2}. \tag{21}$$

Let $\omega^{(c)} = (\eta^{(c)}, \xi^{(c)})$ and $\omega = (\eta, \xi)$ be elements of Ω drawn according to $\mathbb{P}_r^{(c)}$ and \mathbb{P}_r , respectively. We shall show that ω and $\omega^{(c)}$ can be coupled

in such a way that if there is no large p -cluster in η in R_3 , and there is a vertical black path in ξ from $t(S)$ to Γ_h^{w*} that does not leave R_3 , then there is a black path in $\xi^{(c)}$ that does not leave $R_2 \cup R_3 \cup R_4$.

We first couple η and $\eta^{(c)}$ so that they coincide in the exterior of Δ_1 . This is possible because \mathcal{F}_c contains information only about Δ_1 and $\text{int}(\Delta_1)$, and Bernoulli percolation configurations restricted to disjoint sets are independent. Note that the p -clusters of η can extend beyond Δ_1 , therefore each p -cluster of $\eta^{(c)}$ contained in $\text{ext}(\Delta_1)$ is a subset of a p -cluster of η , but they are not necessarily the same.

In order to couple ξ and $\xi^{(c)}$, we consider the collection \mathbb{G} of all (self-avoiding) paths in $[\frac{N}{4}, \frac{3N}{8}] \times [0, 8N]$ from $t(R_3) := [\frac{N}{4}, \frac{3N}{8}] \times \{8N\}$ to Γ_h^{w*} , and give them some deterministic order. We also order the vertices in each path starting from $t(R_3)$ and ending at Γ_h^{w*} . We denote the j -th vertex of the i -th path by v_i^j .

To each vertex $x \in \mathbb{Z}^2$ we assign a vector $(c_1(x), c_2(x))$, where c_1 and c_2 can take three values: black, white or undefined. Let \mathcal{C}_1 and \mathcal{C}_2 be the collections of all values assigned respectively to $c_1(x)$ and $c_2(x)$ for all $x \in \mathbb{Z}^2$, indexed by \mathbb{Z}^2 . We start with $c_1(x)$ being undefined for all $x \in \mathbb{Z}^2$, and $c_2(x)$ being the colour of x given \mathcal{F}_c , or undefined if the colour is not known (note that, in particular, $c_2(x)$ is undefined for all $x \in \text{ext}(\Delta_1)$). We will generate two coupled colour configurations, ξ and $\xi^{(c)}$, according to the correct marginal distributions with the help of the following algorithm. Note that the values assigned to $c_1(x)$ and $c_2(x)$ will change, at least for some x , during the algorithmic construction.

Let c be an ‘‘auxiliary’’ variable that can take the same three values: black, white and undefined. We also use two index variables: i and j .

1. $i := 1, j := 1$.
2. $c := c_1(v_i^j)$.
3.
 - If $c = \text{black}$, $j := j + 1$.
 - If $c = \text{white}$, $i := i + 1$ and $j := 1$. Stop if $i > |\mathbb{G}|$.
 - If $c = \text{undefined}$, with probability p , let $c := \text{black}$, and with probability $1 - p$, let $c := \text{white}$. Then set $c_1(x) := c$ for all $x \in C_{v_i^j}^p(\eta)$ (i.e., for all x in the same η p -cluster of the current vertex), and $c_2(x) := c$ for all $x \in C_{v_i^j}^p(\eta^{(c)})$.
4. Stop if \mathcal{C}_1 contains a black path from $t(R_3)$ to Γ_h^{w*} , otherwise go back to 2.

After the algorithm stops, we let $\xi(x) = c_1(x)$ for all x 's such that $c_1(x)$ is not undefined, and $\xi^{(c)}(x) = c_2(x)$ for all x 's such that $c_2(x)$ is not undefined. Note that, because of the nature of the algorithm, the vertices that have not been assigned a colour are naturally split into p -clusters (e.g., if $c_1(x)$ is undefined, then $c_1(y)$ is undefined for all y in the η p -cluster of x). We then assign colour black with probability p and white with probability $1 - p$ to the p -clusters in η and in $\eta^{(c)}$ that have not yet been assigned a colour, independently of each other.

We now make three important observations.

(1) First of all, it can be easily seen that the configurations ξ and $\xi^{(c)}$ generated in the way described above are distributed according to the correct distributions, \mathbb{P}_r and $\mathbb{P}_r^{(c)}$ respectively.

(2) Moreover, before the very last step of the algorithmic procedure, whenever $c_1(x)$ is black for x in $\text{ext}(\Delta_1)$, $c_2(x)$ is also black for that same x . This follows from the fact that, because of the coupling between η and $\eta^{(c)}$, a difference between the p -clusters of η and those of $\eta^{(c)}$ encountered during the algorithmic construction can only arise when a p -cluster of η ‘‘crosses’’ Δ_1 . In that case, the η p -cluster possibly reaches more vertices in $\text{ext}(\Delta_1)$ than the $\eta^{(c)}$ p -cluster. If such an η p -cluster is coloured white, it makes \mathcal{C}_1 ‘‘more white’’ than \mathcal{C}_2 . If it is coloured black, the algorithm stops because a black path from $t(R_3)$ to Γ_h^{w*} has been generated. Therefore, before the very last step of the algorithmic procedure, for every vertex x in $\text{ext}(\Delta_1)$ such that $c_1(x)$ is black, $c_2(x)$ is also black, and for every vertex such that $c_2(x)$ is undefined, $c_1(x)$ is either undefined or white. This implies that if the algorithm stops because \mathcal{C}_1 contains a black path from $t(R_3)$ to Γ_h^{w*} , then also \mathcal{C}_2 contains a black path from $t(R_3)$ to Γ_h^{w*} .

(3) Finally, at the end of the algorithmic construction described above, $c_2(x)$ can be black only if x is in R_3 or belongs to the $\eta^{(c)}$ p -cluster of a vertex in R_3 .

Now note that, because of the coupling between η and $\eta^{(c)}$, if $x \in R_3$ has dependence range not larger than $N/8$ in η , the same is true for the range of x in $\eta^{(c)}$. Therefore, if no vertex in R_3 has a range larger than $N/8$ in η , when the algorithm stops because it found a black path in \mathcal{C}_1 from $t(R_3)$ to Γ_h^{w*} , by the previous comment and observations (2) and (3) above, there is a black path in \mathcal{C}_2 from $t(R_3)$ to Γ_h^{w*} contained inside $R_2 \cup R_3 \cup R_4$.

It follows that, setting

$$T_{R_3} := \bigcap_{v \in R_3} \left\{ \mathcal{D}(v) < \frac{N}{8} \right\},$$

we obtain a.s.

$$\mathbb{P}_r^{(c)}(E_2) \geq \mathbb{P}_r \left(V_{\frac{N}{8}, 8N}^b | T_{R_3} \right) \nu_p(T_{R_3}).$$

Elementary calculations show that

$$\mathbb{P}_r \left(V_{\frac{N}{8}, 8N}^b | T_{R_3} \right) \geq \mathbb{P}_r \left(V_{\frac{N}{8}, 8N}^b \right) - \nu_p(T_{R_3}^c),$$

where, with a similar computation as in Part 1,

$$\begin{aligned} \nu_p(T_{R_3}^c) &= \nu_p \left(\bigcup_{v \in R_3} \{ \mathcal{D}(v) \geq \frac{N}{8} \} \right) \\ &\leq \left(\frac{N}{8} + 1 \right) (8N + 1) \nu_p \left(\mathcal{D}(0) \geq \frac{N}{8} \right) \\ &\leq (8m_1^2 + 8m_1) e^{-\psi(p) \frac{m_1}{100}} \\ &\leq \zeta, \end{aligned} \tag{22}$$

where $\zeta = \frac{\alpha^{63}}{4(\alpha^{63}+1)}$, and in the last step we used (13). As $\mathbb{P}_r \left(V_{\frac{N}{8}, 8N}^b \right) \geq \alpha^{63}$ (see (19)), and $\zeta < \frac{\alpha^{63}}{2(\alpha^{63}+1)}$, this gives

$$\mathbb{P}_r^{(c)}(E_2) \geq (\alpha^{63} - \zeta)(1 - \zeta) \geq \alpha^{63}(1 - \zeta) - \zeta \geq \frac{\alpha^{63}}{2},$$

proving (21).

Having shown that $\mathbb{P}_r^{(c)}(E_2) > 0$, conditioning on the event E_2 , we call $\tilde{\Pi}_v^b$ the leftmost black (self-avoiding) path from $t(S)$ to Γ_h^{w*} contained in $R_2 \cup R_3 \cup R_4$. We denote by Π_v^b the union of $\tilde{\Pi}_v^b$ and the black r -clusters in S to the left of $\tilde{\Pi}_v^b$ connected to $\tilde{\Pi}_v^b$ (see Figure 3). We denote the region of

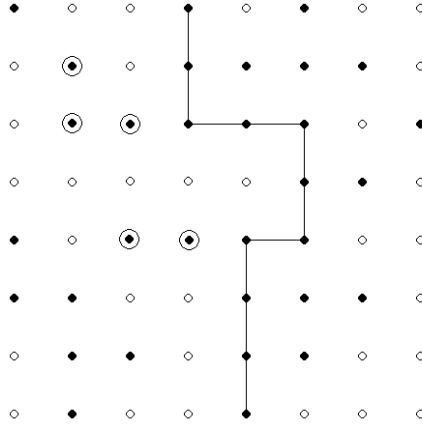


Figure 3: The black vertices joined by a line represent a portion of $\tilde{\Pi}_v^b$. The black vertices to the left side of $\tilde{\Pi}_v^b$ and marked by circles belong to Π_v^b .

S to the right of $\tilde{\Pi}_v^b$ by $J(\tilde{\Pi}_v^b)$. Note that no information on the colours of vertices to the right of $\tilde{\Pi}_v^b$ is needed to determine $\tilde{\Pi}_v^b$ itself and Π_v^b .

We now set

$$C_{L_2} := \bigcup_{x \in (R_2 \cup R_3 \cup R_4) \setminus (\Pi_v^b \cup J(\tilde{\Pi}_v^b))} C_x^p.$$

If there is no vertex in $R_2 \cup R_3 \cup R_4$ on the left side of $\tilde{\Pi}_v^b$ with a dependence range greater than N , then C_{L_2} does not intersect $S_{N,7N} \cap J(\tilde{\Pi}_v^b)$. Therefore, the probability of $\{C_{L_2} \cap S_{N,7N} \cap J(\tilde{\Pi}_v^b) \neq \emptyset\}$ is at most $\frac{\alpha^4}{2L}$, from a computation similar to the one leading to inequality (22). Conditioning on the event $\{C_{L_2} \cap S_{N,7N} \cap J(\tilde{\Pi}_v^b) = \emptyset\}$, we continue with the third part of our construction.

Part 3. In this part, we shall complete our construction which shows that there are “too many” pivotal p -clusters for having a white horizontal $*$ -crossing in S . It is easy to see that if a vertex at distance 1 from some $x \in \Gamma_h^{w*}$ is black and is connected to $t(S)$ by a black path contained in S , then C_x^p is pivotal for $H_{N,8N}^{w*}$. Indeed, due to the fact that every vertex in the lowest horizontal white $*$ -crossing has a black neighbour that is connected to $b(S)$ by a black path, changing the colour of C_x^p would make the existence of a white horizontal $*$ -crossing of S impossible.

Let π be the “upper layer” of Γ_h^{w*} , that is, the set of vertices in Γ_h^{w*} with at least one neighbour in the region $A(\Gamma_h^{w*})$. Denote by x_0 the rightmost vertex of π at distance 1 from Π_v^b . Furthermore, denote the portion of π to the right of x_0 by Π_u^{w*} . We have specified a site x_0 of Γ_h^{w*} that lies in $LH(S) \cap S_{N,4N}$ (i.e. the lower-left quarter of S) with the property that a black path leads from $t(S)$ to a point at distance 1 from x_0 . This implies that $C_{x_0}^p$ is pivotal for $H_{N,8N}^{w*}$.

Recall that we defined m_1, \dots, m_L so that (13)–(15) and (18) hold. Now let us consider annuli $B_i := \{c_i + S_{3m_i, 3m_i} + (-\frac{3m_i}{2}, -\frac{3m_i}{2})\} \setminus \{c_i + S_{m_i, m_i} + (-\frac{m_i}{2}, -\frac{m_i}{2})\}$, $i = 1, \dots, L$, centered at $c_i := x_0$ if m_i is even and at $c_i = x_0 + (1/2, 1/2)$ if m_i is odd. This means that for any i , B_i is an annulus with center at distance at most $1/2$ from x_0 , with inner diameter m_i and outer diameter $3m_i$. We will look for black paths in these annuli between Π_v^b and Π_u^{w*} . Note that even the largest annulus, B_L does not go above $S_{N,7N}$, nor to the right of R_5 , according to (16). Let us denote the bounded region determined by the curves $(\{7\frac{N}{8}\} \times [0, 8N]) \cap A(\Gamma_h^{w*})$ (i.e. the right side of R_5), $[0, N] \times \{7N\}$ (the top of $S_{N,7N}$), Π_v^b , and Π_u^{w*} , by AJ .

Let Δ_2 denote the edge boundary of C_{L_2} , defined at the end of Part 2. We shall look for black paths in the annuli in the region AJ . Let $B_i[\frac{m_i}{100}]$ denote the $\frac{m_i}{100}$ -neighbourhood of B_i . We consider the events $Q_i, i = 1, \dots, L$, that

there is a black path in $B_i[\frac{m_i}{100}] \cap AJ$ between Π_v^b and Π_u^{w*} , and $Q := \{\text{there are at least } \frac{32}{(r_2-r_1)\alpha^{63}\gamma} + 1 \text{ indices } i \text{ such that } Q_i \text{ holds}\}$. Let us denote the σ -algebra generated by the information we have so far by \mathcal{F}_{c_2} , and let us denote the conditional measure $\mathbb{P}_r(\cdot|\mathcal{F}_{c_2})$ by $\mathbb{P}_r^{(c_2)}$. We shall show that for any $r \in [r_1, r_2]$, a.s.,

$$\mathbb{P}_r^{(c_2)}(Q) \geq \frac{1}{4}. \quad (23)$$

Let $\omega^{(c_2)} = (\eta^{(c_2)}, \xi^{(c_2)})$ and $\omega = (\eta, \xi)$ be configurations in the plane, drawn according to $\mathbb{P}_r^{(c_2)}$ and \mathbb{P}_r , respectively. We shall show that ω and $\omega^{(c_2)}$ can be coupled in such a way that if in ω , for all $i = 1, \dots, L$, there is no vertex in B_i with a dependence range larger than $\frac{m_i}{100}$, and for some j there is a black circuit in B_j in ξ , then $\xi^{(c_2)} \in Q_j$.

First, we couple the edge configurations η and $\eta^{(c_2)}$ in $\text{ext}(\Delta_1) \cap \text{ext}(\Delta_2)$. \mathcal{F}_{c_2} contains information that the edges in $\Delta_1 \cup \Delta_2$ are closed, and Π_v^b is coloured black (plus some information about $\text{int}(\Delta_1) \cup \text{int}(\Delta_2)$, but that has no influence on the exterior). This implies that, according to Lemma 2.4, there will be a bias in the configuration $\eta^{(c_2)}$ towards more open edges. In fact, according to Lemma 2.4, in $\text{ext}(\Delta_1) \cap \text{ext}(\Delta_2)$, η and $\eta^{(c_2)}$ can be coupled so that any closed edge in the latter is also closed in the former. We pick two such coupled configurations, and concentrating on $\eta^{(c_2)}$ first, we denote by H the union of Π_v^b with the set of vertices that are connected to Π_v^b by an open path in $\eta^{(c_2)}$. Then ΔH is a closed barrier in $\eta^{(c_2)}$, and by the coupling, this barrier is also closed in η . As a final ingredient in our joint construction, we now redraw the configurations in $\text{ext}(\Delta_1) \cap \text{ext}(\Delta_2) \cap \text{ext}(\Delta H)$ in both configurations, so that in this region the configurations agree, and are (conditionally) independent of $\Delta_1, \Delta_2, \Delta H$ and their interiors. The configurations chosen this way, denoted again by η and $\eta^{(c_2)}$, have the correct marginal distributions.

As in Part 2, we assign a vector $(c_1(x), c_2(x))$ to each vertex $x \in \mathbb{Z}^2$, and let $\mathcal{C}_1, \mathcal{C}_2$ be the collections of all the corresponding values, indexed by \mathbb{Z}^2 . We take $c_2(x) := \text{black}$ for all $x \in \text{int}(\Delta H)$.

We shall perform L algorithms, where the k -th algorithm corresponds to searching for black paths in B_k . We start with $k = 1$. Let \mathbb{G}_k be the collection of all the (self-avoiding) paths in $B_k \cap AJ$, leading from Π_v^b to Π_u^{w*} . We equip \mathbb{G}_k with an arbitrary deterministic ordering. We also order the vertices along each path, starting from Π_v^b , going towards Π_u^{w*} . As before, the j -th vertex in the i -th path in $B_k \cap AJ$ is denoted by ${}_k v_i^j$. The algorithm which generates ξ and $\xi^{(c_2)}$ is the same as in Part 2, using the ‘‘auxiliary’’ variable c and the index variables i and j .

1. $i := 1, j := 1$.

2. $c := c_1({}_k v_i^j)$.
3.
 - If $c = \text{black}$, $j := j + 1$.
 - If $c = \text{white}$, $i := i + 1$ and $j := 1$. Stop if $i > |\mathbb{G}_k|$.
 - If $c = \text{undefined}$, with probability p , let $c := \text{black}$, and with probability $1 - p$, let $c := \text{white}$. Then set $c_1(x) := c$ for all $x \in C_{{}_k v_i^j}^p(\eta)$ (i.e., for all x in the same η p -cluster of the current vertex), and $c_2(x) := c$ for all $x \in C_{{}_k v_i^j}^p(\eta^{(c_2)})$.
4. Stop if \mathcal{C}_1 contains a black path from Π_v^b to Π_u^{w*} in $B_k \cap AJ$, otherwise go back to 2.

When the algorithm terminates, we increase k by one, and if $k \leq L$, we re-run the algorithm with the new value of k . After the last algorithm stops, we set $\xi(x) = c_1(x)$ for all x such that $c_1(x)$ is not undefined, and $\xi^{(c_2)}(x) = c_2(x)$ for all x such that $c_2(x)$ is not undefined. We then assign colour black with probability p and white with probability $1 - p$ to the p -clusters in η and in $\eta^{(c)}$ that have not been assigned a colour yet, independently of each other. Here, we make three important remarks.

(1) Due to the coupling, the bond configurations η and $\eta^{(c_2)}$ are the same in $\text{ext}(\Delta_1) \cap \text{ext}(\Delta_2) \cap \text{ext}(\Delta H)$. Therefore, if for all $k = 1, \dots, L$, there is no vertex in $B_k \cap AJ \cap \text{ext}(\Delta H)$ with a dependence range larger than $\frac{m_k}{100}$ in η , then the same is true in $\eta^{(c_2)}$ as well. By re-writing (15) as $\frac{m_{k+1}}{2} > \frac{3m_k}{2} + \frac{m_k}{100} + \frac{m_{k+1}}{100}$, we see that the $\frac{m_k}{100}$ -neighbourhoods of the annuli B_k are disjoint. Hence, if for all $k = 1, \dots, L$, there is no vertex in B_k with a dependence range larger than $\frac{m_k}{100}$ in η , then any point gets c_1 or c_2 values by at most one of the algorithms.

(2) Similarly to Part 2, the configurations ξ and $\xi^{(c)}$ generated in the way described above are distributed according to the correct distributions, \mathbb{P}_r and $\mathbb{P}_r^{(c_2)}$ respectively. Note that assigning black to $\xi^{(c_2)}(x)$ for all $x \in \text{int}(\Delta H)$ is justified since, by the definition of H , every such x is connected to Π_v^b by an open path in $\eta^{(c_2)}$.

(3) For any k , if there is no vertex in $B_k \cap AJ$ with a dependence range larger than $\frac{m_k}{100}$, then before the very last step of the k -th algorithmic procedure, whenever $c_1(x)$ is black for x in $\text{ext}(\Delta_1) \cap \text{ext}(\Delta_2) \cap B_k$, $c_2(x)$ is also black for that same x .

To see this, we need to notice that due to the coupling between η and $\eta^{(c_2)}$, the p -clusters of a vertex $x \in \text{ext}(\Delta_1) \cap \text{ext}(\Delta_2) \cap B_k$ in η and in $\eta^{(c_2)}$ may differ in the following four cases:

- $x \in \text{int}(\Delta H)$,

- $C_x^p(\eta)$ “crosses” ΔH ,
- $C_x^p(\eta)$ “crosses” Δ_2 ,
- $C_x^p(\eta)$ “crosses” Δ_1 .

The difference in the first case is unimportant since we have $c_2(x) = \text{black}$ for all $x \in \text{int}(\Delta H)$. Recall that ΔH is a closed barrier both in η and in $\eta^{(c_2)}$; hence the second case never happens. The k -th algorithm assigns values to vertices in $B_k \cap AJ$ only. If there is no vertex in $B_k \cap AJ$ with a dependence range larger than $\frac{m_k}{100}$, then the third case does not happen either: ΔH prevents $C_x^p(\eta)$ for $x \in B_k \cap AJ$ from intersecting Δ_2 . The fourth case is handled exactly the same way as in Part 2: if such a p -cluster is coloured white, it makes \mathcal{C}_1 “more white” than \mathcal{C}_2 ; if it is black, the algorithm has found an appropriate black path in \mathcal{C}_1 and therefore terminates.

This shows that, for every $k = 1, \dots, L$, if there are no large η p -clusters in B_k , the presence of a black path in \mathcal{C}_1 from Π_v^b to Π_u^{w*} in $B_k \cap AJ$ implies that there is a black path in \mathcal{C}_2 from Π_v^b to Π_u^{w*} . Remark (1) above shows that this black path in \mathcal{C}_2 is indeed contained in $B_k[\frac{m_k}{100}] \cap AJ$.

This implies that if we let $T_{B_i} := \bigcap_{v \in B_i} \{\mathcal{D}(v) < \frac{m_i}{100}\}$, $B(B_i) := \{\text{there is a black circuit in } B_i \text{ surrounding } x_0\}$ for $i = 1, \dots, L$, and $T := \bigcap_{i=1}^L T_{B_i}$, we obtain a.s.

$$\mathbb{P}_r^{(c_2)}(Q) \geq \mathbb{P}_{p,r}(B(B_i) \text{ holds for at least } \frac{32}{(r_2 - r_1)\alpha^{63\gamma}} + 1 \text{ indices} | T) \nu_p(T).$$

The second factor is very close to one as

$$\begin{aligned} 1 - \nu_p(T) &= \nu_p\left(\bigcup_{i=1}^L T_{B_i}^c\right) \\ &\leq \sum_{i=1}^L \sum_{v \in B_i} \nu_p(\mathcal{D}(v) \geq \frac{m_i}{100}) \\ &\leq \sum_{i=1}^L (8m_i^2 + 8m_i) \nu_p(\mathcal{D}(0) \geq \frac{m_i}{100}) \\ &\leq L \cdot \frac{\alpha^4}{2L} = \alpha^4/2, \end{aligned} \tag{24}$$

where we used translation invariance, the monotonicity of the function $f(x) = (8x^2 + 8x)e^{-\psi(p)\frac{x}{100}}$ above $x = \frac{600}{\psi(p)}$, and inequalities (14) and (13). Note that, conditioned on T , the event $B(B_i)$ depends on the $\frac{m_i}{100}$ -neighbourhood of B_i only. We know the $\frac{m_i}{100}$ -neighbourhoods of the annuli B_i are disjoint.

Therefore, the events $B(B_i)$ ($i = 1, \dots, L$) are conditionally independent, conditioned on T . We also have, for $i = 1, \dots, L$,

$$\mathbb{P}_{p,r}(B(B_i)|T) \geq \mathbb{P}_{p,r}(B(B_i)) - \nu_p(T^c) \geq \alpha^4 - \alpha^4/2,$$

due to (18) and (24). Hence, by the choice of L before inequality (13),

$$\mathbb{P}_{p,r}(B(B_i) \text{ holds for at least } \frac{32}{(r_2 - r_1)\alpha^{63}\gamma} + 1 \text{ indices} | T) \geq 1/2.$$

This shows that

$$\mathbb{P}_r^{(c_2)}(Q) \geq (1 - \frac{\alpha^4}{2}) \frac{1}{2} \geq \frac{1}{4},$$

proving (23).

Note that whenever Q_i happens, there is a pivotal (for the event $H_{N,8N}^{w*}$) p -cluster in or close to B_i . Moreover, for any $i \neq j$, the events Q_i and Q_j give rise to different pivotal clusters. Therefore, conditioning on having reached Part 3, the conditional probability of the event $E_3 := \{\text{there are at least } \frac{32}{(r_2 - r_1)\alpha^{63}\gamma} + 1 \text{ pivotal clusters for } H_{N,8N}^{w*}\}$ is at least the conditional probability of Q , which is at least $\frac{1}{4}$, as we have just concluded.

Since it is easy to see that for any $r \in [r_1, r_2]$ the \mathbb{P}_r -probability of reaching Part 3 is at least $\gamma \cdot (1 - \frac{\alpha^4}{2L}) \cdot \frac{\alpha^{63}}{2} \cdot (1 - \frac{\alpha^4}{2L}) \geq \frac{\gamma\alpha^{63}}{8}$, and we know that $n(H_{N,8N}^{w*})$ is a nonnegative random variable, we have for any $r \in [r_1, r_2]$,

$$\mathbb{E}_r(n(H_{N,8N}^{w*})) \geq \left(\frac{32}{(r_2 - r_1)\alpha^{63}\gamma} + 1 \right) \cdot \frac{1}{4} \cdot \frac{\gamma\alpha^{63}}{8} > \frac{1}{(r_2 - r_1)},$$

finishing the proof of inequality (20), and completing the proof of Theorem 3.2. \square

4 Proofs of the remaining results

For the proof of Theorem 1.3, we need the following result of Russo [20]. Let μ be a probability measure that assigns colours black or white to the vertices of \mathbb{Z}^2 . Let $P_\infty^b(\mu)$ (resp. $P_\infty^{w*}(\mu)$) denote the probability that the black cluster (resp. white $*$ -cluster) of the origin is infinite. Let $S^b(\mu)$ denote the mean size of the black cluster of the origin.

Theorem 4.1. ([20]) *If μ is translation invariant and $S^b(\mu) < \infty$, then $P_\infty^{w*}(\mu) > 0$.*

Proof of Theorem 1.3. First, we shall prove criticality when $p < 1/2$, $r = r_c(p)$. Our argument follows the proof of Proposition 1 in [21]. Fix $p < 1/2$. Take $\varepsilon > 0$ as in Lemma 2.10. By Lemma 2.1 and the monotonicity of the function $f(x) = (x+1)(3x+1)e^{-\psi(p)\frac{x}{3}}$ for x large enough, there exists $N_0 = N_0(p)$ such that, for all $n \geq N_0$,

$$(n+1)(3n+1)\nu_p(\mathcal{D}(0) \geq \frac{n}{3}) \leq \varepsilon. \quad (25)$$

Since $\Theta(p, r) = 0$ for all $r < r_c(p)$, Lemma 2.10 and (25) imply that

$$\mathbb{P}_{p,r}(V_{n,3n}^b) \leq 1 - \varepsilon$$

for all $r < r_c(p)$, $n \geq N_0$.

We claim that for any n , the function $\mathbb{P}_{p,r}(V_{n,3n}^b)$ is continuous in r . To see this, notice that the occurrence of $V_{n,3n}^b$ is completely determined by a partitioning of the vertices in $S_{n,3n}$ in p -clusters, and the colours assigned to these clusters. Let us denote by \mathcal{P}_S the set of partitions of the vertices in $S_{n,3n}$ which are compatible with a bond configuration, and the (random) partition determined by the initial bond percolation by \mathcal{G}_S . Fix an arbitrary partition $g_S \in \mathcal{P}_S$. Since the colours are assigned independently to the p -clusters determined by g_S , it is easy to see that $\mathbb{P}_{p,r}(V_{n,3n}^b | \mathcal{G}_S = g_S)$ is a polynomial function of r , hence continuous in r . This implies that the (finite) linear combination

$$\mathbb{P}_{p,r}(V_{n,3n}^b) = \sum_{g_S \in \mathcal{P}_S} \nu_p(\mathcal{G}_S = g_S) \mathbb{P}_{p,r}(V_{n,3n}^b | \mathcal{G}_S = g_S)$$

is indeed continuous in r .

This shows that for any $n \geq N_0$, if we let $r \rightarrow r_c$, we obtain $\mathbb{P}_{p,r_c}(V_{n,3n}^b) \leq 1 - \varepsilon$. Therefore,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{p,r_c}(V_{n,3n}^b) < 1,$$

which, by Lemma 2.11, implies $\Theta(p, r_c) = 0$, providing the first condition of criticality.

The relation $\Theta^*(p, r_c^*) = 0$ can be proved analogously. Hence, as $r_c^* = 1 - r_c$, we obtain that the \mathbb{P}_{p,r_c} -probability of the origin being in an infinite white $*$ -cluster is 0. Therefore, applying Theorem 4.1 to the measure \mathbb{P}_{p,r_c} yields that the mean size of the black cluster of the origin is infinite, concluding the proof of criticality for $p < 1/2$, $r = r_c(p)$.

The fact that there is no infinite black cluster or white $*$ -cluster at $p = 1/2$, $r \in (0, 1)$, is a straightforward consequence of the fact that $\nu_{1/2}$ -almost every p -configuration contains infinitely many disjoint open circuits

surrounding the origin. These circuits are coloured independently, preventing the possibility of black percolation or white $*$ -percolation. (This idea has been described in [16] already to show that there is no percolation of either colour at $p = 1/2, r = 1/2$.) The infinite mean cluster size follows then from Theorem 4.1, as before.

The supercritical case $p > 1/2$ is obvious: the probability that the origin is in an infinite p -cluster is positive in that case, and so is the probability that the colour assigned to that cluster is black for any $r > 0$. \square

Remark 4.2. The proof of $r_c + r_c^* = 1$ for $p < 1/2$ uses the FKG inequality, exponential decay of correlations and duality. It is easy to see that polynomial decay of correlations of degree strictly greater than 2 would be enough for the proof. The fact that there is no infinite black cluster at $p = 1/2, r \in (0, 1)$, (Theorem 1.3) even though duality and the FKG inequality hold in that case, shows that at $p = 1/2$, for any $c > 2$ and $N \in \mathbb{N}$, there exists $n > N$ such that

$$\nu_{1/2}(\mathcal{D}(0) \geq n) \geq \frac{1}{n^c},$$

i.e., in critical bond percolation on the square lattice, the probability that the origin is connected to ∂B_n by an open path is at least n^{-c} .

Proof of Corollary 1.4. To prove Corollary 1.4, one needs to put together the results in Theorems 1.1–1.3. Strictly speaking, the following three statements need additional clarification: for $p < 1/2$, we have

- (i) $r_c(p) \in [1/2, 1)$,
- (ii) $\Theta^*(p, 1 - r_c(p)) = 0$ and the mean size of the white $*$ -cluster of the origin is infinite, and
- (iii) If $r > r_c(p)$, the size of the white $*$ -cluster has an exponentially decaying tail.

Now $r_c(p) < 1$ follows from Theorem 2.6 in [16]. The other bound $r_c(p) \geq 1/2$ is an easy consequence of $r_c(p) + r_c^*(p) = 1$, since $r_c(p) \geq r_c^*(p)$. We have seen the first half of (ii), i.e. $\Theta^*(p, r_c^*(p)) = 0$, in the proof of Theorem 1.3. We also know that $\Theta(p, r_c(p)) = 0$, which implies, according to Theorem 4.1, that the mean size of the white $*$ -cluster of the origin is infinite. Statement (iii) can be proved the same way as Theorem 1.2. \square

The proof of Corollary 1.5 uses the methods of Russo [20], and van den Berg and Keane [4], based on the following lemma, which may be interesting in itself.

Lemma 4.3. *At $p < 1/2, r > r_c(p)$, the number of infinite black clusters is $\mathbb{P}_{p,r}$ -a.s. equal to 1.*

Proof. For $r_c < r < 1$, similarly to the proof of Theorem 3.1, conditions (1)-(4) of Theorem 2.12 clearly hold for the measure $\mathbb{P}_{p,r}$. Theorem 2.12 states that under these conditions, the number of infinite black clusters is 1. The case $r = 1$ is obvious. \square

Proof of Corollary 1.5. We fix $p < 1/2$, and write $\Theta(r|\eta)$ for the conditional probability that the cluster of the origin is infinite, given that the bond configuration is η . The above mentioned classical arguments and Theorem 1.3 give that $\Theta(r|\eta)$ is for almost all η a continuous function in r .

Now fix $\varepsilon > 0$ and $r \in [0, 1]$. For almost every η , there exists a maximal $\delta(\eta)$ such that if $|r' - r| \leq \delta(\eta)$, then $|\Theta(r|\eta) - \Theta(r'|\eta)| < \varepsilon$. Now choose $\bar{\delta} > 0$ so small that $\nu_p(\eta; \delta(\eta) < \bar{\delta}) < \varepsilon$ and denote the set $\{\eta; \delta(\eta) \geq \bar{\delta}\}$ by A . Since $\Theta(p, r) = \int \Theta(r|\eta) d\nu_p(\eta)$ we then find that for r' such that $|r - r'| < \bar{\delta}$, we have

$$\begin{aligned} |\Theta(p, r) - \Theta(p, r')| &\leq \int_A |\Theta(r|\eta) - \Theta(r'|\eta)| d\nu_p(\eta) + \\ &\quad + \int_{A^c} |\Theta(r|\eta) - \Theta(r'|\eta)| d\nu_p(\eta) \\ &\leq \varepsilon \nu_p(A) + 2\nu_p(A^c) \\ &\leq \varepsilon + 2\varepsilon, \end{aligned}$$

proving the result. \square

5 The DaC model on the triangular lattice \mathbb{T}

On the square lattice, the relationship $r_c(p) + r_c^*(p) = 1$ does not determine the critical value $r_c(p)$. However, on the triangular lattice, percolation is self-dual (i.e., $*$ -paths are the same as ordinary paths), so that the same relationship immediately implies $r_c^*(p) = r_c(p) = 1/2$. In this section, we elaborate a bit on the proof of $r_c(p) + r_c^*(p) = 1$ for $p < p_c(\mathbb{T})$ on the triangular lattice. In this case, the version of the RSW-type theorem of Bollobás and Riordan [6] suffices, and we do not need to use the improvement in [3].

We embed the triangular lattice \mathbb{T} in \mathbb{R}^2 so that its vertices $\mathcal{V}(\mathbb{T})$ are the intersections of the lines $y = -\sqrt{3}x + \sqrt{3}k$ and $y = \frac{\sqrt{3}}{2}l$ for $k, l \in \mathbb{Z}$, and denote the elements of $\mathcal{V}(\mathbb{T})$ by (k, l) . For example, $(0, 0)$ refers to the intersection of $y = -\sqrt{3}x$ and $y = 0$. The edges are given by $\mathcal{E}(\mathbb{T}) := \{(a, b) : a, b \in \mathcal{V}(\mathbb{T}), |a - b| = 1\}$, where $|\cdot|$ denotes the Euclidean norm (see Figure 4). We define and denote paths, circuits, horizontal and vertical crossings exactly as before. Note that $S_{m,n}$ corresponds to a parallelogram in \mathbb{R}^2 of side lengths m and n , as in the example in Figure 4.

Given the equivalence between crossings and $*$ -crossings, we will drop the $*$ from our notation in this section. We note that the definitions and all the preliminary results of Section 2 still apply, modulo the reinterpretation of $*$ -crossings as ordinary crossings and the different value of p_c for bond percolation on the triangular lattice. This observation will be implicitly understood in the rest of the section and we will use the results of Section 2.2 without further comments.

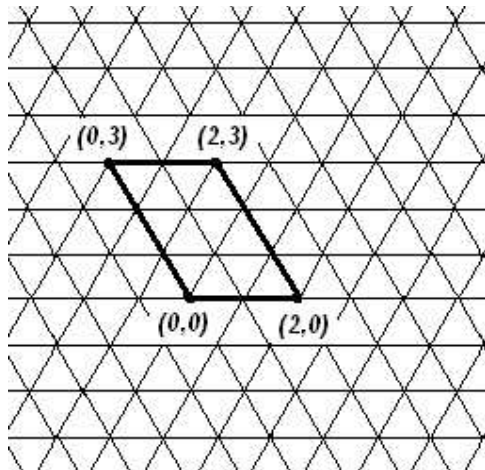


Figure 4: Portion of the triangular lattice \mathbb{T} . The heavy segments are the sides of the parallelogram $S_{2,3}$.

The inequality $r_c(p) \geq 1/2$ can be proved by standard methods. Similar (however somewhat simpler) considerations to those in the proof of Theorem 3.2 lead to $r_c(p) \leq 1/2$ as follows. It is easy to see that for any rhombus S , the probability of having a black vertical crossing is exactly the same as the probability of having a black horizontal crossing in S . This observation, together with Lemma 2.2 and symmetry of black and white at $r = 1/2$ implies the following result.

Lemma 5.1. *For any $p \in [0, 1]$ and any $n \in \mathbb{N}$,*

$$\mathbb{P}_{p,1/2}(V_{n,n}^b) = 1/2.$$

This lemma allows us to use the RSW type theorem of Bollobás and Rioridan [6], which states that $\liminf_{n \rightarrow \infty} \mathbb{P}_{p,1/2}(V_{n,n}^b) > 0$ implies $\limsup_{n \rightarrow \infty} \mathbb{P}_{p,1/2}(V_{n,\rho n}^b) > 0$ for any $\rho > 0$, to obtain the following result.

Lemma 5.2. *For any $p < p_c$, we have*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{p,1/2}(V_{n,3n}^b) > 0. \tag{26}$$

Next we show that certain parallelograms have high crossing probabilities.

Theorem 5.3. *For all $p < p_c$, for any $\varepsilon > 0$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{p, 1/2 + \varepsilon}(V_{n, 3n}^b) = 1.$$

Proof sketch. We assume that there exists a $p < p_c$ and an $\varepsilon > 0$ such that $\limsup_{n \rightarrow \infty} \mathbb{P}_{p, 1/2 + \varepsilon}(V_{n, 3n}^b) < 1$. We denote the measure $\mathbb{P}_{p, r}(\cdot)$ by $\mathbb{P}_r(\cdot)$. The assumption above and Lemma 2.2 gives $\liminf_{n \rightarrow \infty} \mathbb{P}_{1/2 + \varepsilon}(H_{n, 3n}^w) > 0$, which, together with monotonicity, shows that in the interval $r \in [1/2, 1/2 + \varepsilon]$, whenever n is large enough, the \mathbb{P}_r -probability of $H_{n, 3n}^w$ is bounded away from 0.

On the other hand, Lemma 5.2 gives $\limsup_{n \rightarrow \infty} \mathbb{P}_{1/2}(V_{n, 3n}^b) > 0$. Therefore, there exists a sequence of side lengths $n_k \rightarrow \infty$ along which $\mathbb{P}_{1/2}(V_{n_k, 3n_k}^b)$ is also bounded away from 0, which, by monotonicity, gives the same lower bound in the whole interval $r \in [1/2, 1/2 + \varepsilon]$ for $\mathbb{P}_r(V_{n_k, 3n_k}^b)$.

A careful reading of the proof of Theorem 3.2 shows that these lower bounds are enough to determine the parameters L, m_1, \dots, m_L, N of the construction described in the proof of inequality (20), which provides a uniform lower bound on the number of pivotal p -clusters for the event $H_{N, 8N}^w$ in the interval $r \in [1/2, 1/2 + \varepsilon]$, leading to a contradiction. \square

Theorem 5.3 together with Lemma 2.11 implies $r_c \leq 1/2$, establishing the equality $r_c = 1/2$ and completing the proof of Theorem 1.6.

We conclude this section with the proof of Proposition 1.8.

Proof of Proposition 1.8. Let us fix $p < p_c$ and $q \geq 2$ and denote the corresponding probability measure by $\mu_{p, q, r}$. One can check that conditions (1)–(3) of Theorem 2.12 apply to $\mu_{p, q, 1/2}$: condition (1) is obvious, condition (2) can be found, for example, in [14], condition (3) is proved in [15]. If we now assume that for $r = 1/2$ there exists an infinite black cluster with positive probability (meaning that condition (4) is also satisfied by $\mu_{p, q, 1/2}$), colour symmetry implies the existence of an infinite white cluster with positive probability, leading to a contradiction with Theorem 2.12. We then conclude that there exists a.s. no infinite black cluster at $r = 1/2$ and, by colour symmetry again, no infinite white cluster. Since $\mu_{p, q, 1/2}$ is clearly a translation-invariant measure, Theorem 4.1 implies infinite mean size for the black r -cluster of the origin. \square

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