1. Let $\Sigma = \{a, b\}$. Specify (or draw) a TM that accepts exactly all non-empty strings in which the number of $a$’s is exactly twice the number of $b$’s. (For example, $aab$ and $babaa$ are in the language, but $aaabb$ is not.) (20 pts)

2. Let $\Sigma = \{a, b\}$. The TM $M$, with $F = \{q_f\}$, is defined by:

\[
\begin{align*}
\delta(q_0, a) &= (q_1, a, R) \\
\delta(q_0, b) &= (q_0, b, R) \\
\delta(q_1, a) &= (q_0, a, R) \\
\delta(q_1, b) &= (q_f, b, L)
\end{align*}
\]

(a) Transform the question whether string $bab$ is in $L(M)$ into an instance of the MPCP. (8 pts)

(b) Show, by applying the transition function of $M$ to the tape, that $bab \in L(M)$. (2 pts)

(c) Transform this computation of $M$ on input string $bab$ into a solution for the corresponding instance of the MPCP. (15 pts)
3. Argue that NP is closed under union. (That is, if \( L_1 \in \text{NP} \) and \( L_2 \in \text{NP} \), then \( L_1 \cup L_2 \in \text{NP} \).) (15 pts)

4. Consider the bounded tiling problem with the following three tile types

\[
\begin{array}{ccc}
g & t_1 & g \\
 b & & b \\
g & t_2 & r \\
 b & & b \\
r & t_3 & g \\
 g & & g \\
\end{array}
\]

where \( n = 3 \) and the first row is \( t_1 \ t_2 \ t_3 \).

Transform this instance of the bounded tiling problem into an instance of the satisfiability problem. (15 pts)

5. Let \( f : \{0,1\}^4 \rightarrow \{0,1\}^4 \) be defined as follows:

\[
\begin{align*}
f(0000) &= f(0011) = 0000 \\
f(0001) &= f(0010) = 0111 \\
f(0100) &= f(0111) = 0011 \\
f(1000) &= f(1011) = 0001 \\
f(0110) &= f(0101) = 1111 \\
f(1010) &= f(1001) = 1011 \\
f(1101) &= f(1110) = 0110 \\
f(1100) &= f(1111) = 1000
\end{align*}
\]

Apply Simon’s algorithm to determine a (non-trivial) linear dependency between the digits of \( s = 0011 \). (Give one possible scenario.) (25 pts)