Geometric properties of two-dimensional near-critical percolation

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Abstract

Using certain scaling relations for two-dimensional percolation, we study some global geometric properties of "near-critical" scaling limits. More precisely, we show that when the lattice spacing is sent to zero, there are only three possible types of scaling limits for the collection of percolation interfaces: the trivial one consisting of no curves at all, the critical one corresponding to the full scaling limit of critical percolation, and one in which any deterministic point in the plane is surrounded with probability one by a largest loop and by a countably infinite family of nested loops with radii going to zero. All three cases occur. The first one corresponds to the subcritical and supercritical phases. The last one corresponds to the near-critical regime, with a scaling limit which is nontrivial, like the critical one, but which, unlike the critical one, is not scale invariant and thus retains a flavor of supercritical percolation.

Keywords: near-critical percolation, off-critical regime, massive scaling limit
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1 Introduction

The rigorous geometric analysis of the continuum scaling limit of two-dimensional critical site percolation on the triangular lattice has made much progress in recent years. In particular, the work of Schramm [21] and Smirnov [22, 23] has allowed to identify the scaling limit of critical interfaces (i.e., cluster boundaries) in terms of the Schramm-Loewner Evolution (SLE) (see also [10, 24]). Based on that, Camia and Newman have constructed [8] a process of continuum nonsimple loops in the plane, and proved [9] that it coincides with

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the scaling limit of the collection of all percolation interfaces (the *full* scaling limit). The use of the SLE technology and computations [16, 17], combined with Kesten's scaling relations, has also allowed to derive important properties of percolation such as the values of some critical exponents [18, 25].

In later work [6, 7], based on heuristic arguments, Camia, Fontes and Newman have proposed an approach for obtaining a one-parameter family of *near-critical* scaling limits with density of open sites (or bonds) given by

$$p = p_c + \lambda \delta^{\alpha} \tag{1}$$

where p_c is the critical density, δ is the lattice spacing, $\lambda \in (-\infty, \infty)$, and α is set equal to 3/4 to get nontrivial λ-dependence in the limit $\delta \to 0$ (see below and [1, 2, 5]). The approach proposed in [6, 7] is based on the critical full scaling limit and the "Poissonian marking" of some special ("macroscopically pivotal") points, and it leads to a conceptual framework that can in principle describe not only the scaling limit of near-critical percolation but also of related two-dimensional models such as dynamical percolation, the minimal spanning tree and invasion percolation (see [7]).

In [6, 7] some geometric properties of near-critical scaling limits of two-dimensional percolation are conjectured. The main contribution of this note is to prove rigorously those properties. Namely, using ideas and tools that originate in Kesten's work [15], we will prove the following:

In any near-critical scaling limit, every deterministic point of the plane is almost surely surrounded by a largest loop and by countably infinite family of nested loops with radii going to zero.

We will not identify the near-critical scaling limit with the $\delta \to 0$ limit of a percolation model with density of open sites given by (1) with $\alpha = 3/4$, but rather through the property that the correlation length (see Section 2 for a precise definition) remains bounded away from 0 and ∞ in the limit $\delta \to 0$. (For a discussion of why at present it is not possible to identify the near-critical regime with the case $\alpha = 3/4$ of (1), see the paragraph following Corollary 3.5). Nonetheless, our results imply that the near-critical regime is not empty and that when the exponent α in (1) is different from 3/4, the scaling limit is either trivial (for $\alpha < 3/4$) or the same as the critical one (for $\alpha > 3/4$).

More precisely, we will show (see Theorem 3.4 in Section 3.4) that there are only three possible, non-void, types of scaling limits: the trivial one consisting of no curves at all, the critical one corresponding to the full scaling limit of critical percolation, and the near-critical one described above.

The three regimes in Theorem 3.4 correspond to those in Proposition 4 of [20], which contains, among other things, results analogous to some of ours in the context of a single percolation interface and its scaling limit. We learned about [20] while working on this paper, and used part of their Proposition 4 (and its proof) in proving one of our results (namely, case *(2)* in Theorem 3.4).

The above geometric characterization of near-critical scaling limits shows that they are not scale invariant and differ qualitatively from the critical scaling limit at large scales, since in the latter there is no largest loop around any point. At the same time, they resemble the critical scaling limit at short scales because of the presence of infinitely many nested loops with radii going to zero around any given point. Depending on the context, this situation is described as *near-critical*, *off-critical* or *massive* scaling limit (where "massive" refers to the persistence of a macroscopic correlation length, which should give rise to what is known in the physics literature as a "massive field theory").

It is our understanding that significant progress has recently been made [12] in proving the approach proposed in [6, 7]. A consequence would be that the subsequential limits discussed in this paper are in fact limits.

2 Notation and some background

Let $\mathcal{H}_\delta$ denote honeycomb lattice with lattice spacing $\delta > 0$, and its dual, the triangular lattice $\mathcal{T}_\delta$, embedded in $\mathbb{R}^2$ as in Figure 1. A site of the triangular lattice is identified with the face of the honeycomb lattice that contains it.

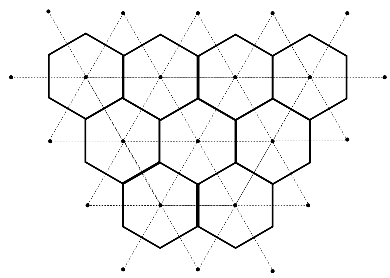

Figure 1: Embedding of the triangular and hexagonal lattices in $\mathbb{R}^2$.

Throughout this paper, we are interested in Bernoulli site percolation on $\mathcal{T}_\delta$, defined as follows. Each site of $\mathcal{T}_\delta$ is independently declared open, and the corresponding hexagon colored white, with probability p. Sites that are not open are declared closed, and the corresponding hexagons are colored black. We denote by $P_{\delta,p}$ the probability measure corresponding to site percolation on $\mathcal{T}_\delta$ with parameter p. It is well known [14] that percolation on the triangular lattice has a phase transition at $p = 1/2$.

A *path* of length n in $\mathcal{T}_\delta$ is a sequence of sites $(x_1, x_2, \ldots, x_n)$ of $\mathcal{T}_\delta$ such that x_k and x_{k+1} are adjacent in $\mathcal{T}_\delta$ for all $k = 1, \ldots, n-1$. A *circuit* of length n is a path $(x_1, x_2, \ldots, x_n)$ such that x_n is adjacent to x_1. We define the diameter of a set $U \subset \mathbb{R}^2$ as

$$\operatorname{diam}(U) := \sup\{|x - y| : x, y \in U\},$$

where $|\cdot|$ denotes Euclidean distance. We call a path or a circuit open or white (respectively, closed or black) if all its sites are open (resp., closed).

The edges between neighboring hexagons with different colors form *interfaces*. A concatenation of such edges will be called a *boundary path* or a *boundary circuit* if it forms a closed curve. Note that boundary curves and circuits are always simple (i.e., no self-touching occurs).

We define the *box* $B(x;r)$ as a parallellogram with center x, Euclidean side-lengths r and sides which are parallel to two of the axes of the triangular lattice, as in Figure 2. For $0 < r < R$, we define the *annulus* $A(x;r,R)$ as

$$A(x;r,R) := B(x;R) \setminus B(x;r).$$

When x is the origin, we will write $B(r)$ and $A(r,R)$, respectively. Note that boxes and annuli are defined in terms of the Euclidean metric and not relative to the lattice spacing.

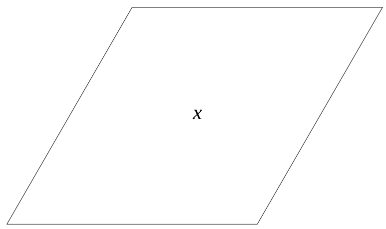

Figure 2: The box $B(x;r)$.

The notion of *correlation length* will be very important. Various equivalent definitions are possible; we choose the one, introduced in [11] and also used in [15], that is most suitable for our purposes. Let n be an integer and $H^w_\delta(n)$ be the event that a percolation configuration on $\mathcal{T}_\delta$ contains an open (white) path inside $B(n\delta)$ starting at its "left side" and ending at its "right side." For each $\epsilon \in (0,1/2)$, the correlation length $L_\epsilon(p)$ is defined as follows.

$$L_\epsilon(p) := \min\{n : P_{\delta,p}(H^w_\delta(n)) > 1/2 + \epsilon\} \text{ when } p > 1/2,$$
$$L_\epsilon(p) := \min\{n : P_{\delta,p}(H^w_\delta(n)) < 1/2 - \epsilon\} \text{ when } p < 1/2.$$

We also define $L_\epsilon(1/2) = \infty$ for all $\epsilon \in (0,1/2)$. Note that in the definition above, the correlation length is measured in lattice spacings (rather than in the Euclidean metric), and is therefore independent of δ.

An important fact about the correlation length is that the ϵ in the definition is unimportant, due to the following result [15]: for any $\epsilon, \epsilon' \in (0,1/2)$ we have

$$L_\epsilon(p) \asymp L_{\epsilon'}(p),$$

where $f \asymp g$ means that the ratio between the functions f and g is bounded away from 0 and ∞ as $p \to 1/2$. We will also need the following three results. The first is a consequence of Theorem 26 of [19] (see also Theorem 1 of [15] for a similar result).

Lemma 2.1 *Consider percolation on $\mathcal{T}_\delta$ with parameter p and let $C^w(r,R)$ (resp., $C^b(r,R)$) be the event that the annulus $A(r,R)$ is crossed (from the inner to the outer boundary) by an open/white (resp., closed/black) path. Then,*

$$P_{\delta,p}(C^w(r,R)) \asymp P_{\delta,p}(C^b(r,R)) \asymp P_{\delta,1/2}(C^b(r,R)) = P_{\delta,1/2}(C^w(r,R))$$

uniformly in p and $0 < r \le R \le \delta L_\epsilon(p)$.

We interpret this result as follows: on a scale not larger than the correlation length, percolation with parameter p looks roughly like critical percolation.

The second result is as follows (see, e.g., [4, 19] for more explanation and references).

Lemma 2.2 *Consider percolation on $\mathcal{T}_\delta$ with parameter $p \ge 1/2$, and let*

$$D_r = \{\exists \text{ closed circuit } S \text{ around the origin with } diam(S) \ge r\}.$$

Then, for each $\epsilon \in (0, 1/2)$ there exist two constants $C_1 = C_1(\epsilon) < \infty$ and $C_2 = C_2(\epsilon) > 0$ such that

$$P_{\delta,p}(D_r) \le C_1 \exp\left(-\frac{C_2 r}{\delta L_\epsilon(p)}\right). \tag{2}$$

The third result is the celebrated power law for the correlation length [25] (see also [19]): as $p \to 1/2$,

$$L_\epsilon(p) = |p - 1/2|^{-4/3+o(1)}. \tag{3}$$

3 Scaling limits

We turn our attention to the main object of study in this article – the scaling limit of the collection of all interface circuits. We will follow the approach of [9], using the topology introduced in [2].

3.1 Compactification of $\mathbb{R}^2$

When taking the scaling limit as the lattice spacing $\delta \to 0$ one can focus on fixed finite regions, $\Lambda \subset \mathbb{R}^2$, or consider the whole $\mathbb{R}^2$ at once. The second option avoids dealing with boundary conditions, but requires an appropriate choice of metric.

A convenient way of dealing with the whole $\mathbb{R}^2$ is to replace the Euclidean metric with a distance function $\Delta(\cdot,\cdot)$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$\Delta(u,v) := \inf_{\varphi} \int (1 + |\varphi|^2)^{-1}\, ds,$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parameterized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere.

3.2 The space of curves

In dealing with the scaling limit we use the approach of Aizenman-Burchard [2]. Denote by $\mathcal{S}_R$ the complete separable metric space of continuous curves in the closure $\overline{\mathbb{D}}_R$ of the disc $\mathbb{D}_R$ of radius R with the metric (4) defined below. Curves are regarded as equivalence classes of continuous functions from the unit interval to $\overline{\mathbb{D}}_R$, modulo monotonic reparametrizations. γ will represent a particular curve and $\gamma(t)$ a parametrization of γ; $\mathcal{F}$ will represent a set of curves (more precisely, a closed subset of $\mathcal{S}_R$). $\mathrm{d}(\cdot,\cdot)$ will denote the uniform metric on curves, defined by

$$\mathrm{d}(\gamma_1,\gamma_2) := \inf \sup_{t\in[0,1]} |\gamma_1(t)-\gamma_2(t)|, \tag{4}$$

where the infimum is over all choices of parametrizations of γ_1 and γ_2 from the interval $[0,1]$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\mathrm{dist}(\mathcal{F},\mathcal{F}') \leq \varepsilon \Leftrightarrow (\forall\, \gamma \in \mathcal{F},\, \exists\, \gamma' \in \mathcal{F}' \text{ with } \mathrm{d}(\gamma,\gamma') \leq \varepsilon, \text{ and vice versa}). \tag{5}$$

The space Ω_R of closed subsets of $\mathcal{S}_R$ (i.e., collections of curves in $\overline{\mathbb{D}}_R$) with the metric (5) is also a complete separable metric space. For each fixed $\delta > 0$, the random curves that we consider are polygonal paths on the edges of the hexagonal lattice $\mathcal{H}_\delta$, dual to the triangular lattice $\mathcal{T}_\delta$.

We will also consider the complete separable metric space $\mathcal{S}$ of continuous curves in $\dot{\mathbb{R}}^2$ with the distance

$$\mathrm{D}(\gamma_1,\gamma_2) := \inf \sup_{t\in[0,1]} \Delta(\gamma_1(t),\gamma_2(t)), \tag{6}$$

where the infimum is again over all choices of parametrizations of γ_1 and γ_2 from the interval $[0,1]$. The distance between two closed sets of curves is again defined by the induced Hausdorff metric as follows:

$$\mathrm{Dist}(\mathcal{F},\mathcal{F}') \leq \varepsilon \Leftrightarrow (\forall\, \gamma \in \mathcal{F},\, \exists\, \gamma' \in \mathcal{F}' \text{ with } \mathrm{D}(\gamma,\gamma') \leq \varepsilon \text{ and vice versa}). \tag{7}$$

The space Ω of closed sets of $\mathcal{S}$ (i.e., collections of curves in $\dot{\mathbb{R}}^2$) with the metric (7) is also a complete separable metric space. We denote by $\mathcal{B}$ its Borel σ-algebra.

When we talk about convergence in distribution of random curves, we always mean with respect to the uniform metrics (4) or (6), while when we deal with closed collections of curves, we always refer to the metrics (5) or (7). In this paper, the space Ω of closed sets of $\mathcal{S}$ is used for collections of boundary circuits and their scaling limits.

3.3 Existence of subsequential scaling limits

Aizenman and Burchard [2] formulate a hypothesis that implies, for every sequence $\delta_j \downarrow 0$, the existence of a scaling limit along some subsequence $\{\delta_{j_i}\}$. The hypothesis in [2] is formulated in terms of crossings of *spherical* annuli, but one can work with the annuli defined in Section 2 just as well. In order to state it, we need one more definition. For

$\delta > 0$, let μ_δ be any probability measure supported on collections of curves that are polygonal paths on the edges of the honeycomb lattice $\mathcal{H}_\delta$.

In our context, the hypothesis is as follows.

Hypothesis 3.1 *For all $k < \infty$ and for all annuli $A(x; r, R)$ with $\delta \le r \le R \le 1$, the following bound holds uniformly in δ:*

$$\mu_\delta \left(A(x; r, R) \text{ is crossed by } k \text{ disjoint curves} \right) \le K_k \left(\frac{r}{R} \right)^{\phi(k)}$$

for some $K_k < \infty$ and $\phi(k) \to \infty$ as $k \to \infty$.

The following theorem follows from a more general result proved in [2].

Theorem 3.2 ([2]) *Hypothesis 3.1 implies that for any sequence $\delta_j \downarrow 0$, there exists a subsequence $\{\delta_{j_i}\}_{i \in \mathbb{N}}$ and a probability measure μ on Ω such that $\mu_{\delta_{j_i}}$ converges weakly to μ as $i \to \infty$.*

It was already remarked in [2] that the above hypothesis can be verified for two-dimensional critical and near-critical percolation (using the BK inequality [3]). The same conclusion follows from results in [19], and is obtained in Proposition 1 of [20]. We will need a slightly more general result, stated and proved below.

Lemma 3.3 *Let $\{\mu_{\delta_j, p_j}\}_{j \in \mathbb{N}}$ be a sequence of measures on boundary paths induced by percolation on $\mathcal{T}_{\delta_j}$ with parameters p_j. For* any *sequence $\delta_j \to 0$ and* any *choice of the collection $\{p_j\}_{j \in \mathbb{N}}$, Hypothesis 3.1 holds.*

Proof. First of all, take $p_j = 1/2$ for all j, and observe that the number of boundary paths crossing an annulus is necessarily even. For $k = 2$, observe that if two disjoint boundary paths cross the annulus, then the annulus must also be crossed by a closed path, and so it cannot contain an open circuit surrounding the x. According to the RSW theorem (see, e.g., [14, 13]), since $p_j = 1/2$ for all j, the probability of this last event is bounded below by a number $1 - g(r/R) > 0$ depending on the ratio r/R only. Therefore, for $k = 2$, the probability of the event we are interested in is bounded above by $g(r/R) < 1$. We now take $K_k = 1$ and define $\phi(2)$ via

$$g(r/R) = (r/R)^{\phi(2)}.$$

For general (even) k, observe that if there are k disjoint boundary paths crossing the annulus, then the annulus must also be crossed by $k/2$ disjoint closed paths. The probability that this happens can be bounded above by $g(r/R)^{k/2}$, using the BK-inequality [3]. Hence we can take $\phi(k) = k\phi(2)/2$ and Hypothesis 3.1 is verified for this choice of $\phi(k)$.

Next consider a decreasing sequence $p_j \downarrow 1/2$. We can use the previous result to obtain the same bound for this case. Indeed, as before, if there are k boundary paths crossing the annulus $A(x;r,R)$, then the annulus must also be crossed by $k/2$ disjoint closed paths. Since $p_j \geq 1/2$ for all j, the probability that this happens is bounded above by

$$P_{\delta,1/2}\left(A(x;r,R) \text{ is crossed by } k/2 \text{ disjoint closed paths}\right) \leq \left(\frac{r}{R}\right)^{\phi(k)},$$

which provides the require bound, independently of j.

The same uniform bound for an increasing sequence $p_j \uparrow 1/2$ follows from swapping black and white in the above argument. Any other sequence $p_j \to 1/2$ can be easily treated by splitting it in two subsequences $\{p'_j\}$ and $\{p''_j\}$ such that $p'_j \downarrow 1/2$ and $p''_j \uparrow 1/2$. □

3.4 Three types of scaling limits

Theorem 3.4 and Lemma 3.3 guarantee the existence of subsequential scaling limits of $\mu_{\delta,p}$ as $\delta \to 0$. With this we are now ready to state the main result of the paper. Due to the black/white symmetry of the model, we can restrict attention to the case $p \geq 1/2$ without loss of generality. For $\epsilon \in (0,1/2)$, we define

$$p^+_\epsilon(n) := \inf\{p : P_{\delta,p}(H^w_\delta(n)) > 1/2 + \epsilon\}.$$

Note that, like the correlation length, $p^+_\epsilon(n)$ is independent of δ. (An analogous $p^-_\epsilon(n)$ can be defined for $p \leq 1/2$).

Theorem 3.4 *Suppose that μ is the weak limit of a sequence $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$, with $\delta_j \to 0$ as $j \to \infty$ and $p_j \geq 1/2$ for all j. Then one of the following non-void scenarios holds.*

(1) Trivial scaling limit: μ-a.s. there are no loops of diameter larger than zero.

(2) Critical scaling limit: μ coincides with the full scaling limit of critical percolation.

(3) Near-critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a largest loop and by a countably infinite family of nested loops with radii going to zero.

Moreover, the third scenario can be realized by taking $0 < \epsilon_1 < \epsilon_2 < 1/2$ and (an appropriate subsequence of) $\{p_j\}_{j\in\mathbb{N}}$ chosen so that $p^+_{\epsilon_1}(1/\delta_j) \leq p_j \leq p^+_{\epsilon_2}(1/\delta_j)$ for every j.

The first case corresponds to the scaling limit of the subcritical and supercritical phases. The second one to sequences $\{p_j\}$ that converge to $1/2$ so fast that the critical scaling limit arises. The last case is particularly interesting because the scaling limit is nontrivial, like the critical one, but unlike the critical one it is not scale invariant and thus retains a flavor of supercritical percolation. This situation is often described as *near-critical* or *off-critical* due to the persistence of a macroscopic correlation length. Theorem 3.4 and (3) readily imply the following result.

Corollary 3.5 *Consider a percolation model on $\mathcal{T}_\delta$ with density of open sites given by $p = 1/2 + \lambda\delta^\alpha$ with $\alpha \neq 3/4$. Then, for every $\lambda \in (-\infty, \infty)$, the scaling limit is unique and moreover*

- *if $\alpha < 3/4$, the scaling limit is trivial,*
- *if $\alpha > 3/4$, the scaling limit is the same as the critical one.*

It is natural to conjecture that the near-critical regime (case *(3)* in Theorem 3.4) corresponds to the case $\alpha = 3/4$. However, the power law (3) only tells us that for $\alpha = 3/4$ the correlation length remains bounded as $\delta \to 0$, it does not guarantee that it also remains bounded away from zero. In order to make the identification between the near-critical regime and the case $\alpha = 3/4$, one would need to replace (3) with $L_\epsilon(p) \asymp |p - 1/2|^{-4/3}$.

4 Proof of Theorem 3.4

We first show how assuming different behaviors for the correlation length leads to the three scenarios described in the theorem. Later we will prove that those three scenarios are non-void and are the only three possibilities.

(1) Suppose that for some $\epsilon \in (0, 1/2)$, $\delta_j L_\epsilon(p_j) \to 0$ as $j \to \infty$. Let E_l denote the event (in $\mathcal{B}$) that there exists a boundary circuit γ surrounding the origin with $\text{diam}(\gamma) > l$. Remember that $\text{diam}(\cdot)$ denotes the diameter in the Euclidean metric. Note that in order to have a boundary circuit of diameter larger than l, there must be a black (closed) circuit with diameter at least l. Therefore, for each fixed l, using Lemma 2.2,

$$\mu_{\delta_j,p_j}(E_l) \leq P_{p_j}^{\delta_j}(D_l) \leq C_1 \exp\left(-\frac{C_2 l}{\delta_j L_\epsilon(p_j)}\right) \longrightarrow 0 \quad \text{as} \quad j \to \infty.$$

Since μ_{δ_j,p_j} converges weakly to μ in the topology induced by (7) and E_l is open in that topology, it follows readily that $\mu(E_l) = 0$ for every $l > 0$.

(2) Suppose that for some $\epsilon \in (0, 1/2)$, $\delta_j L_\epsilon(p_j) \to \infty$ as $j \to \infty$. One can then prove that μ coincides with the critical scaling limit following the same strategy used in [9], combined with (a suitable version of) Theorem 1 of [15] and Proposition 4 of [20]. One consequence of the latter is that, under the above assumption, the scaling limit of a percolation exploration path inside any disc of bounded radius leads to SLE_6. We note that the proof of this fact for a disc with Euclidean diameter L only requires that $\delta_j L_\epsilon(p_j) \geq L$ for each sufficiently large j, and relies entirely on the RSW theorem, on Theorem 1 of [15] (see also [19]), and on bounds on the probabilities of six crossings of an annulus and of three crossings of a semi-annulus contained in a half-plane.

The proof of the existence and properties of the full scaling limit as described in [9] relies on a coupling between an algorithmic construction in the continuum and an analogous construction for discrete percolation on the triangular lattice. The algorithmic construction in the continuum takes place in the unit disc, and each individual step consists in generating an SLE_6 path. Each step of the discrete construction is a percolation exploration process. Using the RSW theorem and bounds on the probabilities of six crossings of an annulus and of three crossings of a semi-annulus contained in a half-plane, it is shown in [9] that the discrete construction converges to the continuum one in the scaling limit, in the topology induced by (7). Using the same tools (and the Harris-FKG inequality for increasing events), it is also shown that the discrete construction finds all boundary circuits of Euclidean diameter larger than any $l > 0$ in a number of steps $N_\delta(l)$ that is bounded in probability as $\delta \to 0$. It follows that the discrete construction generates, as $\delta \to 0$, the scaling limit of all boundary circuits contained in the unit disc, which then must coincide with the process generated by the continuum construction. In order to obtain the full scaling limit in the plane, the unit disc is replaced by a growing sequence of discs of diameter R, with $R \to \infty$, generating family of probability measures corresponding to the scaling limits inside the various discs. Kolmogorov's extension theorem is then used to show the existence of a unique extension of the family of measures to a measure supported on $\dot{\mathbb{R}}^2$.

In the present context, under the above assumption, for any $R < \infty$, for j sufficiently large, the discrete construction can still be coupled to the continuum construction of [9] thanks to Proposition 4 of [20] and (a suitable version of) Theorem 1 of [15]. The first assures the convergence of the individual steps of the discrete construction (i.e., percolation exploration processes) to SLE_6 processes; the second insures the use of the RSW theorem and the necessary bounds on the probabilities of six crossings of an annulus and of three crossings of a semi-annulus contained in a half-plane (for more details on this point, see [19, 20]).

(3) Suppose that for some $\epsilon \in (0, 1/2)$, $\delta_j L_\epsilon(p_j)$ stays bounded away from both 0 and ∞ as $j \to \infty$. That is, there exist $\beta > 0$ and $K < \infty$ such that $\beta \leq \delta_j L_\epsilon(p_j) \leq K$ for each j sufficiently large. To show the a.s. existence of infinitely many boundary circuits around the origin (or any other deterministic point), we proceed as follows. Consider the sequence of annuli $A_1 = A(\beta/2, \beta), A_2 = A(\beta/4, \beta/2), \dots, A_k = A(\beta/2^k, \beta/2^{k-1}), \dots$, and denote by F_m the event (in $\mathcal{B}$) that there is (at least) one boundary circuit in (at least) m of the annuli A_k with k odd. Note that in order to guarantee the presence of a boundary circuit inside the annulus A_k, it suffices to have, for example, an open (white) circuit in $A(3\beta/2^{k+1}, \beta/2^{k-1})$ and a closed (black) circuit in $A(\beta/2^k, 3\beta/2^{k+1})$ (note that those two annuli are disjoint and their union is A_k). Since $\delta_j L_\epsilon(p_j) \geq \beta$ for each large j, it follows from Lemma 2.1 and the RSW theorem that the probability to find an open circuit in $A(3\beta/2^{k+1}, \beta/2^{k-1})$ and a closed circuit in $A(\beta/2^k, 3\beta/2^{k+1})$ is bounded away from 0 as $j \to \infty$, uniformly in k. Therefore, for every $m < \infty$, $\mu_{\delta_j,p_j}(F_m) \to 1$ as $j \to \infty$. Using only annuli A_k with k odd guarantees that when F_m occurs, the m boundary circuits are well separated, which immediately implies that F_m is closed. (Two boundary circuits that are not separated could in principle collapse on each other in the scaling limit, giving rise to a configuration with a smaller number of disjoint circuits. Although one can prove by other means that this cannot happen, our definition of F_m means that we do not need to worry about this possibility). We can thus conclude that $\mu(F_m) \geq \limsup_j \mu_{\delta_j,p_j}(F_m) = 1$.

Our next goal is to prove the a.s existence of a largest boundary circuit surrounding the origin. Let G_L denote the event (in $\mathcal{B}$) that the largest boundary circuit γ surrounding the origin has $\text{diam}(\gamma) \le L$, where $\text{diam}(\cdot)$ denotes the diameter in the Euclidean metric. Then $G := \bigcup_{L=1}^{\infty} G_L$ is the event that there exists a largest circuit surrounding the origin. Note that if all closed (black) circuits around the origin have (Euclidean) diameter smaller than L, then G_L occurs (remember that $p_j \ge 1/2$ for all j). Therefore $\mu_{\delta_j,p_j}(G_L) \ge 1 - P^{\delta_j}_{p_j}(D_L)$, where D_L is the event that the origin is surrounded by a closed circuit of (Euclidean) diameter at least L. Using Lemma 2.2 and the fact that the event G_L is closed in our topology, we can write

$$\begin{aligned}
\mu(G_L) &\ge \limsup_{j\to\infty} \mu_{\delta_j,p_j}(G_L) \\
&\ge 1 - \liminf_{j\to\infty} P^{\delta_j}_{p_j}(D_L) \\
&\ge 1 - \liminf_{j\to\infty} C_1 \exp\left(-\frac{C_2 L}{\delta_j L_\epsilon(p_j)}\right) \\
&\ge 1 - C_1 \exp\left(-\frac{C_2 L}{K}\right).
\end{aligned}$$

Since the events are nested (i.e., $G_{L_1} \subset G_{L_2}$ for $L_1 < L_2$),

$$\mu(G) = \lim_{L\to\infty} \mu(G_L) \ge \lim_{L\to\infty} 1 - C_1 \exp\left(-\frac{C_2 L}{K}\right) = 1.$$

To continue the proof, note that for each $\epsilon \in (0, 1/2)$, as $j \to \infty$,

- either $\delta_j L_\epsilon(p_j) \to 0$,
- or $\delta_j L_\epsilon(p_j) \to \infty$,
- or $\delta_j L_\epsilon(p_j)$ is bounded away from both 0 and ∞.

This is clearly so because we are assuming that $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$ has a limit μ, and we have proved that the three cases above give rise to three incompatible scenarios for μ. Indeed, if we are not in one of the three cases above, then there must be two different subsequences of $\{(\delta_j, p_j)\}_{j\in\mathbb{N}}$ falling in two different cases, which contradicts the existence of a limit μ. We can then conclude that there are no other possible scenarios for μ besides the three described in the theorem.

To conclude the proof, we need to show that all three scenarios are non-void. For the first two, this is obvious. To prove that the third scenario is also non-void, take $0 < \epsilon_1 < \epsilon_2 < 1/2$ and consider any sequence $\{(\delta_j, p_j)\}_{j\in\mathbb{N}}$ such that $\delta_j \to 0$ and $p^+_{\epsilon_1}(1/\delta_j) \le p_j \le p^+_{\epsilon_2}(1/\delta_j)$. This implies that $L_{\epsilon_1}(p_j) \le 1/\delta_j \le L_{\epsilon_2}(p_j)$ for each j. We can assume without loss of generality that the sequence $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$ has a limit μ. (If that is not the case, by Theorem 3.2 and Lemma 3.3 we can extract a subsequence $\{\mu_{\delta_{j_k},p_{j_k}}\}_{k\in\mathbb{N}}$ that does have a limit, and rename it $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$.)

Since $\delta_j L_{\epsilon_1}(p_j)$ remains bounded as $j \to \infty$, $\delta_j L_\epsilon(p_j)$ must remain bounded as $j \to \infty$ for every other $\epsilon \in (0, 1/2)$ because $L_\epsilon(p) \asymp L_{\epsilon_1}(p)$. Analogously, since $\delta_j L_{\epsilon_2}(p_j)$ is bounded away from 0 as $j \to \infty$, $\delta_j L_\epsilon(p_j)$ must be bounded away from 0 as $j \to \infty$ for every $\epsilon \in (0, 1/2)$. Therefore, for each $\epsilon \in (0, 1/2)$, $\delta_j L_\epsilon(p_j)$ remains bounded away from both 0 and ∞ as $j \to \infty$, showing that μ falls in the third scenario. (One could also redo the calculations in case *(3)* above with $\beta = K = 1$ and ϵ replaced by ϵ_1 in the first part and by ϵ_2 in the second.) □

5 Other lattices

We have stated our results for site percolation on the triangular lattice but, except for Corollary 3.5 which relies on the power law (3), and case *(2)* of Theorem 3.4 which relies on results from [9, 10, 20], they also apply to other regular lattices (like the square lattice) and to bond percolation (after replacing 1/2 with p_c when necessary). Indeed, the main tools in our proofs originated in Kesten's work [15] on the square lattice and can be used in both site and bond percolation models on a large class of lattices (see [14]). For a discussion of the range of applicability of Kesten's and related results, and consequently of the results of the present paper (with the above caveat), the reader is referred to Section 8.1 of [19]. The same range of applicability pertains to the following version of case *(2)* of Theorem 3.4, which can be proved using the same methods as in the proofs of cases *(1)* and *(3)* of that theorem.

(2′) Critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a countably infinite family of nested loops with radii going to zero and a countably infinite family of nested loops with radii going to infinity.

This shows that one can distinguish between the three different regimes and corresponding types of scaling limits by means of "classical" scaling relations, without resorting to SLE.

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