

On central limit theorems in the random connection model

Tim van de Brug and Ronald Meester

July 1, 2003

Divisie Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit, De Boelelaan
1081a, 1081 HV Amsterdam, The Netherlands, rmeester@cs.vu.nl.

Abstract

Consider a sequence of Poisson random connection models (X_n, λ_n, g_n) on \mathbb{R}^d , where $\lambda_n/n^d \rightarrow \lambda > 0$ and $g_n(x) = g(nx)$ for some non-increasing, integrable connection function g . Let $I_n(g)$ be the number of isolated vertices of (X_n, λ_n, g_n) in some bounded Borel set K , where K has non-empty interior and boundary of Lebesgue measure zero. Roy and Sarkar (2003) claim that

$$\frac{I_n(g) - \mathbb{E}I_n(g)}{\sqrt{\text{Var}I_n(g)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty,$$

where \rightsquigarrow denotes convergence in distribution. However, their proof is not correct. We correct their proof and extend the result to larger components when the connection function g has bounded support.

1 Introduction

Let (X, λ, g) denote a Poisson random connection model, where X is the underlying Poisson point process on \mathbb{R}^d with density $\lambda > 0$, and where g is a connection function which we assume to be a non-increasing and which satisfies $0 < \int_{\mathbb{R}^d} g(|x|) dx < \infty$. In words, this amounts to saying that any two points x and y of X are connected with probability $g(|x - y|)$, independently of all other pairs, independently of X . Let K be a bounded Borel subset of \mathbb{R}^d with non-empty interior and boundary of Lebesgue measure zero.

Consider a sequence of positive real numbers λ_n with $\lambda_n/n^d \rightarrow \lambda$, let X_n be a Poisson process on \mathbb{R}^d with density λ_n and let g_n be the connection function defined by $g_n(x) = g(nx)$. Consider the sequence of Poisson random connection models (X_n, λ_n, g_n) on \mathbb{R}^d . Let $I_n(g)$ be the number of isolated vertices of (X_n, λ_n, g_n) in K . Roy and Sarkar (2003) claim to prove the following result.

Theorem 1.1

$$\frac{I_n(g) - \mathbb{E}I_n(g)}{\sqrt{\text{Var}I_n(g)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty, \quad (1)$$

where \rightsquigarrow denotes convergence in distribution.

Although we agree with the statement of this result, we do not agree with the proof in Roy and Sarkar (2003). In this note, we explain what went wrong in their proof, and how this can be corrected. In addition, we prove an extension to larger components in case the connection function has bounded support.

2 Truncation and scaling

The central limit theorem (1) is relatively easy to show when g has bounded support, see Roy and Sarkar (2003). Hence, the strategy adopted by Roy and Sarkar (2003) is to truncate the relevant connection functions, and let the truncation go to infinity. This means that there are two operations involved: scaling and truncation. The root of the problem lies in the fact that these two operations do not commute.

Following Roy and Sarkar (2003), we define for $R > 0$ and $n \in \mathbb{N}$ connection functions $g_R, g^R, g_{n,R}, g_n^R : [0, \infty) \rightarrow [0, 1]$ by

$$g_R(x) = 1_{\{x \leq R\}}g(x), \quad g^R(x) = 1_{\{x > R\}}g(x), \quad g_{n,R}(x) = 1_{\{x \leq R\}}g(nx), \quad g_n^R(x) = 1_{\{x > R\}}g(nx).$$

Note that the notation g_R can formally not be used to denote $1_{\{\cdot \leq R\}}g(\cdot)$, since g_n has already been defined as $g(n\cdot)$. Nevertheless we shall adopt this notation, because we think that this will not cause any confusion. Henceforth g_R will always denote $1_{\{\cdot \leq R\}}g(\cdot)$ and g_n will always denote $g(n\cdot)$. Let $L_R(g)$ be the number of isolated vertices of (X, λ, g_R) in K that are not isolated in (X, λ, g) . Let $J_{n,R}(g)$ be the number of isolated vertices of $(X_n, \lambda_n, g_{n,R})$ in K and let $L_{n,R}(g) = J_{n,R}(g) - I_n(g)$ be the number of isolated vertices of $(X_n, \lambda_n, g_{n,R})$ in K that are not isolated in (X_n, λ_n, g_n) .

Roy and Sarkar (2003) claim the following (without proof).

Statement A *If (1) is true when the connection function g has bounded support, then it is the case that*

$$\frac{J_{n,R}(g) - \mathbb{E}J_{n,R}(g)}{\sqrt{\text{Var}J_{n,R}(g)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty, \quad (2)$$

for any connection function g .

They then proceed, via a number of moment estimates involving $J_{n,R}(g)$ and $L_{n,R}(g)$, to show that the truth of (2) for any connection function g , implies the full central limit theorem in (1).

One problem with their argument is that we do not see why Statement A should be true. In particular, the only reason why Statement A would be true, is that we would be able to write $J_{n,R}(g) = I_n(h)$ for some connection function h with bounded support. This would mean that $g_{n,R}$ can be seen as a scaling of h , that is,

$$1_{\{x \leq R\}}g(nx) = h(nx),$$

but this leads to $h(x) = 1_{\{x \leq nR\}}g(x)$, which clearly does not make any sense in general.

It seems then that Roy and Sarkar (2003) interchange truncation and scaling, but these two operations do not commute. This mixing up becomes already apparent when we look

at their Lemma 5 which states (without proof) that

$$\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \mathbb{E} L_{n,R}(g) = p(\lambda, g_R)(1 - p(\lambda, g^R)); \quad (3)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \text{Var} L_{n,R}(g) &= p(\lambda, g_R)(1 - p(\lambda, g^R)) + \lambda \int_{\mathbb{R}^d} (1 - g(|x|)) \\ &\quad \left[p(\lambda, g_R)^2 p_\lambda^{g_R, g_R}(x, 0) - 2p(\lambda, g_R)^2 p(\lambda, g^R) p_\lambda^{g_R, g}(x, 0) + \right. \\ &\quad \left. p(\lambda, g)^2 p_\lambda^{g, g}(x, 0) \right] - p(\lambda, g_R)^2 (1 - p(\lambda, g^R))^2 dx; \end{aligned} \quad (4)$$

where ℓ denotes Lebesgue measure on \mathbb{R}^d and

$$p(\mu, h) = e^{-\mu \int_{\mathbb{R}^d} h(|y|) dy} \quad \text{and} \quad p_\mu^{h_1, h_2}(x_1, x_2) = e^{\mu \int_{\mathbb{R}^d} h_1(|y-x_1|) h_2(|y-x_2|) dy}.$$

However, the following proposition shows that (3) and (4) are not correct; see the forthcoming Lemma 3.3 for a corresponding correct (and useful) statement.

Proposition 2.1 *For $R > \sup\{|x_1 - x_2| : x_1, x_2 \in K\}$ we have*

$$\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \mathbb{E} L_{n,R}(g) = 0; \quad (5)$$

$$\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \text{Var} L_{n,R}(g) = 0. \quad (6)$$

Proof: For $R > 0$ and $n \in \mathbb{N}$ define $k_{nR}, k^{nR} : [0, \infty) \rightarrow [0, 1]$ by

$$k_{nR}(x) = 1_{\{x \leq nR\}} g(x), \quad k^{nR}(x) = 1_{\{x > nR\}} g(x), \quad x \in [0, \infty).$$

We have as $n \rightarrow \infty$,

$$p(\lambda_n, g_{n,R}) = p(\lambda_n/n^d, k_{nR}) \rightarrow p(\lambda, g);$$

$$p(\lambda_n, g_n^R) = p(\lambda_n/n^d, k^{nR}) \rightarrow 1.$$

According to Roy and Sarkar (2003) Lemma 4 we have for $R > \sup\{|x_1 - x_2| : x_1, x_2 \in K\}$,

$$\mathbb{E} L_R(g) = \lambda \ell(K) p(\lambda, g_R)(1 - p(\lambda, g^R)), \quad (7)$$

and therefore,

$$(\lambda_n \ell(K))^{-1} \mathbb{E} L_{n,R}(g) = p(\lambda_n, g_{n,R})(1 - p(\lambda_n, g_n^R)) \rightarrow 0, \quad n \rightarrow \infty,$$

which proves (5).

To prove (6), we use Lemma 4 in Roy and Sarkar (2003) which says that for $R > \sup\{|x_1 - x_2| : x_1, x_2 \in K\}$, we have

$$\begin{aligned} \text{Var}L_R(g) &= \lambda \ell(K) p(\lambda, g_R) (1 - p(\lambda, g^R)) + \lambda^2 \int_K \int_K (1 - g(|x_1 - x_2|)) \\ &\quad \left[p(\lambda, g_R)^2 p_{\lambda}^{g_R, g_R}(x_1, x_2) - 2p(\lambda, g_R)^2 p(\lambda, g^R) p_{\lambda}^{g_R, g}(x_1, x_2) + \right. \\ &\quad \left. p(\lambda, g)^2 p_{\lambda}^{g, g}(x_1, x_2) \right] - p(\lambda, g_R)^2 (1 - p(\lambda, g^R))^2 dx_2 dx_1. \end{aligned} \quad (8)$$

We use (8) with $\lambda = \lambda_n$ and $g = g_n$. Note that as $n \rightarrow \infty$

$$\begin{aligned} p_{\lambda_n}^{g_n, R, g_n, R}(x/n, 0) &= p_{\lambda_n/n^d}^{k_{nR}, k_{nR}}(x, 0) \rightarrow p_{\lambda}^{g, g}(x, 0); \\ p_{\lambda_n}^{g_n, R, g_n}(x/n, 0) &= p_{\lambda_n/n^d}^{k_{nR}, g}(x, 0) \rightarrow p_{\lambda}^{g, g}(x, 0); \\ p_{\lambda_n}^{g_n, g_n}(x/n, 0) &= p_{\lambda_n/n^d}^{g, g}(x, 0) \rightarrow p_{\lambda}^{g, g}(x, 0). \end{aligned}$$

We have

$$\begin{aligned} &\frac{\lambda_n}{\ell(K)} \int_K \int_K (1 - g_n(|x_1 - x_2|)) \left[p(\lambda_n, g_{n,R})^2 p_{\lambda_n}^{g_n, R, g_n, R}(x_1, x_2) - \right. \\ &2p(\lambda_n, g_{n,R})^2 p(\lambda_n, g_n^R) p_{\lambda_n}^{g_n, R, g_n}(x_1, x_2) + p(\lambda_n, g_n)^2 p_{\lambda_n}^{g_n, g_n}(x_1, x_2) \left. \right] - \\ &p(\lambda_n, g_{n,R})^2 (1 - p(\lambda_n, g_n^R))^2 dx_2 dx_1 = \\ &= \frac{\lambda_n}{n^d \ell(K)} \int_K \int_{n(K-x_1)} (1 - g(|x_2|)) \left[p(\lambda_n/n^d, k_{nR})^2 p_{\lambda_n/n^d}^{k_{nR}, k_{nR}}(0, x_2) - \right. \\ &2p(\lambda_n/n^d, k_{nR})^2 p(\lambda_n/n^d, k^{nR}) p_{\lambda_n/n^d}^{k_{nR}, g}(0, x_2) + p(\lambda_n/n^d, g)^2 p_{\lambda_n/n^d}^{g, g}(0, x_2) \left. \right] - \\ &p(\lambda_n/n^d, k_{nR})^2 (1 - p(\lambda_n/n^d, k^{nR}))^2 dx_2 dx_1. \end{aligned}$$

By Lemma 3.1 below with $x = -x_2$ we can apply the dominated convergence theorem. Combining the result with (5) yields (6). \square

In what follows, we proceed along the way that we believe Roy and Sarkar (2003) had in mind.

For this, we introduce for $R > 0$ and $n \in \mathbb{N}$ connection functions $g_{R,n}, g^{R,n} : [0, \infty) \rightarrow [0, 1]$ as follows:

$$g_{R,n}(x) = 1_{\{x \leq R/n\}} g(nx), \quad g^{R,n}(x) = 1_{\{x > R/n\}} g(nx).$$

Note the difference between $g_{R,n}$ and $g_{n,R}$ and between $g^{R,n}$ and g_n^R . Let $J_{R,n}(g)$ be the number of isolated vertices of $(X_n, \lambda_n, g_{R,n})$ in K and let $L_{R,n}(g) = J_{R,n}(g) - I_n(g)$ be the number of isolated vertices of $(X_n, \lambda_n, g_{R,n})$ in K that are not isolated in (X_n, λ_n, g_n) . Note that the notations $g_{R,n}$, $J_{R,n}(g)$ and $L_{R,n}(g)$ can formally not be used here, since $g_{n,R}$, $J_{n,R}(g)$ and $L_{n,R}(g)$ have already been defined. Nevertheless we shall adopt these notations, because henceforth we shall use the function $g_{n,R}$ and the random variables $J_{n,R}(g)$ and $L_{n,R}(g)$ no more. We now claim that the following is true (compare the incorrect Statement A above)

Statement B *If (1) is true when the connection function g has bounded support, then it is the case that*

$$\frac{J_{R,n}(g) - \mathbb{E}J_{R,n}(g)}{\sqrt{\text{Var}J_{R,n}(g)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty, \quad (9)$$

for any connection function g .

To see this, observe that

$$J_{R,n}(g) = I_n(g_R),$$

as can be seen by direct computation. Since g_R has bounded support, Statement B follows. The moral of this is, that we should base the proof on $J_{R,n}(g)$ and $L_{R,n}(g)$ instead of $J_{n,R}(g)$ and $L_{n,R}(g)$. In the next section we show that the proof idea of Roy and Sarkar (2003) can still be carried out, although the computations involved are a little more complicated now.

3 Proof of Theorem 1.1

We start with a technical lemma, needed for applications of dominated convergence.

Lemma 3.1 *There exists N such that for $R > 0$, $n \geq N$ and $x \in \mathbb{R}^d$*

$$\left| (1 - g(|x|)) \left[p(\lambda_n/n^d, g_R)^2 p_{\lambda_n/n^d}^{g_R, g_R}(x, 0) - 2p(\lambda_n/n^d, g_R)^2 p(\lambda_n/n^d, g^R) p_{\lambda_n/n^d}^{g_R, g}(x, 0) + p(\lambda_n/n^d, g)^2 p_{\lambda_n/n^d}^{g, g}(x, 0) \right] - p(\lambda_n/n^d, g_R)^2 (1 - p(\lambda_n/n^d, g^R))^2 \right| \leq Cg(|x|/2), \quad (10)$$

where C is a constant not depending on x , n or R .

Proof: Since $p(\lambda_n/n^d, g_R)p(\lambda_n/n^d, g^R) = p(\lambda_n/n^d, g)$, the expression between the absolute value signs in (10) is equal to

$$\begin{aligned} & -g(|x|) \left[p(\lambda_n/n^d, g_R)^2 p_{\lambda_n/n^d}^{g_R, g_R}(x, 0) - 2p(\lambda_n/n^d, g_R)p(\lambda_n/n^d, g)p_{\lambda_n/n^d}^{g_R, g}(x, 0) + \right. \\ & \quad \left. p(\lambda_n/n^d, g)^2 p_{\lambda_n/n^d}^{g, g}(x, 0) \right] + p(\lambda_n/n^d, g_R)^2 (p_{\lambda_n/n^d}^{g_R, g_R}(x, 0) - 1) - \\ & \quad 2p(\lambda_n/n^d, g_R)p(\lambda_n/n^d, g) (p_{\lambda_n/n^d}^{g_R, g}(x, 0) - 1) + p(\lambda_n/n^d, g)^2 (p_{\lambda_n/n^d}^{g, g}(x, 0) - 1). \end{aligned} \quad (11)$$

Let N be such that $\frac{3}{4}\lambda \leq \lambda_n/n^d \leq \frac{3}{2}\lambda$, $n \geq N$. Then since

$$\int_{\mathbb{R}^d} g_R(|y|) + g(|y|) dy \geq 2 \int_{\mathbb{R}^d} g_R(|y|) dy \geq 2 \int_{\mathbb{R}^d} g_R(|y|)g(|y+x|) dy,$$

we have for $n \geq N$

$$p(\lambda_n/n^d, g_R)p(\lambda_n/n^d, g)p_{\lambda_n/n^d}^{g_R, g}(x, 0) \leq e^{-\frac{3}{4}\lambda \int_{\mathbb{R}^d} g_R(|y|) + g(|y|) dy + \frac{3}{2}\lambda \int_{\mathbb{R}^d} g_R(|y-x|)g(|y|) dy} \leq 1. \quad (12)$$

Also,

$$p(\lambda_n/n^d, g_R)^2 p_{\lambda_n/n^d}^{g_R, g_R}(x, 0) \leq 1, \quad p(\lambda_n/n^d, g)^2 p_{\lambda_n/n^d}^{g, g}(x, 0) \leq 1,$$

which follows from (12) by taking $g = g_R$ or letting $R \rightarrow \infty$ respectively. Hence for $n \geq N$ the absolute value of (11) is bounded by

$$4g(|x|) + 4(p_{2\lambda}^{g, g}(x, 0) - 1). \quad (13)$$

To give an upper bound for the second term in this expression, note that for $y \in \mathbb{R}^d$ either $|y| \geq |x|/2$ or $|y-x| \geq |x|/2$, so

$$\begin{aligned} \int_{\mathbb{R}^d} g(|y-x|)g(|y|) dy & \leq \int_{|y| < |x|/2} g(|y-x|)g(|y|) dy + \int_{|y| \geq |x|/2} g(|y-x|)g(|y|) dy \\ & \leq g(|x|/2) \int_{|y| < |x|/2} g(|y|) dy + g(|x|/2) \int_{|y| \geq |x|/2} g(|y-x|) dy \\ & \leq 2g(|x|/2) \int_{\mathbb{R}^d} g(|y|) dy. \end{aligned}$$

Choose M such that $4\lambda g(M/2) \int_{\mathbb{R}^d} g(|y|) dy \leq 1$. Then since $e^t \leq 1 + et$, $t \leq 1$, we have for $|x| \geq M$

$$e^{4\lambda g(|x|/2) \int_{\mathbb{R}^d} g(|y|) dy} \leq 1 + 4e\lambda g(|x|/2) \int_{\mathbb{R}^d} g(|y|) dy.$$

For $|x| < M$ we have

$$e^{4\lambda g(|x|/2) \int_{\mathbb{R}^d} g(|y|) dy} \leq e^{4\lambda \int_{\mathbb{R}^d} g(|y|) dy} \leq 1 + g(|x|/2)g(M/2)^{-1}[e^{4\lambda \int_{\mathbb{R}^d} g(|y|) dy} - 1].$$

Combining the above inequalities yields

$$p_{2\lambda}^{g,g}(x, 0) - 1 \leq Cg(|x|/2), \quad (14)$$

where C is a constant not depending on x , n or R . We conclude that (13) is bounded by $4(1 + C)g(|x|/2)$. \square

Lemma 3.2

$$\mathbb{E}L_{R,n}(g) = \lambda_n \ell(K) p(\lambda_n, g_{R,n})(1 - p(\lambda_n, g^{R,n})) \quad (15)$$

$$\begin{aligned} \text{Var}L_{R,n}(g) &= \lambda_n \ell(K) p(\lambda_n, g_{R,n})(1 - p(\lambda_n, g^{R,n})) + \lambda_n^2 \int_K \int_K (1 - g_n(|x_1 - x_2|)) \\ &\quad \left[p(\lambda_n, g_{R,n})^2 p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) - 2p(\lambda_n, g_{R,n})^2 p(\lambda_n, g^{R,n}) p_{\lambda_n}^{g_{R,n}, g_n}(x_1, x_2) + \right. \\ &\quad \left. p(\lambda_n, g_n)^2 p_{\lambda_n}^{g_n, g_n}(x_1, x_2) \right] - p(\lambda_n, g_{R,n})^2 (1 - p(\lambda_n, g^{R,n}))^2 dx_2 dx_1 + \\ &\quad \lambda_n^2 p(\lambda_n, g_{R,n})^2 \int_K \int_K g^{R,n}(|x_1 - x_2|) p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) dx_2 dx_1 \end{aligned} \quad (16)$$

Proof: The first statement (15) is proved as in Roy and Sarkar (2003) Lemma 4.

For a Borel subset B of \mathbb{R}^d let $X_n(B)$ be the number of points in $X_n \cap B$. For $t > 0$ denote $K^t = K + \{x \in \mathbb{R}^d : |x| < t\}$. In the model (X_n, λ_n, g_n) let $L_{R,n,t}(g)$ be the number of points ξ in $X_n \cap K$ such that ξ is not connected to any point in $X_n \cap K^t$ at a distance R/n or less from ξ but ξ is connected to some point in $X_n \cap K^t$ at a distance greater than R/n from ξ . Since $L_{R,n,t}(g) \rightarrow L_{R,n}(g)$, $t \rightarrow \infty$, and $L_{R,n,t}(g) \leq X_n(K)$, $t > 0$, and $\mathbb{E}X_n(K)^2 < \infty$, the dominated convergence theorem gives

$$\mathbb{E}L_{R,n,t}(g) \rightarrow \mathbb{E}L_{R,n}(g), \quad \text{Var}L_{R,n,t}(g) \rightarrow \text{Var}L_{R,n}(g), \quad t \rightarrow \infty.$$

In order to compute the moments of $L_{R,n,t}(g)$, note that

$$L_{R,n,t}(g) \sim \sum_{i=1}^{X_n(K^t)} 1_{F_i},$$

where \sim denotes equality in distribution, ξ_i , $i \in \mathbb{N}$ are independent random variables, independent of $X_n(K^t)$, uniformly distributed on K^t and connected to each other according to g_n , and $F_i = \{\xi_i \in K; \xi_i \text{ is not connected to any } \xi_j, j \leq X_n(K^t), \text{ at a distance } R/n \text{ or less from } \xi_i; \xi_i \text{ is connected to some } \xi_j, j \leq X_n(K^t), \text{ at a distance greater than } R/n \text{ from } \xi_i\}$.

Since

$$L_{R,n,t}(g)^2 \sim \sum_{i=1}^{X_n(K^t)} 1_{F_i} + \sum_{i=1}^{X_n(K^t)} \sum_{\substack{j=1 \\ j \neq i}}^{X_n(K^t)} 1_{F_i} 1_{F_j},$$

the variance of $L_{R,n,t}(g)$ can be written as

$$\begin{aligned} \text{Var} L_{R,n,t}(g) &= \tag{17} \\ &= \mathbb{E} L_{R,n,t}(g) + \sum_{m=2}^{\infty} m(m-1) \mathbb{P}(F_1 \cap F_2 \mid X_n(K^t) = m) \mathbb{P}(X_n(K^t) = m) - (\mathbb{E} L_{R,n,t}(g))^2. \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{P}(F_1 \cap F_2 \cap \{\xi_1 \text{ is connected to } \xi_2\} \mid X_n(K^t) = m) = \\ &= \frac{1}{\ell(K^t)^m} \int_K \int_K g^{R,n}(|x_1 - x_2|) \int_{K^t} \dots \int_{K^t} \\ &\quad \prod_{i=3}^m (1 - g_{R,n}(|x_i - x_1|)) (1 - g_{R,n}(|x_i - x_2|)) dx_m \dots dx_3 dx_2 dx_1 \\ &= \frac{1}{\ell(K^t)^m} \int_K \int_K g^{R,n}(|x_1 - x_2|) \\ &\quad \left[\int_{K^t} (1 - g_{R,n}(|y - x_1|)) (1 - g_{R,n}(|y - x_2|)) dy \right]^{m-2} dx_2 dx_1, \end{aligned}$$

whence

$$\begin{aligned} &\sum_{m=2}^{\infty} m(m-1) \mathbb{P}(F_1 \cap F_2 \cap \{\xi_1 \text{ is connected to } \xi_2\} \mid X_n(K^t) = m) \mathbb{P}(X_n(K^t) = m) = \\ &= \lambda_n^2 \int_K \int_K g^{R,n}(|x_1 - x_2|) \sum_{m=0}^{\infty} \frac{e^{-\lambda_n \ell(K^t)} \lambda_n^m}{m!} \\ &\quad \left[\int_{K^t} (1 - g_{R,n}(|y - x_1|)) (1 - g_{R,n}(|y - x_2|)) dy \right]^m dx_2 dx_1 \\ &= \lambda_n^2 \int_K \int_K g^{R,n}(|x_1 - x_2|) e^{\lambda_n \int_{K^t} -g_{R,n}(|y-x_1|) - g_{R,n}(|y-x_2|) + g_{R,n}(|y-x_1|) g_{R,n}(|y-x_2|) dy} dx_2 dx_1 \\ &\rightarrow \lambda_n^2 p(\lambda_n, g_{R,n})^2 \int_K \int_K g^{R,n}(|x_1 - x_2|) p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) dx_2 dx_1, \quad t \rightarrow \infty, \tag{18} \end{aligned}$$

where we use the dominated convergence theorem.

Furthermore,

$$\begin{aligned}
& \mathbb{P}(F_1 \cap F_2 \cap \{\xi_1 \text{ is not connected to } \xi_2\} | X_n(K^t) = m) = \\
& = \frac{1}{\ell(K^t)^m} \int_K \int_K (1 - g_n(|x_1 - x_2|)) \int_{K^t} \cdots \int_{K^t} \\
& \quad \left[1 - \prod_{i=3}^m (1 - g^{R,n}(|x_i - x_1|)) \right] \prod_{i=3}^m (1 - g_{R,n}(|x_i - x_1|)) \\
& \quad \left[1 - \prod_{i=3}^m (1 - g^{R,n}(|x_i - x_2|)) \right] \prod_{i=3}^m (1 - g_{R,n}(|x_i - x_2|)) dx_m \dots dx_3 dx_2 dx_1. \quad (19)
\end{aligned}$$

Exactly as in Roy and Sarkar (2003) Lemma 4, one can now show that

$$\begin{aligned}
& \sum_{m=2}^{\infty} m(m-1) \mathbb{P}(F_1 \cap F_2 \cap \{\xi_1 \text{ is not connected to } \xi_2\} | X_n(K^t) = m) \mathbb{P}(X_n(K^t) = m) = \\
& \rightarrow \lambda_n^2 \int_K \int_K (1 - g_n(|x_1 - x_2|)) \left[p(\lambda_n, g_{R,n})^2 p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) - \right. \\
& \quad \left. 2p(\lambda_n, g_{R,n})^2 p(\lambda_n, g^{R,n}) p_{\lambda_n}^{g^{R,n}, g^n}(x_1, x_2) + p(\lambda_n, g_n)^2 p_{\lambda_n}^{g_n, g_n}(x_1, x_2) \right] dx_2 dx_1, \quad (20)
\end{aligned}$$

as $t \rightarrow \infty$, where we use the dominated convergence theorem.

Combining (17), (15), (18) and (20) yields (16). \square

The following lemma replaces the incorrect Lemma 5 (equation (3) and (4) in our current paper) of Roy and Sarkar (2003).

Lemma 3.3

$$\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \mathbb{E} L_{R,n}(g) = p(\lambda, g_R)(1 - p(\lambda, g^R)) \quad (21)$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \text{Var} L_{R,n}(g) & = p(\lambda, g_R)(1 - p(\lambda, g^R)) + \lambda \int_{\mathbb{R}^d} (1 - g(|x|)) \\
& \quad \left[p(\lambda, g_R)^2 p_{\lambda}^{g_R, g_R}(x, 0) - 2p(\lambda, g_R)^2 p(\lambda, g^R) p_{\lambda}^{g_R, g}(x, 0) + \right. \\
& \quad \left. p(\lambda, g)^2 p_{\lambda}^{g, g}(x, 0) \right] - p(\lambda, g_R)^2 (1 - p(\lambda, g^R))^2 dx + \\
& \quad \lambda p(\lambda, g_R)^2 \int_{\mathbb{R}^d} g^R(|x|) p_{\lambda}^{g_R, g_R}(x, 0) dx \quad (22)
\end{aligned}$$

Proof: Assertion (21) follows from (15) by direct computation. We shall deduce (22) from (16). By the dominated convergence theorem

$$\begin{aligned}
& \frac{\lambda_n}{\ell(K)} p(\lambda_n, g_{R,n})^2 \int_K \int_K g^{R,n}(|x_1 - x_2|) p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) dx_2 dx_1 = \\
& = \frac{\lambda_n}{n^d \ell(K)} p(\lambda_n, g_{R,n})^2 \int_K \int_{n(K-x_1)} g^R(|x_2|) p_{\lambda_n/n^d}^{g_{R,n}, g_{R,n}}(0, x_2) dx_2 dx_1 \\
& \rightarrow \lambda p(\lambda, g_R)^2 \int_{\mathbb{R}^d} g^R(|x|) p_{\lambda}^{g_{R,n}, g_{R,n}}(x, 0) dx, \quad n \rightarrow \infty.
\end{aligned} \tag{23}$$

Furthermore,

$$\begin{aligned}
& \frac{\lambda_n}{\ell(K)} \int_K \int_K (1 - g_n(|x_1 - x_2|)) \left[p(\lambda_n, g_{R,n})^2 p_{\lambda_n}^{g_{R,n}, g_{R,n}}(x_1, x_2) - \right. \\
& \left. 2p(\lambda_n, g_{R,n})^2 p(\lambda_n, g^{R,n}) p_{\lambda_n}^{g_{R,n}, g_n}(x_1, x_2) + p(\lambda_n, g_n)^2 p_{\lambda_n}^{g_n, g_n}(x_1, x_2) \right] - \\
& p(\lambda_n, g_{R,n})^2 (1 - p(\lambda_n, g^{R,n}))^2 dx_2 dx_1 = \\
& = \frac{\lambda_n}{n^d \ell(K)} \int_K \int_{n(K-x_1)} (1 - g(|x_2|)) \left[p(\lambda_n/n^d, g_R)^2 p_{\lambda_n/n^d}^{g_{R,n}, g_{R,n}}(0, x_2) - \right. \\
& \left. 2p(\lambda_n/n^d, g_R)^2 p(\lambda_n/n^d, g^R) p_{\lambda_n/n^d}^{g_{R,n}, g}(0, x_2) + p(\lambda_n/n^d, g)^2 p_{\lambda_n/n^d}^{g, g}(0, x_2) \right] - \\
& p(\lambda_n/n^d, g_R)^2 (1 - p(\lambda_n/n^d, g^R))^2 dx_2 dx_1.
\end{aligned}$$

By Lemma 3.1 with $x = -x_2$, we can apply the dominated convergence theorem. Combining the result with (21) and (23) yields (22). \square

Corollary 3.4

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \mathbb{E} L_{R,n}(g) = 0 \tag{24}$$

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \text{Var} L_{R,n}(g) = 0 \tag{25}$$

Proof: The dominated convergence theorem gives

$$p(\lambda, g_R) \rightarrow p(\lambda, g), \quad p(\lambda, g^R) \rightarrow 1, \quad p_{\lambda}^{g_{R,n}, g_{R,n}}(x, 0) \rightarrow p_{\lambda}^{g, g}(x, 0), \quad p_{\lambda}^{g_{R,n}, g}(x, 0) \rightarrow p_{\lambda}^{g, g}(x, 0),$$

as $R \rightarrow \infty$. Now (24) follows from (21). Another application of the dominated convergence theorem yields

$$\int_{\mathbb{R}^d} g^R(|x|) p_{\lambda}^{g_{R,n}, g_{R,n}}(x, 0) dx \rightarrow 0, \quad R \rightarrow \infty.$$

Finally, the integrand in the first integral on the right hand side of (22) tends to 0 as $R \rightarrow \infty$. By Lemma 3.1 with $\lambda_n = \lambda n^d$, we can apply the dominated convergence theorem to conclude (25). \square

Finally, we can prove the main result:

Theorem 3.5 *If for $R > 0$*

$$\frac{J_{R,n}(g) - \mathbb{E}J_{R,n}(g)}{\sqrt{\text{Var}J_{R,n}(g)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty, \quad (26)$$

then (1) holds.

Proof: Roy and Sarkar (2003) Lemma 3 shows that

$$\lim_{n \rightarrow \infty} (\lambda_n \ell(K))^{-1} \text{Var}I_n(g) = p(\lambda, g) + \lambda p(\lambda, g)^2 \int_{\mathbb{R}^d} (1 - g(|x|)) p_\lambda^{g,g}(x, 0) - 1 \, dx. \quad (27)$$

It follows from (27), Corollary 3.4 and Chebyshev's inequality that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\left| \frac{L_{R,n}(g) - \mathbb{E}L_{R,n}(g)}{\sqrt{\text{Var}I_n(g)}} \right| \geq \varepsilon \right) \leq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\text{Var}L_{R,n}(g)}{\varepsilon^2 \text{Var}I_n(g)} = 0, \quad \varepsilon > 0.$$

Moreover, applying (27) also with g replaced by g_R gives $\lim_{n \rightarrow \infty} \text{Var}J_{R,n}(g)/\text{Var}I_n(g) = \delta_R$, where δ_R is a constant. (This was incorrectly claimed in Roy and Sarkar (2003) with $L_{n,R}(g)$ instead of $L_{R,n}(g)$.) Because

$$(1 - g_R(|x|)) p_\lambda^{g_R, g_R}(x, 0) - 1 \geq (1 - g_R(|x|)) \cdot 1 - 1 \geq -g(|x|)$$

and by (14)

$$(1 - g_R(|x|)) p_\lambda^{g_R, g_R}(x, 0) - 1 \leq 1 \cdot p_\lambda^{g,g}(x, 0) - 1 \leq Cg(|x|/2),$$

where C is a constant not depending on x or R , we have by the dominated convergence theorem $\lim_{R \rightarrow \infty} \delta_R = 1$. Now if (26) holds, then for $x \in \mathbb{R}$

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{I_n(g) - \mathbb{E}I_n(g)}{\sqrt{\text{Var}I_n(g)}} \leq x \right) \leq \\ & \leq \lim_{\varepsilon \downarrow 0} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{J_{R,n}(g) - \mathbb{E}J_{R,n}(g)}{\sqrt{\text{Var}I_n(g)}} \leq x + \varepsilon \right) + \mathbb{P} \left(\left| \frac{L_{R,n}(g) - \mathbb{E}L_{R,n}(g)}{\sqrt{\text{Var}I_n(g)}} \right| \geq \varepsilon \right) \\ & = \Phi(x). \end{aligned}$$

A similar argument yields

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left(\frac{I_n(g) - \mathbb{E}I_n(g)}{\sqrt{\text{Var}I_n(g)}} \leq x \right) \geq \Phi(x),$$

which completes the proof of the theorem. \square

4 Extension to larger components

In this section, we discuss larger components. A central limit theorem for larger components needs another approach, even when the connection function has bounded support. The reason for this is that the exact moment computations of the preceding sections no longer seem possible. At this point, we can only prove a central limit theorem when the connection function g has bounded support. For this, we use a result of Bolthausen (1982), from which it follows that in order to prove a central limit theorem, certain mixing conditions suffice. For convenience, the central limit theorem in this section is stated a little different from the earlier ones, in the sense that we do not scale the connection function and the density, but instead take larger and larger subsets of the space. This is equivalent to the case where $\lambda_n = \lambda n^d$ in the original setup.

For a subset Λ of \mathbb{Z}^d , let the inner boundary of Λ be denoted by $\partial\Lambda$, and its cardinality by $|\Lambda|$. Let the random variable $I^r(\Lambda) = I^r(\Lambda, g)$ be defined as $1/r$ times the number of vertices of (X, λ, g) in $\Lambda + (0, 1]^d$ that are contained in a component of size r . For $z \in \mathbb{Z}^d$ write $I^r(z) = I^r(\{z\})$. We shall prove the following central limit theorem.

Theorem 4.1 *Consider a random connection model with connection function g of bounded support. Then for any increasing sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of finite subsets of \mathbb{Z}^d with $\bigcup_{n \in \mathbb{N}} \Lambda_n = \mathbb{Z}^d$ and $|\partial\Lambda_n|/|\Lambda_n| \rightarrow 0$, $n \rightarrow \infty$, we have*

$$\frac{I^r(\Lambda_n) - \mathbb{E}I^r(\Lambda_n)}{\sqrt{\text{Var}I^r(\Lambda_n)}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty. \quad (28)$$

In order to prove this result, we use the main theorem in Bolthausen (1982). The conditions of his theorem involve three mixing conditions which are trivially satisfied when

g has bounded support, and which we do not repeat here. Under these three mixing conditions, Bolthausen (1982) shows that if in addition

$$\sum_{z \in \mathbb{Z}^d} \mathbb{Cov}(I^r(0), I^r(z)) > 0, \quad (29)$$

then it is the case that

$$\frac{I^r(\Lambda_n) - \mathbb{E}I^r(\Lambda_n)}{\sqrt{|\Lambda_n| \sum_{z \in \mathbb{Z}^d} \mathbb{Cov}(I^r(0), I^r(z))}} \rightsquigarrow N(0, 1), \quad n \rightarrow \infty. \quad (30)$$

Because of the following elementary lemma, which we give without proof, (29) and (30) imply our Theorem 4.1.

Lemma 4.2 *Let $(Y_z)_{z \in \mathbb{Z}^d}$ be a stationary random field with $\mathbb{E}Y_0^2 < \infty$. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of finite non-empty subsets of \mathbb{Z}^d with $|\partial\Lambda_n|/|\Lambda_n| \rightarrow 0$, $n \rightarrow \infty$. If*

$$\sum_{z \in \mathbb{Z}^d} |\mathbb{Cov}(Y_0, Y_z)| < \infty, \quad (31)$$

then

$$\frac{1}{|\Lambda_n|} \text{Var} \sum_{z \in \Lambda_n} Y_z \rightarrow \sum_{z \in \mathbb{Z}^d} \mathbb{Cov}(Y_0, Y_z), \quad n \rightarrow \infty.$$

Note that (31) is satisfied because g has bounded support. It remains to prove (29). We give the proof in the two-dimensional case, but the method clearly generalizes to other dimensions.

With a slight abuse of notation, for a Borel subset B of \mathbb{R}^2 let $I^r(B)$ henceforth be defined as $1/r$ times the number of vertices of (X, λ, g) in B that are contained in a component of size r . According to Lemma 4.2, it suffices to show that there exists $M \in \mathbb{N}$ and $\gamma > 0$ such that for all n ,

$$\text{Var} I^r((0, nM]^2) \geq \gamma n^2. \quad (32)$$

We estimate the variance in (32) with the following general abstract trick, which we learned from J. v.d. Berg (personal communication).

Lemma 4.3 *Let Y be a random variable with finite second moment, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $n \in \mathbb{N}$ and let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n$ be sub- σ -algebras of \mathcal{A} with $\mathbb{E}(Y|\mathcal{F}_0) = \mathbb{E}Y$ and $\mathbb{E}(Y|\mathcal{F}_n) = Y$ a.s. Then we have*

$$\text{Var}Y = \sum_{i=1}^n \mathbb{E} [\mathbb{E}(Y|\mathcal{F}_i) - \mathbb{E}(Y|\mathcal{F}_{i-1})]^2 \quad \text{a.s.}$$

Proof: For $1 \leq i \leq n$, denote $\Delta_i = \mathbb{E}(Y|\mathcal{F}_i) - \mathbb{E}(Y|\mathcal{F}_{i-1})$. We write the variance of Y with a telescoping sum as $\text{Var}Y = \mathbb{E}(\sum_{i=1}^n \Delta_i)^2$. For $1 \leq i < j \leq n$ we have $\mathbb{E}\Delta_i\Delta_j = \mathbb{E}\mathbb{E}(\Delta_i Y|\mathcal{F}_j) - \mathbb{E}\mathbb{E}(\Delta_i Y|\mathcal{F}_{j-1}) = 0$. Hence $\text{Var}Y = \sum_{i=1}^n \mathbb{E}\Delta_i^2$, as required. \square

Let R be such that $g(x) = 0$, $x \geq R$. Define $\mu = \mathbb{E}I^r((0, 1]^2) > 0$. Choose an integer $M > 3\lambda R/\mu$. We shall show that (32) holds for this M , and this is sufficient to prove Theorem 4.1.

Partition the first quadrant of \mathbb{R}^2 into cubes of side length M , and denote these cubes by B_k , $k \in \mathbb{N}$, where the indices run as indicated in Figure 1. For $n \in \mathbb{N}$ let K_n be the set of indices $k \in \{1, \dots, (n-1)^2\}$ that are shaded in Figure 1.

16	15	14	13					
9	8	7	12					
4	3	6	11					
1	2	5	10	17				

Figure 1: The enumeration of cubes in the first quadrant.

For $k \in \bigcup_n K_n$ we define the following sets:

$$\begin{aligned}\lfloor \square_k &= (rR, rR) + B_k; \\ \square_k &= B_k + (-rR, rR]^2; \\ \triangleleft_k &= \square_k \cap \bigcup_{i=1}^{k-1} B_i; \\ \nabla_k &= \square_k \setminus \bigcup_{i=1}^{k-1} B_i;\end{aligned}$$

see Figure 2 and Figure 3.

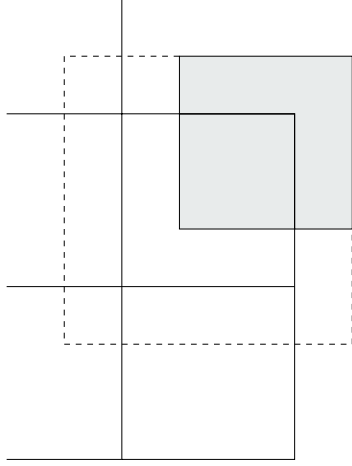


Figure 2

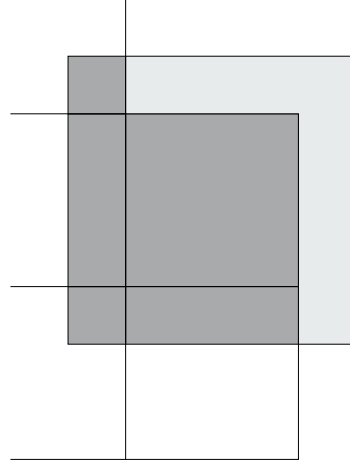


Figure 3

The shaded region on the left is $\lfloor \square_k$. The dark shaded region on the right is \triangleleft_k and the light shaded region on the right is ∇_k .

For $k \in \mathbb{N}$, let \mathcal{F}_k be the σ -algebra generated by the points of X in $\bigcup_{i=1}^k B_i$. We shall first show that for $n \in \mathbb{N}$ and $k \in K_n$ the difference $\mathbb{E}(I^r((0, nM]^2) | \mathcal{F}_{k-1}) - \mathbb{E}(I^r((0, nM]^2) | \mathcal{F}_k)$ is bounded below by a positive uniform constant, with positive probability which is also uniform in k and n .

On the one hand, we have

$$\begin{aligned}\mathbb{E}(I^r((0, nM]^2) | \mathcal{F}_{k-1}) &\geq \mathbb{E}(I^r(\lfloor \square_k) | \mathcal{F}_{k-1}) + \mathbb{E}(I^r((0, nM]^2 \setminus \square_k) | \mathcal{F}_{k-1}) \\ &= \mu M^2 + \mathbb{E}(I^r((0, nM]^2 \setminus \square_k) | \mathcal{F}_k),\end{aligned}\tag{33}$$

since $I^r(\square_k)$ is independent of \mathcal{F}_{k-1} and since the σ -algebra generated by $I^r((0, nM]^2 \setminus \square_k)$ and the points of X in $\bigcup_{i=1}^{k-1} B_i$, is independent of the points of X in B_k .

On the other hand, we also have

$$\begin{aligned} \mathbb{E}(I^r((0, nM]^2)|\mathcal{F}_k) &\leq \\ &\leq (1/r)\mathbb{E}(X(\triangleleft_k)|\mathcal{F}_k) + (1/r)\mathbb{E}(X(\nabla_k)|\mathcal{F}_k) + \mathbb{E}(I^r((0, nM]^2 \setminus \square_k)|\mathcal{F}_k) \\ &= 0 + 2\lambda R(M + rR) + \mathbb{E}(I^r((0, nM]^2 \setminus \square_k)|\mathcal{F}_k), \end{aligned} \tag{34}$$

with probability at least $e^{-\lambda(M+2rR)^2}$, since $X(\triangleleft_k)$ is \mathcal{F}_k -measurable and $X(\nabla_k)$ is independent of \mathcal{F}_k .

Combining (33) and (34) yields for $n \in \mathbb{N}$ and $k \in K_n$,

$$\mathbb{P}(\mathbb{E}(I^r((0, nM]^2)|\mathcal{F}_{k-1}) - \mathbb{E}(I^r((0, nM]^2)|\mathcal{F}_k) \geq \mu M^2 - 2\lambda R(M + rR)) \geq e^{-\lambda(M+2rR)^2}.$$

Now observe that the box $(0, nM]^2$ contains at least αn^2 boxes indexed by an element of K_n , for some $\alpha > 0$. Hence, since $\mu M^2 - 2\lambda R(M + rR) > 0$, we have by Lemma 4.3

$$\begin{aligned} \text{Var} I^r((0, nM]^2) &\geq \sum_{k \in K_n} \mathbb{E}[\mathbb{E}(I^r((0, nM]^2)|\mathcal{F}_k) - \mathbb{E}(I^r((0, nM]^2)|\mathcal{F}_{k-1})]^2 \\ &\geq \alpha n^2 (\mu M^2 - 2\lambda R(M + rR))^2 e^{-\lambda(M+2rR)^2}, \end{aligned}$$

proving the result.

References

- Bolthausen, E. (1982), *On the central limit theorem for stationary mixing random fields*, The Annals of Probability **10**, 1047–1050.
- Roy, R. and Sarkar, A. (2003) *High density asymptotics of the Poisson random connection model*, Physica A **318**, 230–242.