Maximal avalanches in the Bak-Sneppen model

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Abstract

We study the durations of the avalanches in the maximal avalanche decomposition of the Bak-Sneppen evolution model. We show that all the avalanches in this maximal decomposition have infinite expectation, but only 'barely' in the sense that when we make the appropriate threshold a tiny bit smaller (to be made precise below), then the avalanches would have finite expectation. These results are slightly surprising, since simulations suggest finite expectations.

KEYWORDS AND PHRASES: Bak-Sneppen evolution model, avalanche, self-organised criticality.

1 Introduction and main results

The Bak-Sneppen model was originally introduced as a simple model of evolution by Per Bak and Kim Sneppen [2]. Their model can be defined as follows. There are N species arranged on a circle with a random *fitness*, independent and uniformly distributed on $(0, 1)$, assigned to each species. At each discrete time step the system is updated by locating the lowest fitness and replacing this fitness and those of its two neighbours by independent and uniform $(0, 1)$

random variables. We think of the N species as vertices, making a vertex set $\Lambda(N) = \{0, 1, \ldots, N-1\}$, in which all additions are modulo N.

One of the ways to analyse this model is to break it down into a series of *avalanches*. An avalanche at a *threshold* b is said to occur between times s and $s+t$ if at time s all the fitnesses are equal to or greater than b with at most one vertex where equality holds, and time $s+t$ is the next time after s at which this property occurs. In the literature a number different types of avalanches has been proposed. Our definition is consistent with a number of other papers [2, 6, 7, 9]; other types of avalanches have for instance been defined in [4, 5]. Note that if we have a minimum fitness value of b, then we can choose any value up to (and including) b to be our avalanche threshold.

The idea of avalanches gives rise to so called *self-organised critical behaviour* of the system: for large N there appears to be a threshold b_c, close to $2/3$, such that after a while, the dynamics appear to consist of consecutive avalanches at b_c, and in addition, these avalanches seem to exhibit power law behaviour in the sense that both duration and range can be described with a power law [2]. The threshold b_c is not set beforehand; the model seems to organise *itself* into this state. This is the reason to consider this model as an example of self-organised critical behaviour. See Figure 1 for a typical snapshot of the Bak-Sneppen model in stationarity, with $N = 300$. On the horizontal axis we have the 300 vertices, with the dots representing the fitnesses of all vertices.

The Bak-Sneppen model can be thought of as a sequence of consecutive avalanches. Since different avalanche thresholds can be chosen, there are a number of ways to perform such an *avalanche decomposition.* One common approach is to consider the model as a series of avalanches at some *fixed* threshold level b, see for instance [7].

There has also been considerable attention reserved for the so-called *maximal avalanche decomposition.* Here the first avalanche threshold is defined to be the minimum fitness value from the initial fitness values. After this and every

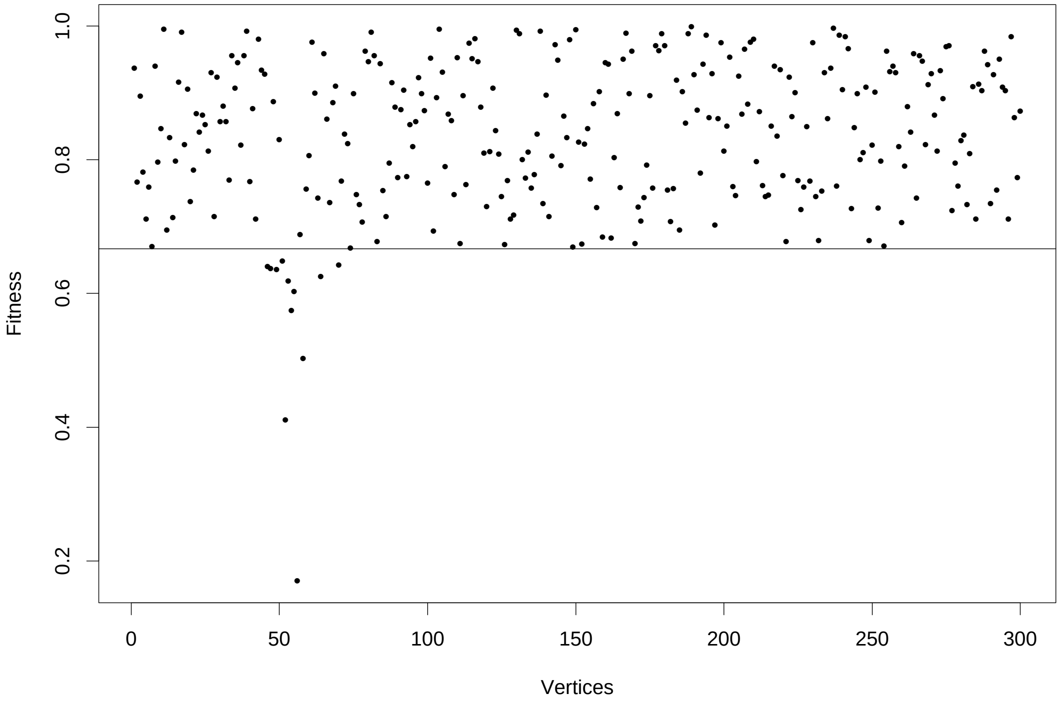

Figure 1: A snapshot of the fitnesses in the Bak-Sneppen model in stationarity.

subsequent avalanche, another avalanche begins with the threshold chosen to be the new minimal value of the model; this is the maximal threshold choice. It is clear that this will lead to the Bak-Sneppen model being seen as a series of avalanches at strictly increasing thresholds. The *gap function* at time s, $G(s)$, is defined to be the avalanche threshold at time s [1]. The gap function is a stepwise increasing function which jumps to a new value each time an avalanche finishes. Figure 2 shows a realisation of the gap function represented by the line, whilst the dots are the minimum fitness values at each time step.

In this report we investigate the expected durations of avalanches in the maximal avalanche decomposition. One reason for looking at the maximal avalanche decomposition (and therefore the gap function also) is that there has been interest in how the Bak-Sneppen model tends towards criticality. One such approach

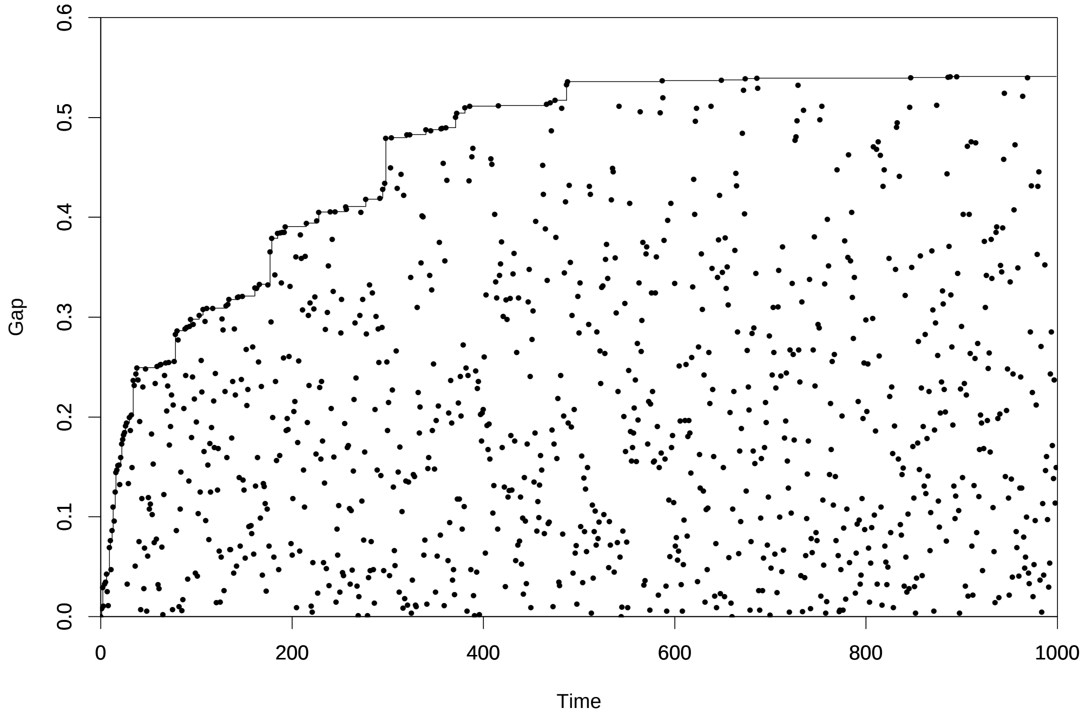

Figure 2: A realisation of the gap function when $N = 100$.

has been the so called *gap equation.* The gap equation is a conjectured differential equation to describe the behaviour of the gap function. It has been commonly referred to in the physics literature [9, 10] and takes the form

$$\frac{dG(s)}{ds} = \frac{1 - G(s)}{L^d \langle S \rangle_{G(s)}}.$$

Here L^d is the size of the system (N by our notation) and $\langle S \rangle_{G(s)}$ is the expected duration of the avalanche taking place from a threshold $G(s)$. The gap equation suffers not only from being non-rigorous, but also from not being well-defined. Surely the function G appearing in the formula cannot refer to the gap function G as defined above (which is the same definition as in [9] and [10] where the above gap equation is discussed), and the correct interpretation of this formula remains somewhat vague and ambiguous.

One of the reason for wanting a gap equation is to yield insight into the behaviour of the infinite limit of the model when close to criticiality. The idea is that criticality (or stationarity) is characterised by avalanche thresholds close to the critical point b_c. However, since the gap function tends to 1 rather than b_c, this poses immediate problems when trying to describe the near-critical behaviour. The physics literature has used the gap equation to conjecture how the gap behaves near the critical point [9]. Assuming that the expected avalanche duration diverges at b_c (only true in the thermodynamic limit) and has power law form with exponent γ, it is claimed that the gap function (only defined on finite systems) behaves as a function of $(\frac{s}{N})^{-\rho}$, where $\rho = (\gamma - 1)^{-1}$. As you can see, all such methods are somewhat speculative and mathematically non-rigorous. For this and other reasons we choose not to look at the gap equation, but concentrate on individual avalanches in the maximal avalanche decomposition instead.

Previous mathematical literature on the model concentrated on the expected duration of an avalanche at a fixed and non-random threshold b [8]. Their results include a number of useful monotonicity results, as well as an explicit differential equation relating the expected duration of avalanches to their expected range. In this paper, we study the avalanches at random thresholds which appear in, or are strongly related to, the thresholds in the maximal avalanche decomposition. The threshold is the only variable we need to know in order to determine the distribution of an avalanche's duration on $\Lambda(N)$. By this we mean that the durations of two avalanches on $\Lambda(N)$ are identically distributed if their thresholds are the same.

Concentrating first on the *initial* avalanche in the maximal decomposition, we see that the initial threshold is the minimum of N independent uniform $(0, 1)$ random variables. To be more explicit, we have an avalanche with random threshold B on $\Lambda(N)$ whose density $h_N(b)$ is given by

$$h_N(b) = N(1-b)^{N-1}, \qquad 0 \leq b \leq 1.$$

Letting D denote the random duration of the initial avalanche, we prove the following theorem.

Theorem 1.1 *The expected duration of the first avalanche is infinite, i.e.* $\mathbb{E}(D) = \infty$.

One consequence of this result is that any subsequent avalanche also has infinite expected duration, as its threshold is stochastically larger. Hence the gap function consists of a sequence of steps, each of which has infinite expected length. These results seem somewhat surprising and the divergent behaviour is not typically noticeable under numerical simulations of the model, especially when N is large.

We decided to perturb the avalanche threshold by making it stochastically smaller and see whether this would lead to convergence. It turns out that $\mathbb{E}(D)$ is 'barely infinite' in that making the threshold a tiny bit stochastically smaller (where this reduction tends to 0 as N tends to ∞) yields finite expected durations. To be precise, we denote by D_N^n the expected duration of an avalanche at a threshold which is set by the minimum of n uniform $(0,1)$ random variables; the subscript N still refers to the system size. In this notation, the previously used $\mathbb{E}(D)$ can now be written as D_N^N. We prove the following result.

Theorem 1.2 *An avalanche on* $\Lambda(N)$ *from a threshold chosen as the minimum of* $n > N$ *independent uniform* $(0,1)$ *random variables has finite expectation, i.e.* $D_N^n < \infty$ *for all* $n > N$.

So just adding one uniform random variable when setting the threshold is enough to get a finite expected duration. However, it is possible to show that under certain conditions all *further* avalanches have infinite expected duration. Recall that on $\Lambda(N)$, setting the threshold as the minimum of N independent uniform (0,1) random variables, gives infinite expected duration. If all the fitnesses (except the minimum) are independent and uniformly distributed above the threshold at the start of the avalanche, then at the end of the avalanche all the vertices will again be independent and uniformly distributed above the threshold. So even if you fix b and choose your fitnesses to be uniform above it, the next avalanche will have infinite expected duration. A weaker form of this result applies when we drop the condition that the fitnesses had to be nicely distributed at the start of the avalanche. All the vertices updated by the avalanche will be independent and uniformly distributed above the threshold at the end of the avalanche. So once we have had a spanning avalanche (one that updates every vertex in the system during its duration) all subsequent avalanches (from maximal thresholds) will have infinite expected duration, no matter what initial fitness values we take.

2 Proof of Theorem 1.1

We prove that the initial maximal avalanche has infinite expected duration. It is essentially a trivial result that $D_3^3 = \infty$. Indeed, at each time step the fitnesses are independently and identically distributed and so this is just a record values result [3]. Alternatively you can find an explicit expression for the duration of an avalanche at a fixed threshold and then integrate out over the distribution of B. For $N > 3$ a more subtle approach is required since we have neither independence nor identically distributed minimal fitnesses. In the proof for general N, we use the observation that if at any time all the fitnesses are below the threshold B, then the avalanche cannot stop before at least N update values greater than B have been observed. We calculate the expected time taken for N such updates to be observed between times when all fitnesses are below the threshold.

Before we start on the main body of the proof we need some definitions. Let $f_i(n)$ denote the fitness of vertex i at time step n, where the initial fitnesses, corresponding to $n = 0$, are independent and uniform on $(0, 1)$. We generate our updates from a sequences of independent uniform $(0, 1)$ random variables, $U_k, k = 0, 1, 2, \ldots$ as follows: if vertex i has the minimal fitness at time n, then the fitnesses at positions $(i-1, i, i+1)$ are replaced by $(U_{3n}, U_{3n+1}, U_{3n+2})$, giving the fitnesses at time $n+1$.

We divide the avalanche into time *blocks* of length $2N$, i.e. block m contains the time steps starting with $2mN$ and finishing with $2(m+1)N-1$, for $m = 0, 1, 2, \ldots$ We call a block *bad* if all the updates in the block are below the threshold B and are in decreasing order. So, block m is bad if

$$B > U_{6mN} > U_{6mN+1} > U_{6mN+2} > \cdots > U_{6(m+1)N-1}.$$

We let s_r denote the (random) time just after the r^{th} bad block, for $r = 1, 2, \ldots$ The next lemma tells us that after a bad block, all fitnesses are below B.

Lemma 2.1 *We have that* $f_i(s_r) < B$ *for all* $0 \leq i \leq N-1$ *and* $r = 1, 2, \ldots$

Proof: Consider the state of the system at the beginning of the r^{th} bad block. We update the vertex with the minimal fitness, together with its two neighbours. Since the block is bad, the three updated vertices now have fitness below the threshold. We now have two possible cases, either one of the new fitnesses is the minimum, or another vertex has the minimal fitness. If the minimum is drawn from a newly updated vertex then the avalanche will continue (for the duration of the bad block) round the circle of vertices, since after another update the minimum among the three updated vertices will be lower than the minimum at the previous step and so will in fact be the global minimum. The second case can not occur more than N times before the first case occurs. Once we have the minimum among new fitnesses, it can not take more than N vertices for the first case to cover the entire vertex set. Hence after $2N$ updates all the vertices will have fitnesses generated by new updates. Thus all fitnesses will be below the threshold, as required. $\square$

We call a period between the endpoints of two bad blocks an *epoch*; hence the m^{th} epoch corresponds to the time interval $[s_m, s_{m+1}-1]$, for $m = 1, 2, \ldots$

We call an epoch *good* if among the updating random variables corresponding to that epoch there are at least N updating random variables greater than B. So, the m^{th} epoch is good if

$$\sum_{k=3s_m}^{3s_{m+1}-1} \mathbb{I}\{U_k \geq B\} \geq N,$$

where $\mathbb{I}$ denotes the indicator function. We let S be the index of the first good epoch. Now observe that *given* that block 0 is bad, the first avalanche can only finish during a good epoch. This indicates that the expectation of S plays an important role in our analysis.

Lemma 2.2 $\mathbb{E}(S) = \infty$.

Proof: The key to the proof is to compute the expectation of S given that $B = b$. Indeed, conditioned on $B = b$, S has a geometric distribution with a successful trial being a good epoch. This is because conditional on the threshold, the lengths of the epochs are independent and identically distributed. We let $p(b)$ be the probability that the first epoch (or any other, for that matter) is good, given that $B = b$. Integrating over the range of b, we then have that

$$\mathbb{E}(S) = \int_0^1 \frac{N(1-b)^{N-1}}{p(b)} db,$$

since B is distributed as the minimum of N independent uniform $(0, 1)$ random variables. It remains to estimate $p(b)$. Since we want to bound the integral from below, we have to bound $p(b)$ from above. We let F_n be the event that the first epoch does not finish before or at time $s_1 + n$, and G_n for the event that

$$\min\{\ell; \sum_{k=3s_1}^{\ell} \mathbb{I}\{U_k = B\} \geq N\} = n,$$

that is, G_n is the event that the n^{th} update random variable in the epoch - ordered naturally as $U_{3s_1}, U_{3s_1+1}, \ldots$ - is the N^{th} above b. We can now write:

$$p(b) = \sum_{n=N}^{\infty} \mathbb{P}(G_n | B = b) \mathbb{P}(F_{\lfloor n/3 \rfloor} | G_n, B = b).$$

Now observe that $\mathbb{P}(G_n|B=b) = \binom{n-1}{N-1}(1-b)^N b^{n-N}$ which we bound from above by $\binom{n-1}{N-1}(1-b)^N$. Furthermore, conditioning on $B=b$ and G_n means that we randomly select $N-1$ updates among the $n-1$ first updates after s_1 which are declared to be the updates above b; the only information about all the other updates is that they are below b. No matter the precise realisation, this implies that there are at least (for $n > 6N^2$) $\lfloor \frac{n-1}{6N} \rfloor - (N-1)$ blocks so far for which our only information is that they have all their corresponding updates below b. Hence the (conditional) distribution of the updates in these blocks are i.i.d. uniform on $(0,b)$, and any of these blocks is bad with a (conditional) probability which is uniformly bounded below by a constant $c_1(N)$, say. This then gives:

$$\begin{aligned} p(b) &\leq \sum_{n=N}^{6N^2} \binom{n-1}{N-1}(1-b)^N + \\ &\quad \sum_{n=6N^2+1}^{\infty} \binom{n-1}{N-1}(1-b)^N(1-c_1(N))^{\lfloor \frac{n-1}{6N} \rfloor - (N-1)} \\ &\leq (1-b)^N \left(\sum_{n=N}^{6N^2} \binom{n-1}{N-1} + \sum_{n=6N^2+1}^{\infty} n^N (1-c_1(N))^{\lfloor \frac{n-1}{6N} \rfloor - (N-1)} \right) \\ &= c(N)(1-b)^N, \end{aligned}$$

for a suitable constant $c(N) < \infty$.

We can now substitute our upper bound for $p(b)$ back into our integral for $\mathbb{E}(S)$, giving

$$\begin{aligned} \mathbb{E}(S) &\geq \int_0^1 \frac{N(1-b)^{N-1}}{p(b)} db \\ &\geq \frac{N}{c(N)} \int_0^1 \frac{(1-b)^{N-1}}{(1-b)^N} db \\ &= \frac{N}{c(N)} \int_0^1 \frac{1}{1-b} db \\ &= \infty. \end{aligned}$$

□

Let A be the event that block 0 is bad. We already noticed that if block 0 is bad, the first avalanche can only end during a good epoch. This gives

$$\mathbb{E}(D) \geq \mathbb{E}(D|A)\mathbb{P}(A) \geq \mathbb{E}(S|A)\mathbb{P}(A),$$

and since $\mathbb{P}(A) > 0$ it suffices to prove that $\mathbb{E}(S|A) = \infty$.

Lemma 2.3 *We have that* $\mathbb{E}(S|A) = \infty$.

Proof: Writing $F_{B|A}$ for the conditional distribution function of B given the event A, we have that

$$\mathbb{E}(S|A) = \int_0^1 \mathbb{E}(S|A, B = b)dF_{B|A}(b). \tag{1}$$

We claim now that

$$\mathbb{E}(S|A, B = b) = \mathbb{E}(S|B = b). \tag{2}$$

To see this, observe that when we condition on $B = b$, further knowledge about updates in block 0 is irrelevant for the distribution of S; indeed, the distribution of S depends only on B.

Furthermore, we claim that

$$F_{B|A}(b) \leq F_B(b), \tag{3}$$

where F_B denotes the distribution function of B. To see this, we write $U = \max\{U_0, \ldots, U_{6N-1}\}$ and argue as follows:

$$\begin{aligned} F_{B|A}(b) &= \mathbb{P}(B \leq b|A) \\ &= \mathbb{P}(B \leq b|B > U_0 > U_1 > \cdots > U_{6N-1}) \\ &= \mathbb{P}(B \leq b|B > U, U_0 > U_1 > \cdots > U_{6N-1}). \end{aligned}$$

Since the event $\{U_0 > U_1 > \cdots > U_{6N-1}\}$ is independent of (U, B), we can remove it from the conditioning event, and we find that

$$\begin{aligned} F_{B|A}(b) &= \mathbb{P}(B \leq b|B > U) \\ &\leq \mathbb{P}(B \leq b), \end{aligned}$$

since B and U are independent, and it is well known in general that conditioning a random variable B to be larger than another, independent, random variable U makes B stochastically larger.

Taking together (1), (2) and (3), we finally obtain

$$\mathbb{E}(S|A) \geq \int_0^1 \mathbb{E}(S|B=b)dF_B(b) = \mathbb{E}(S),$$

which by Lemma 2.2 is equal to infinity, proving the result. □

3 Proof of Theorem 1.2

Recall that Theorem 1.2 stated that an avalanche on $\Lambda(N)$ from a theshold distributed as the minimum of $N+1$ independent uniform random variables has finite expected duration. Our approach is to couple the Bak-Sneppen model with a similar and simpler model whose avalanche durations are more easily calculated. In the Bak-Sneppen model we have the vertices arranged on a circle and so each vertex has only two neighbours. If we now alter this graph structure to make the vertices be arranged on a complete graph, then updating the minimal vertex and its neighbours will update the entire graph. This yields a fairly uninteresting model, with no time dependence or self-organised critical behaviour in the infinite limit. However, the expected duration of an avalanche on this model from a threshold b is easy to calculate and is equal to $(1-b)^{-N} < \infty$.

Intuitively one might think that an avalanche on this new model from a threshold b would be longer than an avalanche on the original Bak-Sneppen model from the same threshold, b. This is because the original model eliminates the smallest value and its two neighbours leaving some old fitnesses that are certainly greater than the minimum. It seems plausible that the distributions of these other vertices are stochastically larger than uniform $(0,1)$ random variables. This intuition is correct, as we will now demonstrate.

Let $\hat{D}_N^n$ be the analogue of D_N^n in the new model. Since, as in the Bak-Sneppen model on $\Lambda(3)$, we have an explicit expression for the expected duration of an avalanche at a fixed threshold on this new model, we can calculate $\hat{D}_N^n$ by integration.

$$\hat{D}_N^n = \int_0^1 \frac{n(1-b)^{n-1}db}{(1-b)^N} < \infty, \text{ for all } n > N,$$

where the expression $n(1-b)^{n-1}$ is the density of the threshold which is set by the minimum of n uniform random variables. For $N = 3$, the new model is the same as the Bak-Sneppen model. For $N > 3$, we need to provide a coupling of these two models to show that the expected duration of a b avalanche is shorter for the Bak-Sneppen model and therefore that the initial avalanche has finite expectation when $n > N$.

The main difficulty to overcome when coupling these two models together is that the time dependencies are completely different. The Bak-Sneppen model depends heavily upon the values at the previous step, whereas the new model is completely independent of previous fitness values. We start by defining a sequence of uniform $(0, 1)$ random variables, $X_1, X_2, X_3, ...$ that shall be used to update the fitnesses of the new model. Contrary to earlier notation, for this proof we enumerate the vertices by $1, 2, ..., N$; the fitnesses of the new model will be $X_1, X_2, ..., X_N$ respectively after the first time step, and $X_{N+1}, X_{N+2}, ..., X_{2N}$ after the second time step and so on.

The basis of the approach is to not fix the exact fitness values in the coupled Bak-Sneppen model. We use the fitness values of the new model to update the distributions of the fitnesses in the Bak-Sneppen model, but only use them to fix the values when we wish to check that the coupled avalanche on the Bak-Sneppen model has stopped. The explanation of the exact details of the coupling will consist of two parts, a precise description of the inductive step of updating the coupled Bak-Sneppen model and an example.

Consider a situation where the avalanche, with threshold b, on the new model is in progress and we know the value of the minimal fitness on the Bak-Sneppen model and the distributions of all the other fitnesses. In a non-coupled realisation of the Bak-Sneppen model the vertex with minimal fitness and its two neighbours would receive new independent uniform $(0,1)$ distributed fitness values. So in an intermediary stage between two time steps we disgard the exact values and distributions of these three fitnesses and consider them as merely being uniform $(0,1)$ random variables. Thus after this intermediary step the coupled Bak-Sneppen model has no exact values and we are ready for the next updates.

The sequence (X_i) of random variables is used to update the new model as described above. Two things can happen: either all the fitness values are above b and the avalanche on the new model stops or we have some active vertices and the avalanche is still in progress. In the first instance we need to check that the avalanche on the coupled Bak-Sneppen model will also finish at this time step (if not before). To do this we associate the fitness values on the vertices of the new model to the respective vertices of the Bak-Sneppen model. To do this we have to first transform the uniform random variables so that they have the correct distributions for the fitnesses of the Bak-Sneppen model. This is simply done by taking the inverse of the distribution functions (sometimes referred to as probability integral transformation). This means that the values of the Bak-Sneppen model are all fixed and have the correct distributions and we can check that all the fitnesses on the Bak-Sneppen model are above the threshold b; see the last paragraph of this section.

It is perhaps worth describing the exact nature of these transformations more precisely. If you wish to generate a random variable with any invertible distribution function F then this can be done by taking a uniform $(0,1)$ distributed random variable, X and taking $F^{-1}(X)$. So our approach here is to use this sort of transformation so that the same sequence (X_i) can be used for the new model and the old model.

In the other case that the avalanche on the new model is still in progress, we can't afford to fix the fitness values for the coupled Bak-Sneppen model. We again follow the same approach to find the values for the Bak-Sneppen model, but instead of fixing them we merely use them to find the value and location of the minimal fitness. Once this is done we disgard all the exact values except the minimum fitness and update the distributions of the other fitnesses to take into account the conditioning on them being larger than the new given minimal value. We are now back to the situation at the start of our description of the inductive state.

Before giving an explicit example it is important to examine some of the properties of this coupling. In particular we wish to find out what possible distributions the fitnesses of the coupled Bak-Sneppen model can take. For any uniformly distributed random variable, its conditional distribution given that it is above a specific value within its range is uniform from that value to its prior upper endpoint. Since all our fitness values (except the minimum) are initially distributed uniform $(b, 1)$ and a direct update of a vertex (by this we mean it is one of the three vertices to receive a new fitness value) resets its distribution to be uniform $(0, 1)$. This means that even though conditioning on the fitness not being the smallest (when the smallest value is unknown) yields non-uniform and complex distributions, the fact that we know the exact minimal value keeps everything uniform.

We are now ready to give a specific example, as we know how to update the fitness distributions in the coupled Bak-Sneppen model. Each time step is split into two lines so the intermediary stage can be clearly seen. In the first line we update the new model and write down the marginal distributions for the Bak-Sneppen model at that update. On the next line the fitness values for the new model are used to calculate the exact minimum for the Bak-Sneppen model and to update the remaining marginal distributions. For brevity we use the shorthand $> b$ for $U(b, 1)$ and > 5 for $U(X_5, 1)$. We use an underscore to denote the minimum fitness value. It is important to remember that the minimum of the new model doesn't play an explicit role in determining the behaviour of the Bak-Sneppen model. The underscore is used here only to highlight the types of behaviour that you can observe from this coupling.

New model					Comments	Bak-Sneppen model				
1	2	3	4	5	Vertices	1	2	3	4	5
$> b$	$\underline{b}$	$> b$	$> b$	$> b$	Initial setup	$> b$	$\underline{b}$	$> b$	$> b$	$> b$
X_1	X_2	X_3	X_4	X_5	Update new	> 0	> 0	> 0	$> b$	$> b$
X_1	X_2	$\underline{X_3}$	X_4	X_5	Update Bak-Sneppen	> 3	> 3	$\underline{X_3}$	$> b$	$> b$
X_6	X_7	X_8	X_9	X_{10}	Update new	> 3	> 0	> 0	> 0	$> b$
$\underline{X_6}$	X_7	X_8	X_9	X_{10}	Update Bak-Sneppen	> 9	> 9	> 9	$\underline{X_9}$	$> b$
X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	Update new	> 9	> 9	> 0	> 0	> 0
X_{11}	$\underline{X_{12}}$	X_{13}	X_{14}	X_{15}	Update Bak-Sneppen	$> x$	$\underline{x}$	$> x$	$> x$	$> x$

Since vertex 2 has the minimum value initially, we consider vertices $1, 2$ and 3 being uniform $(0, 1)$ in the first intermediary stage. We also update the fitnesses of the new model. Note that in this example, at all times the minimal fitness in the new model is below b, so the avalanche doesn't stop during the steps we have detailed. So we don't fix the exact fitness values in the couple Bak-Sneppen model, but instead use the random variables, $X_1, X_2, ..., X_5$, to find the location of the minimal fitness in the Bak-Sneppen model. In the new model X_3 takes the minimum value and so vertex 3 in the Bak-Sneppen model is minimal with fitness X_3. This is because out of the transformed values X_3 is necessarily the smallest, as the transformations to X_4 and X_5 make them larger and no transformations are required for X_1, X_2 and X_3. Since the avalanche is still in progress, $X_3 < b$ so the distributions of the fitnesses of vertices 4 and 5 remain the same, whereas for vertices 1 and 2 they are now uniform $(X_3, 1)$.

The next time step follows on in a similar vein, except that after the transformations we find that the transformed value of X_9 is minimal even though before transformation X_6 was smaller. This means that the location of the minimal vertex on the two models is now different. It is also worth noticing that X_9 must be greater than X_3 by the way that the fitness distributions have been updated. The final time step detailed gives an example of the more complicated behaviour that can be observed. Here vertex 2 is minimal in both models, but in the Bak-Sneppen model the fitness value at vertex 2 is x rather than X_{12}, where $x = X_9 + (1 - X_9)X_{12}$. This is because we have had to transform the value of X_{12} so that it had the correct distribution.

To conclude this example let us consider the case when the avalanche in the new model stops. This means that $\min(X_{11}, X_{12}, X_{13}, X_{14}, X_{15}) > b$. We would then fix the fitnesses for the Bak-Sneppen model giving $X_9 + (1 - X_9)X_{11}, X_9 + (1 - X_9)X_{12}, X_{13}, X_{14}, X_{15}$.

It is clear that if the avalanche on the new model stops then so will the coupled avalanche, since we transfrom the fitnesses of the new model into larger values for the Bak-Sneppen model. This is because the marginal fitness distributions of the Bak-Sneppen model are always uniform above some value greater than 0. This coupling shows that $\hat{D}_N^n < \infty \Rightarrow D_N^n < \infty$ and hence we have the desired result. $\square$

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