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Connections Between 'Self-Organised' and 'Classical' Criticality

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Abstract. We investigate the nature of the self-organised critical behaviour in the Abelian sandpile model and in the Bak – Sneppen evolution model. We claim that in either case, the self-organised critical behaviour can be explained by the careful choice of the details of the model: they are *designed* in such a way that the models are necessarily attracted to the critical point of a conventional parametrised equilibrium system. In the case of the Abelian sandpile we prove this connection to conventional criticality rigorously in one dimension, and provide evidence for a similar result in higher dimensions. In the case of the Bak – Sneppen evolution model, we give an overview of the current results, and explain why these results support our claim.

We conclude that the term self-organised criticality is somewhat confusing, since the tuning of parameters in a model has been replaced by the careful choice of a suitable model. Viewed as such, we can hardly call this critical behaviour spontaneous.

Keywords: self-organised criticality, Bak – Sneppen evolution model, Abelian sandpile, classical criticality, particle system

AMS Subject Classification: 82C22, 60K35

1. Introduction and motivation

In recent years, there has been a growing interest in stochastic dynamics in connection with so called *self-organised criticality* (SOC). To explain the terms: the label *self-organised* is used in the literature for ordering or pattern formation amongst many interacting units. Implicit is the notion that the phenomenon of interest appears spontaneously. For instance, in the Ising model this is exactly how magnetisation appears. In fact, this self-organisation is precisely what we seek, to explain or describe the emergence of order in nature.

The expression self-organised *criticality* carries greater specificity though, because criticality usually does *not* happen spontaneously. When we use the term *critical* we refer to properties of conventional (equilibrium) models in statistical physics (for instance percolation, the Ising model or the contact process), *at criticality*, notably power law decay of spatial and/or temporal correlations, and scale-invariance of certain quantities, like crossing probabilities in percolation. The term SOC has been used to describe systems which exhibit spontaneous critical behaviour, without tuning of parameters. This would seem to embrace certain aspects of random walks, but one typically restricts the term SOC to systems that are attracted, in the course of time, to a critical *stationary* state, the main examples being sandpile models (introduced in [3]), and the Bak – Sneppen model (introduced in [2]). The long range character of SOC systems is implicit in the definition: small local changes can have dramatic effects, sometimes leading to a complete re-organisation of the entire configuration. In [12], [8] and [1], a lot of motivation for studying SOC is given: it turns out that many natural phenomena exhibit scale-invariance and power law behaviour, and the idea is that nature can not tune parameters for all these different phenomena; the system organises *itself* into a critical state.

Assertions in the literature about spontaneous or parameter-free criticality have tended to obscure the picture somewhat, fostering the impression that SOC is a phenomenon *sui generis*, inhabiting a different world than that of standard and conventional critical phenomena. The main reason to write this paper is to point out however, that SOC and conventional criticality are strongly related, in fact.

First of all, we think that the term SOC is confusing in itself, since it is often not clear what (the absence of) tuning means exactly. More importantly, we believe that there are many connections between SOC systems and the conventional equilibrium models from statistical mechanics in the sense that the critical behaviour of the SOC system can be directly related to the behaviour at criticality of such a classical equilibrium system. In a way you can say that systems which exhibit SOC behaviour are *carefully designed* in such a way that critical behaviour is to be expected.

An early example of a result in this direction goes back to [6], where the following situation is considered. Let $X(e)$ be independent uniform $(0,1)$ random variables, indexed by the nearest neighbour edges of the usual d-dimensional integer lattice. We construct a sequence $C = (C_0, C_1, \ldots)$ of random connected subgraphs of the integer lattice in the following way. The graph C_0 contains the origin and no edges. Given C_i, we obtain C_{i+1} by adding to C_i the edge e_{i+1} for which $X(e_{i+1}) = \min\{X(e); e \text{ is a boundary edge of } C_i\}$, together with the endvertex of this edge (if not already belonging to C_i). Consider

$$Q_n(y) = \frac{1}{n} \sum_{j=1}^{n} 1_{\{X(e_j) \le y\}}.$$

In [6] it is proved that

$$Q_n(y) \to 1 \qquad \text{if } y > p_c,$$

and

$$Q_n(y) \to y/p_c \quad \text{if } y < p_c,$$

where p_c is the critical probability for independent bond percolation on the d-dimensional integer lattice. That is to say, the dynamics 'discovers' the critical probability p_c without any tuning of parameters. A little reflection reveals why this result is to be expected: indeed, when our process hits the infinite percolation cluster corresponding to a certain level, it will never leave this cluster. So of course, the sequential construction itself does not involve any parameters, but the *choice* of the process is careful and designed as to reconstruct some aspects of critical bond percolation.

The goal of the present paper is to present similar ideas in two important contexts: the Abelian sandpile model and the Bak – Sneppen evolution model. The analogy with the above-mentioned percolation example is not perfect, but the ideas are similar, namely showing that the models themselves are carefully chosen as to be able to direct themselves into a critical state. In the case of the Abelian sandpile, we first explain on an intuitive level why the system organises itself into the critical state corresponding to a particular classical equilibrium particle system. We formally prove this relation in the one-dimensional case, and also discuss the situation in higher dimensions. In the case of the Bak – Sneppen model, we give an overview of the results that are rigorously available at this point, together with an explanation of how the apparent spontaneous critical behaviour follows easily from considerations about the underlying parametrised classical system.

2. The Abelian sandpile

Consider a d-dimensional box with side length L, and suppose that at each integer point z of the box, we have a pile of sand with height $h(z) \in \{1, 2, \ldots, 2d\}$. The dynamics proceeds as follows. We pick a random vertex z in the box, and increase the height of this vertex by 1. If the new height exceeds $2d$, we distribute $2d$ grains of sand among its $2d$ neighbouring vertices, one for each neighbour; if z happens to be at the boundary of the box, then one or more of the neighbours of z do not belong to the box, and the corresponding grains are lost; they leave the system. As a result of this procedure, $h(z)$ is down at 1, and the heights of the neighbouring vertices have increased by 1. It is possible that one (or more) neighbouring vertices now have more than $2d$ grains of sand. These vertices do the same thing: distribute $2d$ grains among their neighbours, et cetera, until all vertices have a height which is at most $2d$. The dynamics which follow upon adding one grain of sand at a randomly chosen vertex z until all heights are back to at most $2d$, is known as an *avalanche*, and it is precisely these avalanches that seem to obey power law (critical) behaviour. So, for instance, the (random) number of vertices involved in a single avalanche in the stationary state is believed to have a tail which is described by a power law. The long range nature of this system becomes apparent when you realise that a single avalanche can span a significant proportion of the whole system, and this happens with a reasonably large probability, as is articulated in the power law (as opposed to exponential decay). There are many variations of the above theme. Models like this are known as *Abelian sandpile models* (ASM). The Abelian sandpile gives rise to an interesting Abelian group of operators [9], and the algebraic analysis of this group has become a serious branch of research, see for instance [4, 7] and also the discussion below.

2.1. Connection to an equilibrium system at criticality

The ideas that we want to discuss here are inspired by [10]. We first explain what we have in mind in an intuitive fashion. We introduce a particle system which we call *synchronised random walk* (SRW). Each site z of the d-dimensional integer lattice initially harbours a number $\eta(z) \in \{1, 2, \ldots\}$ of particles, with density ξ, say. Associated to each site z there is a Poisson process at rate one. These Poisson processes are independent of each other and of the current state of the particle system. When the Poisson clock 'rings' at site z and $\eta(z) > 2d$, then $2d$ particles at site z are transferred to the $2d$ neighbours of z, one particle for each neighbour. When the Poisson clock rings at site z and $\eta(z) \leq 2d$, nothing happens. Since the rates are bounded, the above description gives rise to a well-defined Markov process.

The idea is that this SRW system should posses a conventional critical density ξ_c in the following sense: if the initial density of the particles is below ξ_c, then all motion will stop locally; on the other hand, there are initial distributions with density larger than ξ_c for which there is no local extinction of dynamics.

The relation between the SRW and ASM is that they essentially have the same local dynamics. There is, however, a fundamental difference between the ASM and SRW: the former allows addition and loss of particles; in the sandpile, particles may exit from one of the boundary sites. If we allow particles to leave, then eventually the system will reach a stable configuration. When this happens in the sandpile, we add a new particle at a randomly chosen site. It is precisely this innocent-sounding prescription — add a particle when and only when all other activity ceases — which carries the infinite time scale separation essential to the appearance of SOC in sandpiles. The boundary of the system is crucial for setting up the critical state.

So we see that the SRW, which has the same local dynamics as the sandpile, shows critical behaviour if and only if we *tune* the initial density to ξ_c, while the ASM *attracts itself* to a stationary state with critical behaviour. How do these two facts relate to each other? Evidently, there must be a connection between the two phenomena. In [10], the following heuristic explanation is offered. Consider the ASM. In the presence of activity, that is, when topplings occur, we should typically have $\xi > \xi_c$, and particles can only *leave* the system. In the absence of activity, we typically have $\xi < \xi_c$, and there is only *addition* of particles. Hence the density of particles typically changes in the direction of ξ_c and therefore the only *stationary* value for the sandpile is ξ_c. The sandpile necessarily attracts itself to the critical density of the SRW.

This is a very attractive intuitive explanation of the appearance of self-organised criticality in sandpiles. In the next subsection, we actually prove the described connection between ASM and SRW rigorously in dimension one.

2.2. The connection in one dimension

We consider the one-dimensional ASM on $\Omega_n := \{1,2\}^{\{1,2,\ldots,n\}}$, and formalise the informal earlier descriptions. The addition operator $a_i : \Omega_n \to \Omega_n$ represents the addition of a grain of sand at position i and the subsequent toppling of the system until a stable configuration has been reached, i.e. for $\omega \in \Omega_n$, $a_i(\omega)$ is the state of the one-dimensional ASM after adding a grain to site i and toppling to a stable configuration.

Let p be a probability measure on $\{1,2,\ldots,n\}$. The one-dimensional ASM starting from state $\omega_0 \in \Omega_n$ is the discrete time Markov chain $\{\omega_k : k \in \mathbf{N}\}$ with transition probabilities

$$\mathsf{P}(\omega_k = a_i(\omega) \mid \omega_{k-1} = \omega) = p_i,$$

for $i \in \{1,2,\ldots,n\}$. There is only one recurrent class, which is independent of p. We denote this class by R_n and

$$R_n = \{\omega \in \Omega_n : |\{i : \omega(i) = 1\}| \leq 1\}.$$

According to [9], the Abelian group acting on R_n, generated by the operators

$$a_i, i = 1, \ldots, n,$$

is cyclic. We denote the stationary distribution of the ASM on Ω_n by μ_n. The measure μ_n is uniform on R_n for all p (see [13] for a proof of this statement), which implies that in stationarity, the expected number of particles per site is given by

$$\frac{1}{n+1}\cdot 1+\frac{n}{n+1}\cdot 2=\frac{2n+1}{n+1};$$

this tends to 2 as $n\to\infty$.

Next consider the SRW in dimension one, starting at time zero in some state η. Associated to each site $i\in\mathbf{Z}$, there is a Poisson process at rate one. These Poisson processes are independent of each other and of the current state of the particle system. When the Poisson clock 'rings' at site i and $\eta(i)\geq 3$, then

$$\begin{array}{rcl}\eta(i) & \to & \eta(i)-2,\\ \eta(i-1) & \to & \eta(i-1)+1,\\ \eta(i+1) & \to & \eta(i+1)+1.\end{array}$$

When the Poisson clock rings at site i and $\eta(i)\leq 2$, nothing happens. We denote the realisation of the system at time t starting from $\eta\in\Omega$ by η_t; we write P^ν for the distribution of the SRW when the initial distribution is ν, and E^ν for the corresponding expectation operator. Writing A_j for the event that site j changes its state infinitely often, we prove the following theorem:

Theorem 2.1. *Let ν be a stationary ergodic measure on Ω. Write*

$$\int_\Omega \eta(0)\,d\nu(\eta)=:\xi.$$

Then for $\xi<2$,

$$\mathsf{P}^\nu(A_j)=0\quad\textit{for all } j\in\mathbf{Z},$$

and for $\xi>2$,

$$\mathsf{P}^\nu(A_j)=1\quad\textit{for all } j\in\mathbf{Z}.$$

The interesting point of this result is that the one-dimensional sandpile model and the SRW have in some sense the same 'critical density'. When we consider larger and larger one-dimensional sandpiles in equilibrium, the expected number of grains of sand per site tends to 2. In the particle system above we see that when there are on average less than 2 particles per site the process 'freezes' locally, while we keep seeing changes when there are on average more than 2 particles per site. This confirms the intuitive connection described in Section 2.1 above. Our proof actually uses the correspondence between ASM and SRW explicitly.

Before we prove Theorem 2.1, we show that the expected number of particles per site in the SRW is constant as a function of time.

Proposition 2.1. *Let ν be a stationary (i.e. translation invariant) measure on Ω. Then for all $t\geq 0$ and $z\in\mathbf{Z}$, $\mathsf{E}^\nu(\eta_t(z))=\mathsf{E}^\nu(\eta_0(z))$.*

Proof. Consider the SRW starting from configuration $\eta \in \Omega$. Let, for $z \in \mathbf{Z}$ and $t \geq 0$, $n(z, \eta, t)$ be the number of moments during the period $[0, t]$ that there were at least two particles present at site z and the Poisson clock at site z rang. That is, $n(z, \eta, t)$ is the number of times until time t that two particles from site z were transferred to the neighbouring sites. Observe that

$$\eta_t(z) = \eta_0(z) - 2n(z, \eta, t) + n(z+1, \eta, t) + n(z-1, \eta, t). \tag{2.1}$$

If η_0 is chosen according to a stationary measure ν, then $n(z, \eta, t)$ is also distributed according to a stationary measure. Observe that $E^{\nu}\left(n(z, \eta, t)\right) < \infty$ since $n(z, \eta, t)$ is dominated by a rate 1 Poisson process. Taking expectations in (2.1) leads to the desired result. $\square$

Proof of Theorem 2.1. By stationarity, it suffices to consider $j = 0$. Observe that A_0 is translation invariant, since if A_0 occurs, then infinitely many times, two particles jump from site 0 to both the sites -1 and 1, which implies that also A_1 occurs. By ergodicity we have that either $\mathsf{P}^{\nu}(A_0) = 0$ or $\mathsf{P}^{\nu}(A_0) = 1$.

Let ν be a stationary ergodic measure with $\xi = \int_{\Omega} \eta(0)\, d\nu(\eta) < 2$. We will assume $\mathsf{P}^{\nu}(A_0) = 1$ and derive a contradiction. The idea is that we compare the number of particles in the SRW in a certain block to the right of site 0 with the number of grains in a one-dimensional ASM of the same length. We will see that after some time, with large probability, the number of particles in the block to the right of site 0 in the SRW is not smaller than the number of particles in a (minimal) recurrent configuration of the corresponding ASM. Since the average number of grains per site in a recurrent configuration of the sandpile is arbitrarily close to 2 if we take the sandpile large enough (and after enough additions) this will lead to a contradiction, since the expected number of particles per site in the SRW is strictly smaller than two.

We make this precise. Suppose that $\mathsf{P}^{\nu}(A_0) = 1$. Fix $M \in \mathbf{N}$ such that $(2M-1)/M > \xi$ and consider the ASM on Ω_M. Let G_M be the group generated by the addition operators $\{a_i : i \in \{1, 2, \ldots, M\}\}$. As mentioned before, G_M is a cyclic group. We can see as follows that this implies that if enough grains have been added at the left most site in the ASM, the ASM must be in a recurrent state (this is well known but we like to present the details): Write $k_1, k_2, \ldots, k_M$ for the smallest constants for which $a_1^{k_i} = a_i$, that is, $k_1 = 1$ and for $i \in \{2, 3, \ldots M\}$:

$$k_i = \min\{n \in \mathbf{N} : a_1^n = a_i\}.$$

When at least one grain is added at each site in the ASM on Ω_M, the process must be in a recurrent configuration, since for all $\omega \in \Omega_M$,

$$a_M \circ a_{M-1} \circ \cdots \circ a_1(\omega)$$

is recurrent. Indeed, we can re-order the a_i such that the process has visited the maximal configuration somewhere (this can be done by first applying the a_i for those i with $\omega(i) = 1$). This re-ordering has no effect on the final configuration, and once the process has visited the maximal configuration it can only change into other recurrent states. This also implies that after there have been $k_1 + \cdots + k_M$ additions at the first site, the ASM must be in a recurrent state since

$$a_M \circ \cdots \circ a_1 = a_1^{k_1+\cdots+k_M}.$$

Write B_T for the event that, in the SRW, there have been at least $2(k_1 + k_2 + \cdots + k_M)$ moments before time T at which there was a change of state at site zero, such that after this change there were three or more particles present at this site. Assuming $\mathsf{P}^\nu(A_0) = 1$, we have

$$\lim_{T\to\infty} \mathsf{P}^\nu(B_T) = 1.$$

Choose $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$ such that

$$(1-\varepsilon_1)(1-\varepsilon_2)\frac{2M-1}{M} > \xi;$$

choose T_1 such that

$$\mathsf{P}^\nu\left(A_{T_1}\right) > 1-\varepsilon_1,$$

and choose T_2 such that the probability that a rate 1 Poisson process has rung more than $k_1 + \cdots + k_M$ times before time T_2 is at least $1 - \varepsilon_2$.

We will show that at time $T_1 + T_2$, the expected number of particles in the SRW at the sites 1 up to M (inclusive) is strictly larger than $M\xi$, which contradicts Proposition 2.1.

At time $T_1 + T_2$, the probability that at least $k_1 + \cdots + k_M$ particles have been transferred from site 0 to site 1 is larger than $(1-\varepsilon_1)(1-\varepsilon_2)$. In the ASM on Ω_M, if we add at least $k_1 + \cdots + k_M$ particles to site 1, the system must be in a recurrent state (no matter what other additions have been made), which implies that there are at least $2M-1$ particles present in the ASM. In the SRW, if at least $k_1+\cdots+k_M$ particles have been transferred from site 0 to site 1, then also at least $2M-1$ particles must be present at the sites 1 up to M inclusive. Since, in that case, if we would just relax the configuration in the SRW at the sites 1 up to M as if it were an ASM, we would arrive at a recurrent state for the ASM, with at least $2M-1$ particles and this relaxation can only decrease the total number of particles in the SRW at the sites 1 up to M. From this we conclude that

$$\sum_{i=1}^{M} \mathsf{E}^\nu(\eta_{T_1+T_2}(i)) > (1-\varepsilon_1)(1-\varepsilon_2)(2M-1) > M\xi,$$

which is the desired contradiction.

For $\xi > 2$, the assumption that $\mathsf{P}^{\nu}(A_0) = 0$ will lead to a contradiction, as follows. Suppose that $\mathsf{P}^{\nu}(A_0) = 0$. Then there exist a (non-random) time $T > 0$ such that

$$\mathsf{P}^{\nu}\left(\eta_T(0) \text{ is constant for } t \geq T\right) > 0.$$

Fix a T as above and define

$$D_T(\eta)(i) = \begin{cases} 0 & \text{if } \eta_t(i) \text{ is not constant for } t \geq T, \\ 1 & \text{if } \eta_t(i) \text{ is constant for } t \geq T. \end{cases}$$

Let us call the sites with $D_T(\eta)(i) = 1$ *special sites.* Since the probability of seeing a special site is positive, ergodicity implies that there are arbitrarily large (in both the negative and positive direction) special sites.

Observe that if we relax the configuration of the SRW at time T between any two special sites as if it were a sandpile, no grains would leave that sandpile. So if we compute the average number of particles in the SRW between two special sites we get at most 2. By ergodicity and Proposition 2.1 we know that the average number of particles per site at time T in larger and larger blocks tends to $\xi > 2$. But above, we a.s. found a sequence of larger and larger blocks along which the average number of particles is at most two, which is the desired contradiction. □

2.3. A similar correspondence in two dimensions?

As mentioned earlier, we conjecture that also in higher dimensions the SRW and the ASM have the same critical density. Unfortunately, the proof for the one-dimensional case cannot simply be carried over to higher dimensions. In one dimension, we used that (if we look at larger and larger sandpiles) the average number of particles per site in *any* recurrent configuration tends to the critical density. In higher dimensions this is simply not true. Let us explain this for the two-dimensional case.

Consider the $n \times n$ ASM. Using the limiting height probabilities as computed in [17], the expected number of particles per site in equilibrium tends to about 3.12, for $n \to \infty$ (let us denote this limit by ξ_c^2). However, there exist both recurrent configurations with an average number of grains per site that is smaller than ξ_c^2 (e.g. the configuration with three grains on every site) and recurrent configurations with on average more than ξ_c^2 grains per site (e.g. the configuration with four grains on every site).

However, we expect the following conjecture to be true. We write $\Omega := \{1, 2, \ldots\}^{\mathbf{Z}^2}$ for the state space of the two-dimensional SRW process, η_t for the (random) configuration of the SRW at time t if its initial configuration was $\eta \in \Omega$, P^{ν} for the distribution of the process when the initial distribution is ν, E^{ν} for the corresponding expectation operator, and A_z for the event that in this process, site z changes its state infinitely often.

Conjecture 2.1. *Consider the SRW in dimension two. Let* ν *be a stationary ergodic measure on* Ω *and write* $\mathsf{E}^\nu(\eta_0(0,0)) := \xi$. *Then*

1) *for* $\xi < \xi_c^2$, $\mathsf{P}^\nu(A_z) = 0$ *for all* $z \in \mathbf{Z}^2$;
2) *for* $\xi_c < \xi \leq 4$, *there exists a* ν *with* $\mathsf{E}^\nu(\eta_0(0,0)) = \xi$ *and* $\mathsf{P}^\nu(A_z) = 1$ *for all* $z \in \mathbf{Z}^2$;
3) *for* $\xi > 4$ *and any* ν, $\mathsf{P}^\nu(A_z) = 1$ *for all* $z \in \mathbf{Z}^2$.

Observe that it is certainly not true that $\mathsf{P}^\nu(A_z) = 1$ for all ν with $\xi_c < \xi \leq 4$, since when the process starts from a stationary ergodic measure concentrating at $\{3,4\}^{\mathbf{Z}^2}$, no changes will occur at all.

We prove the following weaker result.

Proposition 2.2. *Consider the SRW in dimension two. Let* ν *be a stationary ergodic measure on* Ω *with* $\mathsf{E}^\nu(\eta_0(0,0)) := \xi$. *Then for* $\xi < 3$,

$$\mathsf{P}^\nu(A_z) = 0 \quad \textit{for all } z \in \mathbf{Z}^2.$$

Proof. Again, by ergodicity, either $\mathsf{P}^\nu(A_z) = 0$ for all z or $\mathsf{P}^\nu(A_z) = 1$ for all z. We use the same idea as in the one-dimensional case, observe that the higher dimensional analogue of Proposition 2.1 is also true. Let $\xi < 3$. According to [9], in an $n \times n$ ASM, the group formed by the n^2 addition operators of the sandpile is generated by the n operators which correspond to adding grains to one of the four sides of the $n \times n$ square. Assume that $\mathsf{P}^\nu(A_z) = 1$ for all z. This implies that when we consider the configuration of the SRW in an $n \times n$ box after a long enough time and we relax the configuration in the box as if it were a sandpile, then with a very high probability we get a recurrent (sandpile) configuration (this can be made precise in the same way as in the one-dimensional situation). The number of grains in any recurrent (sandpile) configuration is at least equal to the sum of the number of sites and the number of bonds in the ASM (see [9]), so for the $n \times n$ ASM, in a recurrent configuration, there are at least $n^2 + 2n(n-1) = 3n^2 - 2n$ grains. This implies that after a long enough time, we get a lower bound for the expected number of particles per site in the $n \times n$ box of the SRW that is arbitrarily close to 3 (by taking n large enough), which contradicts the fact that $\xi < 3$. □

3. The Bak – Sneppen evolution model

Consider a system with N species. These species are represented by N vertices on a circle, evenly spaced, say. Each of the species is assigned a so called *fitness*, a number between 0 and 1. The dynamics of evolution is modelled as follows. Every discrete time step, we choose the vertex with minimal fitness, and we think of the corresponding species as disappearing completely. This species is then replaced by a new one, with a fresh and independent fitness, uniformly distributed on $[0, 1]$.

So far, the dynamics does not have any interaction between the species, and does not result in an interesting process. Indeed, if we only replace the species with the lowest fitness, then it is easy to see that the system converges to a situation with all fitnesses equal to 1. Interaction is introduced by also replacing the two neighbours of the vertex with lowest fitness by new species with independent fitnesses. This interaction represents co-evolution of related species. This neighbour interaction makes the model very interesting from a mathematical point of view.

It is very simple to run this model on a computer. Simulations then suggest the following behaviour. For large N (see [12] and [1] for simulation results), it appears that the one-dimensional marginals are uniform (in the limit for $N \to \infty$) on $(f, 1)$ for some f whose numerical value is supposed to be close to 2/3. We refer to Figure 1 for a snapshot of the fitnesses in the Bak – Sneppen model in stationarity.

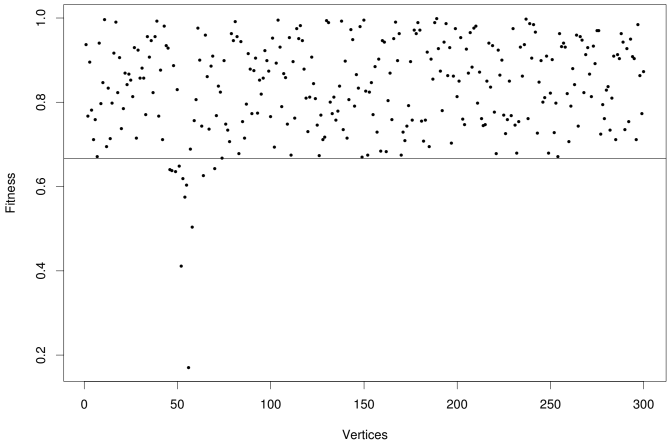

Figure 1. A snapshot of the fitnesses in the Bak – Sneppen model in stationarity.

This threshold value f is the basis for self-organised critical behaviour, according to [2], [1] and [12], as follows. Since in the limit there is no mass below f, one can look at so called *avalanches* of fitnesses below this threshold: start counting at the moment all fitnesses are above f and finish the counting at the first next moment that all fitnesses are above f again. The random number of updates, for instance, counted this way, is supposed to follow a power law, and so is the number of vertices involved in such an avalanche. This is in accordance with the paradigm of self-organised criticality, since there seems to be power law behaviour without any tuning of parameters.

Motivated by biological considerations concerning evolution, the question arises as to whether the Bak – Sneppen model has any direct concrete impact on biology. Per Bak considered the model as mainly of philosophical interest [1], the main point being that evolutionary rules can in fact explain the phenomenon of *punctuated equilibrium.* This last notion refers to the idea that evolution is not a gradual process, but takes places in bursts of activities, some bursts being small and others being huge. These bursts find their analogues in the various avalanches in the Bak – Sneppen model. However, scientists have tried to apply the Bak – Sneppen model to biological data, see for instance [5] and [11]. In these papers, one concentrates on a huge number of subsequent generations in a bacterial colony; the number of generations you should think of here are in the tens of thousands. The relative fitness of a generation is then defined as the relative growth rate of a typical member of this generation, when competing with a first-generation bacteria. To explain how the Bak – Sneppen model can play a role here, we should introduce the so-called *maximal avalanche decomposition.* Here the first avalanche threshold is defined to be the minimum fitness value from the initial fitness values. After this and every subsequent avalanche, another avalanche begins with the threshold chosen to be the new minimal value of the model; this is the maximal threshold choice. It is clear that this will lead to the Bak – Sneppen model being seen as a series of avalanches at strictly increasing thresholds. The *gap function* at time s is defined to be the avalanche threshold at time s. The gap function is a stepwise increasing function which jumps to a new value each time an avalanche finishes. Figure 2 shows a realisation of the gap function represented by the line, whilst the dots are the minimum fitness values at each time step.

The idea is that the increase of the fitness of subsequent generations, can be described by means of such a gap function; according to the cited papers, the fit is very good. Whether or not it is necessary to invoke the Bak – Sneppen model in these explanations is perhaps not for us to say.

Going back to the mathematics of the Bak – Sneppen model, it is not difficult to show that in the limit there can be no probability mass of the fitnesses below 1/3. The following result from [15] establishes that the one-dimensional marginals do not concentrate on 1 as $N \to \infty$. We denote by F_N the distribution function of the one-dimensional marginal in the stationary regime, in the system with N vertices.

Theorem 3.1. *If $q < 1$ is close enough to 1, then there exists $c_q > 0$, independent of N, such that*

$$F_N(q) > c_q.$$

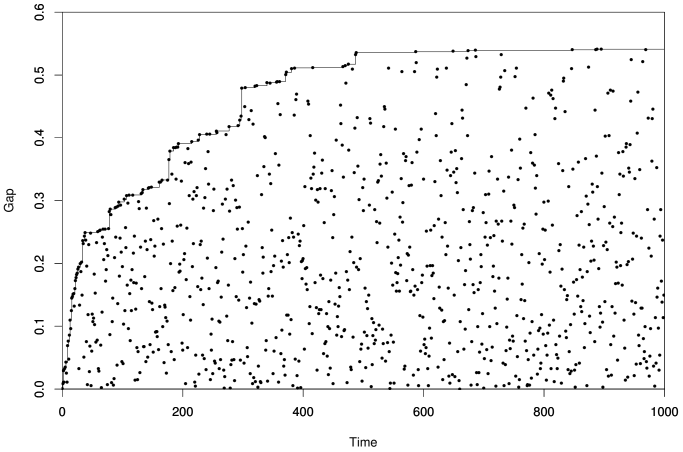

Figure 2. A realisation of the gap function.

3.1. Connection to a conventional system at criticality

As in the case of the Abelian sandpile model, the parametrised system to which we want to relate our model of self-organised criticality, lives in infinite volume. Clearly, the rules of the Bak – Sneppen model do not make immediate sense on an infinite line, since there need not be a smallest fitness among a collection of infinitely many fitnesses. However, the notion of an avalanche can be naturally extended to an avalanche on $\mathbf{Z}$, as follows.

Assume that every vertex $x \in \mathbf{Z}$ accommodates a fitness, a random variable with value in $[0, 1]$. Choose an arbitrary threshold $b > 0$. If the number of vertices with fitness below b is finite and positive, then the update rules of Bak – Sneppen model are still well-defined. Suppose, therefore, that at time 0 we have a configuration with all fitnesses above b. Irrespective of the various fitnesses, we start by updating 0 and its two neighbours, -1 and 1. If, after that, among $-1, 0, 1$ there are vertices with fitnesses below b, we choose the one with minimal fitness, and update it together with its two neighbours, and so on. As soon as we get a configuration with all fitnesses above b, we stop the procedure. Let us call the model described above the *Infinite Volume Bak – Sneppen model*, abbreviated IVBS. We define the *duration* $\eta(b)$, and the *range set* $\xi(b)$ of the avalanche in the IVBS as the number of time steps and the collection of vertices updated in the avalanche, respectively. Note, however, that it is perhaps possible (depending on the threshold b) that we never stop and that there always is at least one fitness below the threshold. In this case, we say that an *infinite avalanche* occurs. For an infinite avalanche, we set $\eta(b) = \infty$, and define $\xi(b)$ as the union of the vertices updated during $[0,m]$, $m \in \mathbf{N}$. It is easy to see that for an infinite avalanche, $|\xi(b)| = \infty$ a.s.

Now we can vary the threshold b and study the avalanche characteristics $\eta(b)$ and $\xi(b)$ as a function of b. To this end, we define

$$\begin{aligned} R_\infty(b) &= \mathsf{E}(|\xi(b)|), \\ D_\infty(b) &= \mathsf{E}(\eta(b)), \\ P_\infty(b) &= \mathsf{P}(|\xi(b)| = \infty). \end{aligned}$$

In [16] it is shown that $R_\infty(b)$, $D_\infty(b)$ and $P_\infty(b)$ are non-decreasing in b, and therefore it makes sense to define the critical thresholds:

$$\begin{aligned} b_c^p &= \inf\{b > 0\,;\, P_\infty(b) > 0\}, \\ b_c^r &= \inf\{b > 0\,;\, R_\infty(b) = \infty\}, \\ b_c^d &= \inf\{b > 0\,;\, D_\infty(b) = \infty\}. \end{aligned}$$

Since $P_\infty(b) > 0$ implies that $R_\infty(b) = \infty$, we have $b_c^r \leq b_c^p$. Since for any b it follows that $D_\infty(b) \geq R_\infty - 2$, we also have $b_c^d \leq b_c^r$. Thus the three critical thresholds are naturally ordered by $b_c^d \leq b_c^r \leq b_c^p$. It is not hard to see that $b_c^d > 0$, and in [15] it is shown that $b_c^p < 1$, which together imply that all critical values are non-trivial, that is, bounded away from 0 and 1.

It is natural to conjecture that these three critical thresholds are all equal. At this moment we do not know this, although it is proved in [16] that $b_c^d = b_c^r$.

One of the reasons to be interested in these critical thresholds, is the following result, also taken from [16]. This result relates the 'conventional' critical thresholds of the IVBS to the limit (for the number of species tending to infinity) of the one-dimensional marginals of the stationary distributions of the (finite) Bak – Sneppen models. It states that if all three critical thresholds in the IVBS are equal, then indeed the one-dimensional mariginals of the Bak – Sneppen models tend to the uniform distribution above the common value of the three critical thresholds.

Theorem 3.2. *If $b_c^d = b_c^r = b_c^p$ and their common value is denoted by b_c, say, then $F_N \to U(b_c, 1)$ in distribution, where $U(a,b)$ denotes the distribution function of a uniform random variable on the interval (a,b).*

Now that we have set up a parametric model and reviewed the main results in the literature, we can discuss the apparent self-organised critical behaviour of the Bak – Sneppen model in connection with the critical thresholds of the IVBS defined above. In fact the Bak – Sneppen model 'recovers' the critical threshold of the IVBS. We will explain why this happens; the idea is very similar to the discussion around the Abelian sandpile model of the previous section.

It is to be expected that in the IVBS, when $b < b_c$ (we assume here that all three critical thresholds are equal), the size of a b-avalanche (in both duration and range) decays exponentially, whilst for $b = b_c$ we should have ordinary critical behaviour, that is, power law decay of range and duration. Finally, for $b > b_c$ there is no decay to zero at all. We will explain how this classical behaviour of the IVBS becomes 'visible' in

the SOC behaviour of the Bak – Sneppen model on N vertices, provided that N is sufficiently large. That is, we will *assume* that the IVBS behaves as described above and show that it is very plausible that this leads to the SOC behaviour of the Bak – Sneppen model.

Indeed, since the range and duration of a b-avalanche in the IVBS on a level $b < b_c$ decay exponentially, fitnesses in the Bak – Sneppen model can not be below b for a very long time, and most fitnesses will very soon be larger than this particular b (apart from a localised region where the current action of replacing fitnesses takes place). Since, for N large enough, this region is very small compared to the size of the system, in the limit as N tends to infinity, we will not see fitnesses below any $b < b_c$. On the other hand, in the IVBS, when b approaches b_c from below, the exponents of the exponential decay will become smaller and smaller, and accordingly, avalanches will tend to be larger and larger in both duration and range, up to the point b_c where avalanches (seen on the infinite line) will have infinite expected duration and range. Hence, in the Bak – Sneppen model, when we take $b > b_c$ and N large, then the b-avalanches that we see will take very long, and fitnesses that end up being larger than this b will take uniform values on $(b, 1)$. This explains that the avalanches that we observe in the Bak – Sneppen model, for N large at least, are b-avalanches for b close to the critical threshold b_c. This explains the apparent power law behaviour of the avalanches in the Bak – Sneppen model, in the sense that we see avalanches corresponding to the critical threshold in the IVBS.

4. Conclusion

It appears that self-organised criticality is not a big surprise once you see the construction in its proper context. In a conventional equilibrium system, one needs to carefully tune a set of parameters as to obtain critical behaviour. In a model which exhibits self-organised critical behaviour, this tuning of parameters doesn't seem to be necessary, and the critical behaviour appears to arise for free. However, from our point of view, in the case of the models for self-organised critical behaviour which are here discussed (the ASM and the Bak – Sneppen model), the tuning of parameters has been replaced by the *very careful* choice of the model. The characteristics of the model need to be defined in such a way that the model automatically attracts itself to the critical point of a suitable equilibrium system.

In this paper, we have explained the reason for self-organised critical behaviour in two of the main models, *given* the classical behaviour of suitable related conventional systems. To explain SOC behaviour from this point of view then, is to argue that the model behaves as a conventional system in criticality. This, clearly, is much simpler than proving power law behaviour rigorously in the model for self-organised criticality itself. We expect that further research will yield similar rigorous verifications of these ideas.

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