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BOUNDING BASIC CHARACTERISTICS OF SPATIAL EPIDEMICS WITH A NEW PERCOLATION MODEL

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We introduce a new percolation model to describe and analyze the spread of an epidemic on a general directed and locally finite graph. We assign a two-dimensional random weight vector to each vertex of the graph in such a way that the weights of different vertices are i.i.d., but the two entries of the vector assigned to a vertex need not be independent. The probability for an edge to be open depends on the weights of its end vertices, but conditionally on the weights, the states of the edges are independent of each other. In an epidemiological setting, the vertices of a graph represent the individuals in a (social) network and the edges represent the connections in the network. The weights assigned to an individual denote its (random) infectivity and susceptibility, respectively. We show that one can bound the percolation probability and the expected size of the cluster of vertices that can be reached by an open path starting at a given vertex from above and below by the corresponding quantities for respectively independent bond and site percolation with certain densities; this generalizes a result of Kuulasmaa [18]. Many models in the literature are special cases of our general model.

1. Introduction, background and main results. We consider an extension of the standard SIR (Susceptible $\to$ Infectious $\to$ Removed) epidemic [1, 2, 12] on a directed graph $G = (V, E)$. Here V is the (countable) vertex set of the graph, and E consists of directed edges between vertices in V. An edge from u to v is denoted by uv, and we say that v is a (directed) neighbour of u. Unless specified otherwise, we assume that the graph $G = (V, E)$ is simple, that is, for any $u, v \in V$ there is at most one edge from u to v. Furthermore, we assume that the graph is locally finite, in the sense that both the in-degree and out-degree of every vertex are finite.

In a standard SIR epidemic, a vertex is identified with an individual which makes (asymmetric) contacts with each of its neighbours at rate τ. If an infectious individual u contacts a susceptible individual v, then v becomes infectious itself. If an individual becomes infectious it will stay infectious for

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a random time; the infectious periods of different individuals are i.i.d. After the infectious period an individual is removed, which can either mean that it is recovered and immune or that the individual has died. A removed individual never becomes susceptible or infectious again. Usually, one assumes that there is one initially infectious individual v_0, and that all other individuals in the network are initially susceptible. Furthermore, one assumes that demography plays no role, in the sense that we ignore births, deaths not caused by the infectious disease and migration. This is a reasonable assumption if we consider emerging infectious diseases for which the time-scale of the spread is much smaller than the time-scale of demography.

In the model just described, one implicitly assumes that all individuals in the network are the same (at least with respect to the epidemic) apart from their position in the network. In particular all individuals will have the same total infectivity and susceptibility. In real-life however, infectivity and susceptibility show individual variation, notably because of immunological polymorphism, or due to polymorphic reactions to vaccination [5, 6].

In this paper, we model heterogeneity of the population by assigning a random infectivity W_v and susceptibility $\bar{W}_v$ to each vertex v in V, where the vectors $(W_v, \bar{W}_v)$ are assumed to be i.i.d. and distributed as $(W, \bar{W})$. We do not assume that W_v and $\bar{W}_v$ are independent. Conditionally on the weights, if u becomes infected and $uv \in E$, then v becomes infected (if it wasn't already) with probability $\kappa(W_u, \bar{W}_v)$, where κ is some connection function, specified in the model. In percolation terms, this means that the directed edge uv is open with (conditional) probability $\kappa(W_u, \bar{W}_v)$; the states of edges sharing an endpoint are not independent (given the weights).

In this paper we are mainly interested in (i) the probability of a large outbreak, (ii) the probability that a disease spreads from one given individual to another one and (iii) the expected final size od an epidemic. Percolation models have served before as useful tools to analyze these quantities; see for instance [11, 13, 16, 18, 19, 20, 21, 23] for related material. We denote by $\mathbb{P}$ the probability measure governing the full process of assigning weights, and making the edges open or closed. (We do not really need to formally define the full sample space of the process.) One necessary property is that $\mathbb{P}(\kappa(W_u, \bar{W}_v) \in [0, 1]) = 1$, since $\kappa(\cdot, \cdot)$ represents a probability. Sometimes it is useful to discuss the induced measure of $\mathbb{P}$ on the space $\Omega := \{\text{open}, \text{closed}\}^E$, that is the induced measure on configurations of open and closed edges.

A general class of connection functions is formed by the so called *factorisable* connection functions. We say that the connection function $\kappa(x, y)$ is factorisable if there exists functions $\kappa_1(x)$ and $\kappa_2(y)$ such that $\kappa(x, y) =$

$\kappa_1(x)\kappa_2(y)$. We claim that for factorisable κ we can without loss of generality assume that $\kappa(x,y) = xy$ and that $(W, \bar{W})$ takes values in $[0,1]^2$. To see this we define

$$\begin{aligned} a &:= \inf\{x \in [0,\infty]; \mathbb{P}(\kappa_1(W) > x) = 0\}, \\ b &:= \inf\{x \in [0,\infty]; \mathbb{P}(\kappa_2(\bar{W}) > x) = 0\}, \end{aligned}$$

and we consider four possibilities for a and b.

(1) If either $a = 0$ or $b = 0$, then $\mathbb{P}(\kappa(W_u, \bar{W}_v) = 0) = 1$. Replacing $(W, \bar{W})$ by $(0,0)$ and $\kappa(x,y)$ by xy does not change the induced measure on Ω, so the claim follows for this case.

(2) If $a = \infty$, then b cannot be positive, since if $a = \infty$ and $b > 0$ then by the definition of a and b, $\mathbb{P}(\kappa_1(W_u) > 2/b, \kappa_2(\bar{W}_v) > b/2) > 0$, which contradicts $\mathbb{P}(\kappa(W_u, \bar{W}_v) > 1) = 0$. Similarly, if $b = \infty$, then a cannot be positive.

(3) If $0 < a < \infty$, then b cannot exceed $1/a$, since if $0 < 1/a < b < \infty$, then there exists $\epsilon > 0$, such that $(a-\epsilon)(b-\epsilon) > 1$. However, by the definition of a and b, $\mathbb{P}(\kappa_1(W_u) > a - \epsilon, \kappa_2(\bar{W}_v) > b - \epsilon) > 0$, which contradicts $\mathbb{P}(\kappa(W_u, \bar{W}_v) > 1) = 0$.

(4) If $0 < a < \infty$ and $0 < b \leq 1/a$, then replacing $\kappa(x,y)$ by xy and $(W, \bar{W})$ by $(a^{-1}\kappa_1(W), a\kappa_2(\bar{W}))$ does not change the induced measure on Ω. By the definition of a and b, $\mathbb{P}((a^{-1}\kappa_1(W), a\kappa_2(\bar{W})) \in [0,1]^2) = 1$, so the claim follows.

A factorisable connection function is appropriate in situations in which there is at most one contact from an infectious individual to a given neighbour during its infectious period, or when only at the first contact of an infective individual with a given neighbour the infection may be transmitted. This last assumption is proposed in some models for the spread of HIV [17, 24], where the number of sexual contacts per couple can be ignored and only the number of partners is of importance. In those models the probability of an infectious contact from u to v is given by $\kappa(W_u, \bar{W}_v) = W_u\bar{W}_v$, with $W_u = 1 - \exp[-\int_0^{\Lambda_u} \tau_u(x)dx]$, where Λ_u is the length of the infectious period of individual u and $\tau_u(x)$ is the (possibly inhomogeneous) rate at which individual u makes infectious contacts at time x after its infection, and where $\bar{W}_v$ is the probability that individual v (if still susceptible) becomes infected at an infectious contact. In large randomly mixing populations it is usually assumed that the contact rate of a pair of individuals scales with n^{-1}, where n is the number of individuals in the population. With this assumption, multiple contacts of a pair of individuals are rare, and the difference between $\kappa(x,y) = 1 - e^{-xy/n}$ and $\kappa(x,y) = xy/n$, is of order n^{-2}.

There are, however, natural and important non-factorisable choices for κ. We will discuss two examples, namely

$$\kappa_a(x,y) = \frac{xy}{\beta + xy} \tag{1}$$

and

$$\kappa_b(x,y) = 1 - e^{-\alpha xy}, \tag{2}$$

where α and β are positive constants. The choice in (1) can be found in [8], and (2) is discussed in [22], both in the context of complete graphs. In neither of these two papers epidemiological interpretations of the connection functions are given. To shed some light on a possible epidemiological interpretation of κ_a and κ_b, we write

$$\kappa_a(x,y) = \frac{xy}{\beta + xy} = 1 - \sum_{k=0}^{\infty} \frac{\beta}{x+\beta}\left(\frac{x}{x+\beta}\right)^k (1-y)^k$$

and

$$\kappa_b(x,y) = 1 - e^{-\alpha xy} = 1 - \sum_{k=0}^{\infty} \frac{(\alpha x)^k}{k!} e^{-\alpha x}(1-y)^k.$$

(Sometimes we abuse notation, and write $\kappa_a(z) = z/(\beta + z)$ and $\kappa_b(z) = 1 - e^{-\alpha z}$.) When $0 < y < 1$, we see that we can interpret these connection functions as follows: for κ_a, an infectious individual has a geometric-$\beta/(x+\beta)$ number of contacts with a neighbour, and each time, the probability that the infection is accepted is equal to y. For κ_b, a similar interpretation is possible, replacing the geometric number of attempts by a Poisson-αx number. In both cases, the number of attempts stochastically increases when x grows, in accordance to our interpretation of W as infectivity.

It should be noted that (1) arises when the infectious period of every individual is exponentially-β distributed and the per neighbour infection rate of individual u is W_u, while the probability that an infectious contact with susceptible individual v leads to an infection is given by $\bar{W}_v$. If the infectious period of individual u is Λ_u, and during its infectious period it makes infectious contacts with every neighbour at rate $\alpha\tau_u$, then the number of attempts will have a Poisson-$\alpha\tau_u\Lambda_u$ distribution; hence both κ_a and κ_b arise naturally. Note that replacing the combination κ_a and $(W, \bar{W})$ by κ_b and $(\Lambda W/\alpha, \bar{W})$, where Λ is exponentially distributed with parameter β and independent of W, does not change the induced measure on Ω.

We define the usual independent bond percolation measure $\mathbb{P}_p^{bond}$ as the product measure on Ω in which edges are independently open with probability p [14]. If $\kappa(x,y) = xy$, $\mathbb{P}(W = p_1, \bar{W} = p_2) = 1$ and $p_1 p_2 = p$, then the induced measure of $\mathbb{P}$ on Ω is just $\mathbb{P}_p^{bond}$. If $\kappa(x, y) = xy$ and $\mathbb{P}(W = \bar{W} = 1) = 1 - \mathbb{P}(W = \bar{W} = 0) = p$, we denote the corresponding measure by $\mathbb{P}_p^{site}$. Indeed, this measure corresponds to the edge representation of independent site percolation with parameter p [14], in which an edge is open if and only if both its starting and ending vertex are open. Note that although $\mathbb{P}_p^{bond}$ is defined on Ω, $\mathbb{P}_p^{site}$ is defined on the full space. This is perhaps slightly confusing, but it works best this way.

In order to state our results, we need a few definitions. An ordered set of edges in E, $\xi = (v_0v_1, v_1v_2, \ldots, v_{n-1}v_n)$ is a *self-avoiding (directed) path of length n* from v_0 to v_n, if $v_i \neq v_j$ for all $0 \le i < j \le n$. It is straightforward to extend this definition to self-avoiding paths of infinite length. All paths in this paper are implicitly assumed to be self-avoiding. We say that a path is open if all edges in the path are open and we use the notation $v_i \rightsquigarrow v_j$ if there is at least one open path from v_i to v_j. If the final vertex of a path ξ_1 is the first vertex of a path ξ_2, we write (ξ_1, ξ_2) for the *conjunction* of ξ_1 and ξ_2, that is, if $\xi_1 = (v_0v_1, v_1v_2, \ldots, v_{n-1}v_n)$ and $\xi_2 = (v_nv_{n+1}, v_{n+1}v_{n+2}, \ldots, v_{n+m-1}v_{n+m})$, then

$$(\xi_1, \xi_2) = (v_0v_1, v_1v_2, \ldots, v_{n-1}v_n, v_nv_{n+1}, \ldots, v_{n+m-1}v_{n+m}).$$

Note that the conjunction of two self-avoiding paths is not necessarily self avoiding.

For a finite or infinite path $\xi = (v_0v_1, v_1v_2, \ldots, v_{n-1}v_n, v_nv_{n+1}\cdots)$, we say that $\xi^s(v) := (v_0v_1, v_1v_2, \ldots, v_{n-1}v_n)$ is the *truncation of ξ at v_n* and $\xi^t(v_n) := (v_nv_{n+1}, \ldots)$ is the *tail of ξ starting at v_n*. Both the truncation at v_n and the tail starting at v_n may be the empty path (that is, a path of length 0). We are now ready to specify the collections of paths we consider in this paper.

Definition 1.1 *We say that a collection of paths Ξ is* weakly hoppable*, if for any v and any two paths $\xi, \phi \in \Xi$ going through v and for which the conjunction $(\xi^s(v), \phi^t(v))$ is self-avoiding, it is the case that $(\xi^s(v), \phi^t(v)) \in \Xi$.*

In words, we need to be able to "hop" from one path to another if they cross.

Furthermore, let $E^{(n)}$ be the collection of the first n edges in E, according to some given enumeration of the edges for which $\cup_{n\in\mathbb{N}} E^{(n)} = E$. We "approximate" Ξ by sets Ξ_n defined as follows: Ξ_n is the collection of paths consisting of all finite paths in Ξ that are completely contained in $E^{(n)}$, together with all infinite paths which start in $E^{(n)}$, truncated at the first moment they leave $E^{(n)}$.

Finally, for a collection of paths Ξ we denote by $\mathcal{C}^{\Xi}$ the event that at least one path in Φ is open.

Definition 1.2 *We say that a collection of paths Ξ is* hoppable *if it is weakly hoppable and if in addition, for some enumeration of the edges,*

$$\mathcal{C}^{\Xi} = \lim_{n\to\infty} \mathcal{C}^{\Xi_n}.$$

Most natural and useful collections of paths are hoppable. For instance, the collection of infinite paths starting at a given vertex (or, more generally, in a finite set) is hoppable, as is the collection of all paths from a given vertex u to a given vertex v (or, more generally, from a finite set to another finite set).

The main results of this paper are the following, generalizing results in [18] (see also [19] for related results for the case in which W and $\bar{W}$ are independent).

Theorem 1.3 *Let $\kappa(x,y) = xy$. **(a)** For all $p \leq \mathbb{E}(W\bar{W})$ and for any hoppable collection Ξ of paths in E, we have*

$$\mathbb{P}(\mathcal{C}^{\Xi}) \geq \mathbb{P}_p^{site}(\mathcal{C}^{\Xi}).$$

***(b)** For all $p \geq \max[\mathbb{E}(W\bar{W}), \mathbb{E}(W)\mathbb{E}(\bar{W})]$ and for any hoppable collection Ξ of paths in E, we have*

$$\mathbb{P}(\mathcal{C}^{\Xi}) \leq \mathbb{P}_p^{bond}(\mathcal{C}^{\Xi}).$$

Theorem 1.4 *Suppose that $\kappa(x,y) = 1 - e^{-\alpha xy}$ or $\kappa(x,y) = \frac{xy}{\beta+xy}$. Then, for every*

$$p \geq \kappa(\max[\mathbb{E}(W\bar{W}), \mathbb{E}(W)\mathbb{E}(\bar{W})])$$

and any hoppable collection of paths Ξ, we have

$$\mathbb{P}(\mathcal{C}^{\Xi}) \leq \mathbb{P}_p^{bond}(\mathcal{C}^{\Xi}).$$

Theorems 1.3 and 1.4 are corollaries of the forthcoming Theorems 3.1, 4.1 and Corollary 4.2.

We will first illustrate these results by a numerical example, and then by discussing the situation on a tree. In this latter example, we also shed some light on the reason why $E(W\bar{W})$ and $E(W)E(\bar{W})$ play an important role in our analysis.

Example 1.5 Suppose that $W = \bar{W}$ is uniformly distributed on $(a,1)$ for some $a \in (0,1)$, $G = \mathbb{L}^2$ is the square lattice with nearest neighbour (directed) edges and $\kappa(x,y) = xy$. If $a \leq \sqrt{3/4} - 1/2 \approx 0.37$, then

$\mathbb{E}(W\bar{W}) = \mathbb{E}(W^2) \leq 1/2$. Because the critical value for independent bond percolation is 1/2 [14], the probability that the cluster of individuals that can be reached by an open path from the origin is infinite is 0 for this model. Let $p_c^{site}(\mathbb{L}^2) \approx 0.59$ be the critical value for site percolation on $\mathbb{L}^2$. If $a > \frac{\sqrt{12p_c^{site}(\mathbb{L}^2)-3}-1}{2} \approx 0.51$ then the probability that the cluster of individuals that can be reached by an open path from the origin is infinite is positive for this model.

Example 1.6 Let $G = (V, E)$ be the rooted tree in which all vertices have out-degree d, and in-degree 1, apart from the root v_0 which has in-degree 0. We say that the root is the generation-0 vertex; if there is a path of length n from the root to vertex v, then v is a said to be a generation-n vertex.

Let $\kappa(x, y) = xy$, and let Ξ_k be the set of all paths of length k starting at the root. Furthermore, let Z_k be the number of open paths in Ξ_k, i.e., Z_k is the number of generation-k vertices that can be reached by an open path from the origin. For $k \geq 1$, and generation-1 vertex v_1, we define $Y_k(v_1)$ as the number of generation-k vertices v_k, for which there is an open path from v_1 to v_k, conditioned on $W_{v_0} = 1$ and $\bar{W}_u = 1$ for all generation-k vertices u.

We claim that $Y_k(v_1)$ is distributed as the size of the $(k-1)$-th generation of a Galton-Watson process [15] starting with one individual, in which individuals have no offspring with probability $\mathbb{E}(1 - \bar{W}) + \mathbb{E}(\bar{W}(1 - W)^n)$ and offspring of size j with probability $\mathbb{E}(\bar{W}\binom{n}{j}W^j(1 - W)^{n-j})$, for $0 < j \leq n$. Indeed, we can adopt the point of view that for a vertex to have any offspring, it first has to accept the disease from its predecessor in the tree, and if they do so, in addition need to send the disease to the next generation. This leads to an offspring mean of $d\mathbb{E}(\bar{W}W)$.

It now seems quite natural, given the computation above, to consider a class of probability measures for which $\mathbb{E}(W\bar{W})$ is constant (compare [5, 6]). However, in order to proof the inequality $\mathbb{P}(\mathcal{C}^{\Xi}) \leq \mathbb{P}_p^{bond}(\mathcal{C}^{\Xi})$ for all hoppable collections of paths in E, we need the additional assumptions that $p \geq \mathbb{E}(W)\mathbb{E}(\bar{W})$. This can be seen by assuming that Ξ consists of a single edge. The marginal probability that this edge is open, is given by $\mathbb{E}(W)\mathbb{E}(\bar{W})$. This explains, to some extent, the importance of the quantities $\mathbb{E}(W\bar{W})$ and $\mathbb{E}(W)\mathbb{E}(\bar{W})$.

2. Discussion.

Before we start proving the results, we collect in this section a number of remarks.

- The (directed) edge density in our percolation model is $\mathbb{E}(W)\mathbb{E}(\bar{W})$. It is not hard to check that under $\mathbb{P}_p^{site}$, the directed edge density is

p^2, and under $\mathbb{P}_p^{bond}$ it is p. In Theorem 1.3(a) therefore, we compare a model with edge density $\mathbb{E}(W)\mathbb{E}(\bar{W})$ with a model with edge density at most $(\mathbb{E}(W\bar{W}))^2$. Note that the latter density is at most the former density. On the other hand, in Theorem 1.3(b), the edge density at the right is at least as large as the edge density at the left.

- In fact, the class of possible collections Ξ for which the result is true, is larger than the class of hoppable collections. We have chosen for this formulation because it contains most collections of interest, and also because of its elegance. One class of paths that can be seen to satisfy the results is the class of infinite *backwards* paths ending at a given vertex v. Indeed, by simply interchanging the role of W and $\bar{W}$ (see also [19] for this trick), it is easy to see that our results are valid for this class as well.
- Suppose that $\kappa(x,y) = xy$. By choosing Ξ as the collection of paths from u to v, we obtain that among all measure with $\mathbb{E}(W\bar{W}) \geq \mathbb{E}(W)\mathbb{E}(\bar{W})$, it is the case that $\mathbb{E}[\mathbb{1}(u \rightsquigarrow v)] = \mathbb{P}(u \rightsquigarrow v)$ is at most $\mathbb{P}^{bond}_{\mathbb{E}(W\bar{W})}(u \rightsquigarrow v)$ and at least $\mathbb{P}^{site}_{\mathbb{E}(W\bar{W})}(u \rightsquigarrow v)$. Let C_u be the set of vertices that can be reached by an open path from vertex u, and let S_v be the susceptibility set of v [3], that is, the set of vertices from which there is an open path to v. The observations above give

$$\mathbb{E}(|C_u|) = \sum_{v\in V} \mathbb{P}(u \rightsquigarrow v) \leq \sum_{v\in V} \mathbb{P}^{bond}_{\mathbb{E}(W\bar{W})}(u \rightsquigarrow v) = \mathbb{E}^{bond}_{\mathbb{E}(W\bar{W})}(|C_u|).$$

and

$$\mathbb{E}(|S_v|) = \sum_{u\in V} \mathbb{P}(u \rightsquigarrow v) \leq \sum_{u\in V} \mathbb{P}^{bond}_{\mathbb{E}(W\bar{W})}(u \rightsquigarrow v) = \mathbb{E}^{bond}_{\mathbb{E}(W\bar{W})}(|S_v|).$$

- It is not possible to use straightforward stochastic domination arguments to prove the theorems in their full generality. This can be seen by considering uncorrelated W and $\bar{W}$. In that case the marginal probability that any edge $uv \in E$ is open is the same for all measures for which $\mathbb{E}(W\bar{W})$ is constant. However, the edge density in $\mathbb{P}^{bond}_{\mathbb{E}(W\bar{W})}$ is also $\mathbb{E}(W\bar{W})$ and hence stochastic domination cannot be used to prove Theorem 1.3(b). To illustrate this we use an example from a model that is among the models already studied in [18]. Consider a measure $\mathbb{P}^{(a)}$ under which $\bar{W} = 1$ $\mathbb{P}^{(a)}$-a.s. and $\mathbb{P}^{(a)}(W = p) = 1$, and a measure $\mathbb{P}^{(b)}$ under which $\bar{W} = 1$ $\mathbb{P}^{(b)}$-a.s. and $\mathbb{P}^{(b)}(W = 1) = p = 1 - \mathbb{P}^{(b)}(W = 0)$. Note that $\mathbb{P}^{(a)}$ induces $\mathbb{P}^{bond}_p$ on Ω. Consider a vertex v with two neighbours, and let Ξ consist of the two paths of length 1 from v to its neighbours. The probability of $\mathcal{C}^{\Xi}$ is indeed higher under $\mathbb{P}^{(a)}$ than under $\mathbb{P}^{(b)}$. However, the probability that *both* edges are open is higher under $\mathbb{P}^{(b)}$.

- If E is symmetric, i.e., $uv \in E \Leftrightarrow vu \in E$, and if $\mathbb{P}(W = \bar{W}) = 1$ and $\kappa(x,y) = \kappa(y,x)$, then the law of the cluster of vertices that can be reached by open paths from $v \in V$ on G is the same as the law of the open cluster containing v on the undirected counterpart of G (the graph obtained by replacing the two edges connecting the same vertices by 1 undirected edge) [11]. Hence many questions on undirected graphs can be addressed as questions on directed graphs.
- We can compare some of our results with bounds given in [4]: any undirected 1-dependent edge percolation model on the 2-dimensional square lattice, with marginal probability for an edge to be open at least 0.8639 is supercritical. Since this bound holds for all 1-dependent measures, it is to be expected that our bounds improve on this in our specific model. Since we can only compare undirected models, we can only compare in the symmetric case (which is perhaps not the most interesting one). Still, since $\mathbb{P}_p^{site}$ percolates above $p = 0.68$ [25] (this is a rigorous bound, the correct value of the critical probability is around 0.59), we see that our model percolates when $\mathbb{P}(W = \bar{W}) = 1$, κ is symmetric and $\mathbb{E}(W\bar{W}) = \mathbb{E}(W^2) \geq 0.68$. Hence we improve on the general bound when $\mathbb{E}(W^2) \geq 0.68$ and $(\mathbb{E}(W))^2 < 0.8639$.
- Our model is a generalization of many other percolation processes. Some examples are:

  1. If $\kappa(x,y) = x$, then the model corresponds to the locally dependent random graph model introduced by Kuulasmaa [18]. This model was introduced as a model for the spread of epidemics in which individuals have variable infectivity, but homogeneous susceptibility.
  2. For an undirected graph G, $\kappa(x,y) = xy$ and $b, s \in [0,1]$, we define $\mathbb{P}_{b,s}^{mixed}$ by

     $$\mathbb{P}_{b,s}^{mixed}(W = \bar{W} = 0) = 1 - \mathbb{P}_{b,s}^{mixed}(W = \bar{W} = \sqrt{b}) = 1 - s. \tag{3}$$

     This is the mixed percolation measure with parameters b and s of [9]. This model can be interpreted as a model in which vertices are open with probability s, and two open vertices are connected by an open edge with probability b.
  3. If G is the complete undirected graph of n vertices, and

     $$\kappa(x,y) := \kappa_n(x,y) = xy/(n + xy),$$

then the generalized random graphs from [8] are obtained. In this and the following examples, people are mainly interested in asymptotic properties of a series of graphs for $n \to \infty$. Therefore, the reference to the number of vertices in G is kept explicit through the notation $\kappa_n(x, y)$.

4. If G is a complete undirected graph of n vertices and $\kappa_n(x, y) = 1 - e^{-xy/n}$, then the Poissonian random graphs from [22] are obtained.
5. The generalized random graph and the Poissonian random graph are special cases of the inhomogeneous random graphs studied by Bollobás, Janson and Riordan [7], where G is a complete undirected graph and $\kappa_n(x, y) = \kappa(x, y)/n$, for general κ. Their methods heavily rely on branching process techniques. Because these techniques are not useful in general on graphs that are nor complete nor tree-like, we will use techniques in the spirit of the ones introduced by Kuulasmaa [18], which lead to less extensive results.
6. If G is the complete undirected graph of n vertices and the probability that uv is open, conditioned on the weights, is equal to

$$W_u W_v (\sum_{w \in V} W_w)^{-1},$$

then the model introduced by Chung and Lu [10] is obtained. This model does not fit our proposal, because the state (open or closed) of the edge uv depends on weights other than W_u and W_v. However, if the random variables W_w are i.i.d. and have finite mean $\mathbb{E}(W)$, then we may approximate $W_u W_v (\sum_{w\in V} W_w)^{-1}$ by $W_u W_v (n\mathbb{E}(W))^{-1}$, which is of the correct form.

- Our results show that the claim by Newman in [20, 21] that bond percolation corresponds to infectious diseases with any distribution of the infectious period is not correct (see [13, p.87], [16] or [23] for a discussion of this claim), but conservative, in the sense that using the bond percolation model may lead to too large predictions for the expected outbreak size and the probability of a large outbreak of an epidemic.
- Theorem 1.3 can be generalised to models where the vectors $(W_v, \bar{W}_v)$ are independent, but not necessarily identically distributed. The theorem then reads:

Theorem 2.1 *For any hoppable collection Ξ of paths in E, we have*
(a) *For all $\mathbf{p} = (p_v; v \in V)$, for which $p_v \leq \mathbb{E}(W_v \bar{W}_v)$ holds for all $v \in V$, we have*

$$\mathbb{P}(\mathcal{C}^\Xi) \geq \mathbb{P}_p^{site}(\mathcal{C}^\Xi),$$

where $\mathbb{P}_{\mathbf{p}}^{site}(W_v = \bar{W}_v = 1) = p_v = 1 - \mathbb{P}_{\mathbf{p}}^{site}(W_v = \bar{W}_v = 0)$.
(b) *For all $(\mathbf{p}, \bar{\mathbf{p}}) = ((p_v, \bar{p}_v); v \in V)$ for which $p_v \geq \mathbb{E}(W_v)$, $\bar{p}_v \geq \mathbb{E}(\bar{W}_v)$ and $p_v \bar{p}_v \geq \mathbb{E}(W_v \bar{W}_v)$ holds for all $v \in V$, we have*

$$\mathbb{P}(\mathcal{C}^\Xi) \leq \mathbb{P}_{\mathbf{b}}^{bond}(\mathcal{C}^\Xi),$$

where $\mathbf{b} = (b_{uv}; uv \in E)$, $b_{uv} \geq p_u \bar{p}_v$ and $\mathbb{P}_{\mathbf{b}}^{bond}$ is the product measure on Ω for which $\mathbb{P}_{\mathbf{b}}^{bond}(uv$ is open$) = b_{uv}$.

Note that under the $\mathbb{P}_{\mathbf{b}}^{bond}$-measure, edge uv is open with probability b_{uv}, independently of the other edges.

3. Proof of Theorem 1.3. Let E_v be the set of all edges starting at v, and E_v^* the set of all edges ending at v. For any pair of sets $A \subset E_u$ and $B \subset E_u^*$, we define the *zero function* $z_u(A, B; \mathbb{P})$ by

$$\begin{aligned}
z_u(A, B; \mathbb{P}) &:= \mathbb{E}(1 - [1 - (1 - W_u)^{|A|}][1 - (1 - \bar{W}_u)^{|B|}]), \text{ if } |A||B| > 0, \\
z_u(A, \emptyset; \mathbb{P}) &:= \mathbb{E}((1 - W_u)^{|A|}), \text{ if } |A| > 0, \\
z_u(\emptyset, B; \mathbb{P}) &:= \mathbb{E}((1 - \bar{W}_u)^{|B|}), \text{ if } |B| > 0, \\
z_u(\emptyset, \emptyset; \mathbb{P}) &:= 1.
\end{aligned}$$

When the graph is transitive we do not need the reference to the vertex u in the zero function. The name zero-function is taken from [18]; if $\kappa(x, y) = x$, which can be interpreted as a model with $\mathbb{P}(\bar{W} = 1) = 1$ and, of course, factorisable connection function. Hence, in that case, for $|A| > 0$, we have

$$z_v(A, B; \mathbb{P}) = \mathbb{E}((1 - W_v)^{|A|}) = \mathbb{P}(\text{none of the edges in } A \text{ is open}).$$

In the epidemiological setting, we can think of W_u and $\bar{W}_u$ as probabilities of “sending” and “accepting” the disease to (from) a neighbour, and the zero function $z_u(A, B; \mathbb{P})$ is the (conditional) probability that if all endpoints of edges in B become infected and “send the disease” to u, either u will not “accept the disease” via an edge in B, or u will not “send the disease” to any of the endpoints of edges in A.

We write $z_v(\mathbb{P}^{(a)}) \geq z_v(\mathbb{P}^{(b)})$ if $z_v(A, B; \mathbb{P}^{(a)}) \geq z_v(A, B; \mathbb{P}^{(b)})$ for all $A \subset E_u$ and $B \subset E_u^*$. The following result is interesting in its own right, and will be the main tool to prove Theorem 1.3.

Theorem 3.1 *If $\kappa(x,y) = xy$ and $z_v(\mathbb{P}^{(a)}) \le z_v(\mathbb{P}^{(b)})$ for all $v \in V$, then for any hoppable collection Ξ of paths,*

$$\mathbb{P}^{(b)}(\mathcal{C}^{\Xi}) \le \mathbb{P}^{(a)}(\mathcal{C}^{\Xi}).$$

Remark 3.2 Theorem 3.1 does not hold if we would allow for all collections of paths Ξ. Indeed, here is a counterexample. Let G be the subgraph of $\mathbb{L}^2$ consisting of the origin and its nearest neighbours (with nearest neighbour edges). The weights assigned to the neighbours of the origin are all equal to 1. We consider two measures $\mathbb{P}^{(a)}$ and $\mathbb{P}^{(b)}$ on the weights assigned to the origin:
$\mathbb{P}^{(a)}(W = \bar{W} = 0) = 3/5$, $\mathbb{P}^{(a)}(W = 1/2, \bar{W} = 1) = \mathbb{P}^{(a)}(W = 1, \bar{W} = 1/2) = 1/5$, and
$\mathbb{P}^{(b)}(W = 0, \bar{W} = 1/2) = \mathbb{P}^{(b)}(W = 0, \bar{W} = 1) = \mathbb{P}^{(b)}(W = 1/2, \bar{W} = 0) = \mathbb{P}^{(b)}(W = 1, \bar{W} = 0) = \mathbb{P}^{(b)}(W = 1, \bar{W} = 1) = 1/5$.

A small computation shows that for $|A||B| \ge 1$,

$$z_0(A, B; \mathbb{P}^{(a)}) = 1 - (1/5)(2^{-|A|} + 2^{-|B|}) \ge 4/5 = z_0(A, B; \mathbb{P}^{(b)}),$$

and for $|A| > 0$,

$$z_0(A, \emptyset; \mathbb{P}^{(a)}) = 3/5 + (1/5)2^{-|A|} > 2/5 + (1/5)2^{-|A|} = z_0(A, \emptyset; \mathbb{P}^{(b)}),$$

while for $|B| > 0$,

$$z_0(\emptyset, B; \mathbb{P}^{(a)}) = 3/5 + (1/5)2^{-|B|} > 2/5 + (1/5)2^{-|B|} = z_0(\emptyset, B; \mathbb{P}^{(b)}).$$

Hence we have that $z_0(\mathbb{P}^{(a)}) \ge z_0(\mathbb{P}^{(b)})$.

Now let ξ be the path from $(0,-1)$ to $(0,1)$, let ϕ be the path from $(0,1)$ to $(0,-1)$ in G and let $\Xi = \{\xi, \phi\}$; this is not a hoppable collection. Note that $\mathbb{P}(\mathcal{C}^{\Xi}) = 1 - \mathbb{E}([1 - W\bar{W}]^2)$, and a quick computation yields

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi}) = 3/10 > 1/5 = \mathbb{P}^{(b)}(\mathcal{C}^{\Xi}).$$

Proof of Theorem 3.1: In the proof we do not assume that the vectors $(W_v, \bar{W}_v)$ for $v \in V$ are identically distributed. For the proof it will be helpful to introduce coloured edges, as follows. We colour all (directed) edges in the graph *yellow*. Furthermore, for each yellow edge we draw a *blue* edge in the opposite direction. Given the weights of the vertices, a yellow edge uv is open with probability W_u, and a blue edge yz is open with probability $\bar{W}_y$. The state of different edges are independent, given the weights. If, for some uv the yellow edge from u to v is open, and also the blue edge from v to u is open, we say that the edge from u to v is *green*. With this terminology, $\mathcal{C}^{\Xi}$ is the event that Ξ contains at least one green path. We denote by E' the collection of all yellow and blue edges, by $E^{(n)'}$ the collection of all yellow and blue edges corresponding with edges in $E^{(n)}$ and by E'_v the collection of yellow and blue edges starting at v. We denote by $E'_{v,y}$ and $E'_{v,b}$ the collection of yellow respectively blue edges starting at v.

The epidemiological interpretation of these colours is as follows. If the edge uv is yellow, then, if u becomes infected, it tries to infect v. Similarly, if the edge vu is blue, v will "accept" an infection coming from u. Hence for the infection to go from u to v, both edges must be open, that is, uv must be green. It is clear that this is a correct representation of the spread of the disease through the graph.

We can interpret the zero function z_v as acting on nonempty subsets A and B of $E'_{v,y}$ and $E'_{v,b}$ respectively:

$$\begin{aligned} z_v(A,B,\mathbb{P}) &= \mathbb{P}(\text{all edges in } A \text{ are closed} \cup \text{ all edges in } B \text{ are closed}),\\ z_v(A,\emptyset,\mathbb{P}) &= \mathbb{P}(\text{all edges in } A \text{ are closed}),\\ z_v(\emptyset,B,\mathbb{P}) &= \mathbb{P}(\text{all edges in } B \text{ are closed}). \end{aligned}$$

(i) The first step is to prove that for all $n \geq 1$, $\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}) \geq \mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n})$, if there is a $u \in V$ such that $z_u(\mathbb{P}^{(a)}) \leq z_u(\mathbb{P}^{(b)})$, and such that $z_v(\mathbb{P}^{(a)}) = z_v(\mathbb{P}^{(b)})$ for all $v \in V \setminus u$.

For $J \subset I \subset E'$, let $\mathcal{E}_{I,J}$ denote the event that all edges in J are open and all edges in $I \setminus J$ are closed.

Observe that for $i = a, b$,

$$\mathbb{P}^{(i)}(\mathcal{C}^{\Xi_n}) = \sum_{J \subset E^{(n)'} \setminus E'_u} \mathbb{P}^{(i)}(\mathcal{C}^{\Xi_n} | \mathcal{E}_{E^{(n)'} \setminus E'_u, J}) \mathbb{P}^{(i)}(\mathcal{E}_{E^{(n)'} \setminus E'_u, J}).$$

By the assumption $\mathbb{P}^{(a)}(\mathcal{E}_{E^{(n)'} \setminus E'_u, J}) = \mathbb{P}^{(b)}(\mathcal{E}_{E^{(n)'} \setminus E'_u, J})$, this implies that

$$\begin{aligned} \mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n}) &= \sum_{J \subset E^{(n)'} \setminus E'_u} \mathbb{P}^{(a)}(\mathcal{E}_{E^{(n)'} \setminus E'_u, J}) \\ &\quad \times (\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n} | \mathcal{E}_{E^{(n)'} \setminus E'_u, J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n} | \mathcal{E}_{E^{(n)'} \setminus E'_u, J})), \end{aligned}$$

Now we consider five cases for $J \subset E^{(n)'} \setminus E'_u$:

1. $\mathcal{E}_{E^{(n)'}, J} \in \mathcal{C}^{\Xi_n}$, in which case

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n} | \mathcal{E}_{E^{(n)'} \setminus E'_u, J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n} | \mathcal{E}_{E^{(n)'} \setminus E'_u, J}) = 0,$$

since both probabilities are equal to 1.

2. $\mathcal{E}_{E^{(n)'},J\cup E'_u} \notin \mathcal{C}^{\Xi_n}$, in which case

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) = 0$$

   since both probabilities are equal to 0.

3. There are maximal non-empty sets $A \subset E'_{u,y} \cap E^{(n)'}$ and $B \subset E'_{u,b} \cap E^{(n)'}$ such that

$$\mathcal{E}_{E^{(n)'},J\cup e} \notin \mathcal{C}^{\Xi_n},$$
$$\mathcal{E}_{E^{(n)'},J\cup e'} \notin \mathcal{C}^{\Xi_n}$$

   and

$$\mathcal{E}_{E^{(n)'},J\cup e\cup e'} \in \mathcal{C}^{\Xi_n}$$

   for all $e \in A$ and $e' \in B$. Since $z_u(A,B,\mathbb{P}^{(a)}) \leq z_u(A,B,\mathbb{P}^{(b)})$ for all non-empty A and B and because Ξ is hoppable, this implies

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) \geq 0.$$

4. There is a maximal non-empty set $A \subset E'_{u,y} \cap E^{(n)'}$ such that

$$\mathcal{E}_{E^{(n)'},J} \notin \mathcal{C}^{\Xi_n}$$

   and

$$\mathcal{E}_{E^{(n)'},J\cup e} \in \mathcal{C}^{\Xi_n}$$

   for all $e \in A$. Since $z_u(A,\emptyset,\mathbb{P}^{(a)}) \leq z_u(A,\emptyset,\mathbb{P}^{(b)})$ and because Ξ is hoppable, this implies

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) \geq 0.$$

5. There is a maximal non-empty set $B \subset E'_{u,b} \cap E^{(n)'}$ such that

$$\mathcal{E}_{E^{(n)'},J} \notin \mathcal{C}^{\Xi_n}$$

   and

$$\mathcal{E}_{E^{(n)'},J\cup e'} \in \mathcal{C}^{\Xi_n}$$

   for all $e' \in B$. By $z_u(\emptyset,B,\mathbb{P}^{(a)}) \leq z_u(\emptyset,B,\mathbb{P}^{(b)})$ and because Ξ is hoppable, this implies

$$\mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) - \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n}|\mathcal{E}_{E^{(n)'}\setminus E'_u,J}) \geq 0.$$

Note that the fourth case can only occur if u is in the set of first vertices of paths in Ξ and the fifth case can only occur if u is in the set of final vertices of paths in Ξ. We claim that these five cases cover all possibilities; indeed it is at precisely this point where we use the assumption that Ξ is hoppable. In case 3 for instance, let $\Xi_n^{(t)}(u)$ be the maximal set of paths starting in u and $\Xi_n^{(s)}(u)$ be the maximal set of paths ending in u, such that for any $\xi^{(t)} \in \Xi_n^{(t)}(u)$ and any $\xi^{(s)} \in \Xi_n^{(s)}(u)$, $(\xi^{(s)}, \xi^{(t)}) \in \Xi_n$, which are well defined because Ξ is hoppable. Now A is the set of yellow edges from u in $E^{(n)'}$ which are the first edge in a path $\Xi_n^{(t)}(u)$, of which the blue counterpart is open and that are green with possible exception of the edge in A. Similarly, B is the set of blue edges from u in $E^{(n)'}$ which correspond to the final edge in a path $\Xi_n^{(s)}(u)$, of which the yellow counterpart is open and that are green with possible exception of the edge in B. Note that the open paths obtained by the conjunction of paths from $\Xi_n^{(s)}(u)$ and $\Xi_n^{(t)}(u)$ are self avoiding, because otherwise $\mathcal{E}_{E',J} \in \mathcal{C}^{\Xi_n}$.

Similar conclusions hold for case 4 and case 5. From the counterexample in Remark 3.2, it can be seen that A and B as described here need not exist if Ξ is not hoppable.

(ii) In this second step, we relax the condition that the zero functions differ in one place only. Since the event $\mathcal{C}^{\Xi_n}$ depends on the weights of at most $2n$ vertices in V, it is straightforward to construct a sequence of probability measures, $(\mathbb{P}^{(i)}; 1 \leq i \leq 2n)$, such that $\mathbb{P}^{(1)}(\mathcal{C}^{\Xi_n}) = \mathbb{P}^{(a)}(\mathcal{C}^{\Xi_n})$, $\mathbb{P}^{(2n)}(\mathcal{C}^{\Xi_n}) = \mathbb{P}^{(b)}(\mathcal{C}^{\Xi_n})$ and such that two subsequent zero functions $z_v(\mathbb{P}^{(i)})$ and $z_v(\mathbb{P}^{(i+1)})$ for $1 \leq i \leq 2n-1$ differ at only one vertex $v_i \in V$. Repeatedly applying part (i) finishes this part of the proof.

(iii) To finish the proof, we simply note that from the definition of hoppable, we have for $i = a, b$,

$$\mathbb{P}^{(i)}(\mathcal{C}^{\Xi}) = \lim_{n\to\infty} \mathbb{P}^{(i)}(\mathcal{C}^{\Xi_n}).$$

and the result follows. □

Proof of Theorem 1.3: We divide the proof in three parts: first we prove part (a) of the theorem, then we proof part (b) for the cases where W and $\bar{W}$ are positively and negatively correlated separately.

(a) We prove that for $p \leq \mathbb{E}(W\bar{W})$, and all $A, B \in E_v$, we have that

$$z_v(A, B, \mathbb{P}) \leq z_v(A, B, \mathbb{P}_p^{site}).$$

Theorem 3.1 then gives the site percolation lower bound in Theorem 1.3. We need to distinguish between the cases where $|A| = 0$ or $|B| = 0$ and the case where $|A| \geq 1$ and $|B| \geq 1$. If $|B| > 0$, then

$$\begin{aligned} z_v(\emptyset, B, \mathbb{P}) &= \mathbb{E}((1-\bar{W})^{|B|}) \\ &\leq \mathbb{E}(1-\bar{W}) \\ &\leq 1-p \\ &= z_v(\emptyset, B, \mathbb{P}_p^{site}). \end{aligned}$$

In a similar fashion one shows that for $|A| > 0$,

$$z_v(A, \emptyset, \mathbb{P}) \leq z_v(A, \emptyset, \mathbb{P}_p^{site}).$$

If $|A||B| \geq 1$, then

$$\begin{aligned} z_v(A, B, \mathbb{P}) &:= \mathbb{E}\Big(1 - [1-(1-W)^{|A|}][1-(1-\bar{W})^{|B|}]\Big) \\ &= \mathbb{E}\Big(1 - W\bar{W} \sum_{k=0}^{|A|-1} (1-W)^k \sum_{l=0}^{|B|-1} (1-\bar{W})^l\Big) \\ &\leq \mathbb{E}(1-W\bar{W}) \\ &\leq 1-p \\ &= z_v(A, B, \mathbb{P}_p^{site}). \end{aligned}$$

(b1) Suppose that $p = \mathbb{E}(W\bar{W}) \geq \mathbb{E}(W)\mathbb{E}(\bar{W})$. Chose W^* and $\bar{W}^*$ such that $\mathbb{E}(W) \leq W^*$, $\mathbb{E}(\bar{W}) \leq \bar{W}^*$ and $p = W^*\bar{W}^* = \mathbb{E}(W\bar{W})$. Let $\hat{\mathbb{P}}_p^{bond}$ be a measure that satisfies $\hat{\mathbb{P}}_p^{bond}(W = W^*, \bar{W} = \bar{W}^*) = 1$.

We proceed by proving

$$z_v(A, B, \mathbb{P}) \geq z_v(A, B, \hat{\mathbb{P}}_p^{bond}).$$

From Theorem 3.1 and the observation that the induced measure of $\hat{\mathbb{P}}_p^{bond}$ on Ω is just $\mathbb{P}_p^{bond}$, Theorem 1.3 will then follow for $p = \mathbb{E}(W\bar{W})$. By straightforward stochastic domination arguments, this implies that the statement then also holds for $p \geq \mathbb{E}(W\bar{W})$.

If $|B| > 0$, then $z_v(\emptyset, B, \mathbb{P}) = \mathbb{E}((1-\bar{W})^{|B|})$. By Jensen's inequality this is larger than or equal to $(1-\mathbb{E}(\bar{W}))^{|B|}$, which is larger than or equal to $(1-\bar{W}^*)^{|B|} = z_v(\emptyset, B, \hat{\mathbb{P}}_p^{bond})$, because $\mathbb{E}(\bar{W}) \leq \bar{W}^*$. For $|A| > 0$, we can use similar arguments to show that $z_v(A, \emptyset, \mathbb{P}) \geq z_v(A, \emptyset, \hat{\mathbb{P}}_p^{bond})$.

If $|A||B| \geq 1$, then we need to show that

$$\mathbb{E}\Big((1-(1-W)^{|A|})(1-(1-\bar{W})^{|B|})\Big) \leq (1-(1-W^*)^{|A|})(1-(1-\bar{W}^*)^{|B|}).$$

When $|A| = |B| = 1$, this is immediate, but already for $|A| = 1$ and $|B| = 2$, this is not completely straightforward. A computation shows that in this case we need to prove that

$$\mathbb{E}(W\bar{W}^2) \geq W^*(\bar{W}^*)^2,$$

which follows from applying the Cauchy-Schwarz inequality to $\bar{W}\sqrt{W}$ and $\sqrt{W}$.

In the general case, it turns out that one can apply the following version of Hölder's inequality: for all $n > 1$ and non-negative random variables X and Y, it is the case that

$$\mathbb{E}(X^n) \geq \frac{[\mathbb{E}(XY)]^n}{[\mathbb{E}(Y^{n/(n-1)})]^{n-1}} = \mathbb{E}(Y^{n/(n-1)})\Big(\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^{n/(n-1)})}\Big)^n. \tag{4}$$

We now write

$$\begin{aligned} z_v(A,B,\mathbb{P}) &:= \mathbb{E}\Big(1-[1-(1-W)^{|A|}][1-(1-\bar{W})^{|B|}]\Big) \\ &= \mathbb{E}\Big((1-W)^{|A|}\Big) + \mathbb{E}\Big([1-(1-W)^{|A|}](1-\bar{W})^{|B|}\Big). \end{aligned}$$

By Hölder's inequality (4) with $X = (1-\bar{W})(1-(1-W)^{|A|})^{1/|B|}$, $Y = (1-(1-W)^{|A|})^{(|B|-1)/|B|}$ and $n = |B|$, this is bounded below by

$$\mathbb{E}\Big((1-W)^{|A|}\Big) + \mathbb{E}[1-(1-W)^{|A|}]\Big[1 - \frac{\mathbb{E}(\bar{W}[1-(1-W)^{|A|}])}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^{|B|}$$

and again by using Hölder's inequality (4) with $X = \bar{W}^{1/|A|}(1-W)$, $Y = \bar{W}^{(|A|-1)/|A|}$ and $n = |A|$ we see that this quantity is bounded below by

$$\mathbb{E}\Big((1-W)^{|A|}\Big) + \mathbb{E}[1-(1-W)^{|A|}]\Big[1 - \frac{\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^{|B|}.$$

This is first seen to be equal to

$$1 - \mathbb{E}[1-(1-W)^{|A|}]\Big(1 - \Big[1 - \frac{\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^{|B|}\Big) \tag{5}$$

and then to

$$1-\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big] \times \sum_{k=0}^{|B|-1}\Big[1-\frac{\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^k.$$

Then we use Jensen's inequality and $E(W) \leq W^*$ to bound this below by

$$1-\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big] \times \sum_{k=0}^{|B|-1}\Big[1-\frac{\mathbb{E}(\bar{W})\Big[1-\Big(1-\frac{\bar{W}^*W^*}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{1-(1-W^*)^{|A|}}\Big]^k.$$

This equals

$$1-(1-(1-W^*)^{|A|})\Big(1-\Big[1-\frac{\mathbb{E}(\bar{W})\Big[1-\Big(1-\frac{W^*\bar{W}^*}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{1-(1-W^*)^{|A|}}\Big]^{|B|}\Big)$$

which in turn equals

$$1-(1-(1-W^*)^{|A|})\Big(1-\Big[1-\frac{W^*\bar{W}^*\sum_{k=0}^{|A|-1}\Big(1-\frac{W^*\bar{W}^*}{\mathbb{E}(\bar{W})}\Big)^k}{1-(1-W^*)^{|A|}}\Big]^{|B|}\Big).$$

Finally, we use that $\mathbb{E}(W) \leq W^*$ and $\mathbb{E}(\bar{W}) \leq \bar{W}^*$ to bound this below by

$$1-(1-(1-W^*)^{|A|})\Big(1-\Big[1-\frac{W^*\bar{W}^*\sum_{k=0}^{|A|-1}(1-W^*)^k}{1-(1-W^*)^{|A|}}\Big]^{|B|}\Big)$$

which is just

$$1-(1-(1-W^*)^{|A|})(1-(1-\bar{W}^*)^{|B|})$$

which is at least

$$z_v(A,B,\hat{\mathbb{P}}_p^{bond}).$$

(b2) Suppose now that $\mathbb{E}(W)\mathbb{E}(\bar{W}) \geq \mathbb{E}(W\bar{W})$. We prove that for all measures $\mathbb{P}$ that satisfy $p = \mathbb{E}(W)\mathbb{E}(\bar{W}) \geq \mathbb{E}(W\bar{W})$, it holds that

$$z_v(A,B,\mathbb{P}) \geq z_v(A,B,\hat{\mathbb{P}}_p^{bond}),$$

where we now, with some abuse of notation, define the measure $\hat{\mathbb{P}}_p^{bond}$ by $\hat{\mathbb{P}}_p^{bond}(W = \mathbb{E}(W), \bar{W} = \mathbb{E}(\bar{W})) = 1$. Again, restricted to Ω, $\hat{\mathbb{P}}_p^{bond}$ corresponds to $\mathbb{P}_p^{bond}$.

The cases $|A| = 0$ and $|B| = 0$ are immediate consequences of the main result in [18]. The computations for the case where both $|A|$ and $|B|$ are strictly positive start similar to the computations in part (b1). Indeed, up to (5) we did not use the positive correlation in part (b1). So, we start with

$$z_v(A,B,\mathbb{P}) \geq 1-\mathbb{E}[1-(1-W)^{|A|}]\Big(1-\Big[1-\frac{\mathbb{E}(\bar{W})\Big[1-\Big(\frac{\mathbb{E}(\bar{W}(1-W))}{\mathbb{E}(\bar{W})}\Big)^{|A|}\Big]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^{|B|}\Big).$$

Since $\mathbb{E}(W\bar{W}) \leq \mathbb{E}(W)\mathbb{E}(\bar{W})$ this quantity is bounded from below by

$$1 - \mathbb{E}[1-(1-W)^{|A|}]\Big(1 - \Big[1 - \frac{\mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^{|B|}\Big),$$

which is equal to

$$1 - \mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}] \sum_{k=0}^{|B|-1} \Big[1 - \frac{\mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}]}{\mathbb{E}(1-(1-W)^{|A|})}\Big]^k.$$

By Jensen's inequality this is at least

$$1 - \mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}] \sum_{k=0}^{|B|-1} \Big[1 - \frac{\mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}]}{1-(1-\mathbb{E}(W))^{|A|}}\Big]^k.$$

This quantity is equal to

$$1 - \mathbb{E}(\bar{W})[1-(1-\mathbb{E}(W))^{|A|}] \sum_{k=0}^{|B|-1} [1-\mathbb{E}(\bar{W})]^k,$$

which in turn equals

$$z_v(A,B,\hat{\mathbb{P}}_p^{bond}) = 1 - [1-(1-\mathbb{E}(W))^{|A|}][1-(1-\mathbb{E}(\bar{W}))^{|B|}].$$

This completes the proof. □

4. Proof of Theorem 1.4. In order to deal with the connection functions mentioned in Theorem 1.4, we start by considering models for which there exist constants m and $\bar{m}$ such that $\mathbb{P}(0 \leq W \leq m, 0 \leq \bar{W} \leq \bar{m}) = 1$ and connection functions of the form

$$\kappa(x,y) = 1 - \sum_{k=0}^{\infty} q_k (1-(m\bar{m})^{-1}xy)^k,$$

for numbers $q_k \geq 0$, for which $\sum_{k=0}^{\infty} q_k = 1$ (that is, $\kappa(0,0) = 0$). First we provide a theorem in the spirit of Theorem 1.3, then we formulate a corollary which leads to Theorem 1.4.

In the first theorem we use the mixed percolation measure $\mathbb{P}_{b,s}^{mixed}$ as defined in (3).

Theorem 4.1 *Suppose that* $\mathbb{P}(0 \leq W \leq m, 0 \leq \bar{W} \leq \bar{m}) = 1$ *and let*

$$\kappa(x,y) = \kappa(xy) = 1 - \sum_{k=0}^{\infty} q_k (1-(m\bar{m})^{-1}xy)^k,$$

for non-negative q_k for which $\sum_{k=0}^{\infty} q_k = 1$. Then we have
(a) *for $b \leq 1 - q_0$, $s \leq (m\bar{m})^{-1}\mathbb{E}(W\bar{W})$ and for any hoppable collection of paths Ξ, it is the case that*

$$\mathbb{P}(\mathcal{C}^{\Xi}) \geq \mathbb{P}_{b,s}^{mixed}(\mathcal{C}^{\Xi});$$

(b) *for $p \geq \kappa(\max[\mathbb{E}(W\bar{W}), \mathbb{E}(W)\mathbb{E}(\bar{W})])$, and for any hoppable collection of paths Ξ, it is the case that*

$$\mathbb{P}(\mathcal{C}^{\Xi}) \leq \mathbb{P}_{p}^{bond}(\mathcal{C}^{\Xi}).$$

Proof of Theorem 4.1: In the proof of Theorem 1.3 we did not use that G is a simple graph. In the current proof we introduce a random graph $\hat{G}$ whose realisations need not be simple, and use our results for factorisable connection functions on realisations of $\hat{G}$.

Let $\hat{G}$ be the random graph obtained from G by replacing an edge $uv \in E$ by a random number X_{uv} of edges, in such a way that the collection $\{X_{uv}; uv \in G\}$ is i.i.d. with $\mathbb{P}(X_{uv} = k) = q_k$. We use $uv_i \in \hat{E}$ to denote the i-th edge (if present) from vertex u to vertex v in a realisation of the random graph $\hat{G}$. Similarly we define the collection $\hat{E}^{(n)} \subset \hat{E}$ by $uv_i \in \hat{E}^{(n)}$ if and only if $uv \in E^{(n)}$.

For $\xi \in \Xi$, let $\hat{\Xi}(\xi) = \hat{\Xi}(\xi, \hat{G})$ be the collection of self-avoiding paths in $\hat{G}$ that visit the same vertices as ξ, in the same order as in ξ. Note that if $q_0 > 0$ then $\hat{\Xi}(\xi)$ may be empty. We define $\hat{\Xi} := \hat{\Xi}(\hat{G}) := \cup_{\xi \in \Xi}\hat{\Xi}(\xi)$ and observe that this collection of paths in $\hat{E}$ is hoppable if Ξ is hoppable.

Now consider percolation on a realisation $\hat{g}$ of $\hat{G}$, as follows: we use connection function $\kappa(x, y) = xy$, and assign weights distributed as $(W/m, \bar{W}/\bar{m})$ to each vertex. It now follows from Theorem 1.3, applied to $\hat{g}$, that for all $r \geq \max(\mathbb{E}((m\bar{m})^{-1}W\bar{W}), (m\bar{m})^{-1}\mathbb{E}(W)\mathbb{E}(\bar{W}))$ we have (in the obvious notation)

$$\mathbb{P}_r^{bond}(\mathcal{C}^{\hat{\Xi}} | \hat{G} = \hat{g}) \geq \mathbb{P}(\mathcal{C}^{\hat{\Xi}} | \hat{G} = \hat{g}), \tag{6}$$

and for all $r \leq \mathbb{E}((m\bar{m})^{-1}W\bar{W})$ we have

$$\mathbb{P}_r^{site}(\mathcal{C}^{\hat{\Xi}} | \hat{G} = \hat{g}) \leq \mathbb{P}(\mathcal{C}^{\hat{\Xi}} | \hat{G} = \hat{g}). \tag{7}$$

We claim that by integrating out $\hat{G}$, we obtain the required inequalities. To see this note that for any given edge uv, the probability that one of the uv_i's is open in $\hat{G}$, equals

$$1 - \sum_{k=0}^{\infty} q_k(1 - (m\bar{m})^{-1}W_u\bar{W}_v)^k = \kappa(W_u, \bar{W}_v). \tag{8}$$

Also, different edges behave independently. Hence, if we define an edge uv to be open in G if and only if one of the uv_i's in $\hat{G}$ is open, then we obtain a realisation from the percolation process on G, with weight vector $(W, \bar{W})$ and connection function as stated in the theorem. It follows that integrating out the right hand sides in (6) and (7) gives $\mathbb{P}(C^{\Xi})$.

Similarly, the left hand side in (6) leads to $\mathbb{P}_p^{bond}(C^{\Xi})$, with p defined as

$$p = 1 - \sum_{k=0}^{\infty} q_k (1-r)^k.$$

Since $r \geq \max(\mathbb{E}((m\bar{m})^{-1} W\bar{W}), (m\bar{m})^{-1}\mathbb{E}(W)\mathbb{E}(\bar{W}))$, we see that the required inequality is valid for the stated values.

When we integrate the left hand side of (7), we introduce the possibility that no edge remains between two vertices u and v; this happens with probability q_0. This is the reason that in this case we obtain a mixed model. □

We can now prove the following corollary, which immediately leads to Theorem 1.4.

Corollary 4.2 *Suppose that a given connection function $\kappa(x,y)$ can, for all large enough c, be written as*

$$\kappa(x,y) = \kappa(xy) = 1 - \sum_{k=0}^{\infty} q_k^{(c)} \left(1 - \frac{xy}{c}\right)^k \tag{9}$$

where for all c, $(q^{(c)} \geq 0)$ and $\sum_{k=0}^{\infty} q_k^{(c)} = 1$. Then for every

$$p \geq \kappa(\max[\mathbb{E}(W\bar{W}), \mathbb{E}(W)\mathbb{E}(\bar{W})])$$

and any hoppable collection of paths Ξ, it is the case that

$$\mathbb{P}(\mathcal{C}^{\Xi}) \leq \mathbb{P}_p^{bond}(\mathcal{C}^{\Xi}).$$

The important and previously discussed connection functions

$$\kappa(x,y) = 1 - e^{-\alpha xy} = 1 - \sum_{k=0}^{\infty} \frac{(c\alpha)^k}{k!} e^{-c\alpha}(1 - xy/c)^k,$$

with $\alpha > 0$ and

$$\kappa(x,y) = \frac{xy}{\beta + xy} = 1 - \sum_{k=0}^{\infty} (\beta/c)(1+\beta/c)^{-(k+1)}(1 - xy/c)^k,$$

with $\beta > 0$, both satisfy condition (9) and hence Theorem 1.4 immediately follows.

Proof of Corollary 4.2: For $m > 0$, $\bar{m} > 0$ and $0 \le a < 1$, we define the probability measure $\mathbb{P}^{(m,\bar{m},a)}$ on Ω as follows. Assign i.i.d. uniform$(0,1)$ random variables to the vertices in V, and denote the random variable assigned to $v \in V$ by U_v. If $U_v \le a$, then all edges in E with v as start or end vertex are open. If $U_v > a$, we assign independent 2-dimensional weights $(W_v^{(m)}, \bar{W}_v^{(\bar{m})})$ to v, with distribution function

$$F(x,y) = \mathbb{P}(W \le x, \bar{W} \le y | W \le m, \bar{W} \le \bar{m}).$$

If $U_u > a$ and $U_v > a$, then $uv \in E$ is open with probability $\kappa(W_u^{(m)}, \bar{W}_v^{(\bar{m})})$ and conditioned on the weights assigned to the vertices and on $\{U_v; v \in V\}$ the states of the edges are independent.

Observe that for every $\epsilon > 0$, we may chose $m > 0$ and $\bar{m} > 0$ such that $\mathbb{P}(W < m, \bar{W} < \bar{m}) > 1 - \epsilon$. Using these ϵ, m and $\bar{m}$, it is straightforward to prove by a coupling argument that for every n

$$\mathbb{P}^{(m,\bar{m},0)}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{(m,\bar{m},\epsilon)}(\mathcal{C}^{\Xi_n}). \tag{10}$$

From Theorem 4.1 we conclude that

$$\mathbb{P}^{(m,\bar{m},0)}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{bond}_{\kappa(\max(\mathbb{E}(W^{(m)}\bar{W}^{(\bar{m})}),\mathbb{E}(W^{(m)})\mathbb{E}(\bar{W}^{(\bar{m})}))}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi_n}), \tag{11}$$

for $p \ge \kappa(\max(\mathbb{E}(W\bar{W}), \mathbb{E}(W)\mathbb{E}(\bar{W}))$.

Since the event $\mathcal{C}^{\Xi_n}$ depends on only a finite number of edges and weights assigned to vertices, $\mathbb{P}^{(m,\bar{m},\epsilon)}(\mathcal{C}^{\Xi_n})$, varies continuously with ϵ, which implies that for any $\delta > 0$, there exists $\epsilon > 0$ such that $\mathbb{P}^{(m,\bar{m},\epsilon)}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{(m,\bar{m},0)}(\mathcal{C}^{\Xi_n}) + \delta$. Combined with (10) and (11) this gives that for any $\delta > 0$,

$$\mathbb{P}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi_n}) + \delta,$$

which implies that $\mathbb{P}(\mathcal{C}^{\Xi_n}) \le \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi_n})$.

The proof is completed by observing that for hoppable Ξ, we have

$$\mathbb{P}(\mathcal{C}^{\Xi}) = \lim_{n\to\infty} \mathbb{P}(\mathcal{C}^{\Xi_n}) \le \lim_{n\to\infty} \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi_n}) = \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi}).$$

□

With the extra assumption that W and $\bar{W}$ are independent, Miller [19] proved for a broad class of convex connection functions $\kappa(x,y)$ and Ξ the set of infinite paths starting at vertex v, that

$$\mathbb{P}(\mathcal{C}^{\Xi}) \le \mathbb{P}^{bond}_p(\mathcal{C}^{\Xi}).$$

where $p \geq \mathbb{E}(\kappa(W, \bar{W})) \geq \kappa(\mathbb{E}(W\bar{W}))$. The second inequality follows from the convexity of κ and Jensen's inequality.

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