Example 1.24 Using LEM, we show that $p \rightarrow q \vdash \neg p \vee q$ is valid:

1	$p \rightarrow q$	premise
2	$\neg p \vee p$	LEM
3	$\neg p$	assumption
4	$\neg p \vee q$	$\vee i_1$ 3
5	p	assumption
6	q	$\rightarrow$e 1, 5
7	$\neg p \vee q$	$\vee i_2$ 6
8	$\neg p \vee q$	$\vee$e 2, 3–4, 5–7

It can be difficult to decide which instance of LEM would benefit the progress of a proof. Can you re-do the example above with $q \vee \neg q$ as LEM?

1.2.3 Natural deduction in summary

The proof rules for natural deduction are summarised in Figure 1.2. The explanation of the rules we have given so far in this chapter is *declarative*; we have presented each rule and justified it in terms of our intuition about the logical connectives. However, when you try to use the rules yourself, you'll find yourself looking for a more *procedural* interpretation; what does a rule do and how do you use it? For example,

- $\wedge$i says: to prove $\phi \wedge \psi$, you must first prove ϕ and ψ separately and then use the rule $\wedge$i.
- $\wedge e_1$ says: to prove ϕ, try proving $\phi \wedge \psi$ and then use the rule $\wedge e_1$. Actually, this doesn't sound like very good advice because probably proving $\phi \wedge \psi$ will be harder than proving ϕ alone. However, you might find that you *already have* $\phi \wedge \psi$ lying around, so that's when this rule is useful. Compare this with the example sequent in Example 1.15.
- $\vee i_1$ says: to prove $\phi \vee \psi$, try proving ϕ. Again, in general it is harder to prove ϕ than it is to prove $\phi \vee \psi$, so this will usually be useful only if you've already managed to prove ϕ. For example, if you want to prove $q \vdash p \vee q$, you certainly won't be able simply to use the rule $\vee i_1$, but $\vee i_2$ will work.
- $\vee$e has an excellent procedural interpretation. It says: if you have $\phi \vee \psi$, and you want to prove some χ, then try to prove χ from ϕ and from ψ in turn. (In those subproofs, of course you can use the other prevailing premises as well.)
- Similarly, $\rightarrow$i says, if you want to prove $\phi \rightarrow \psi$, try proving ψ from ϕ (and the other prevailing premises).
- $\neg$i says: to prove $\neg\phi$, prove $\bot$ from ϕ (and the other prevailing premises).

The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules:

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{PBC} \qquad\qquad \dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Figure 1.2. Natural deduction rules for propositional logic.

At any stage of a proof, it is permitted to introduce any formula as assumption, by choosing a proof rule that opens a box. As we saw, natural deduction employs boxes to control the scope of assumptions. When an assumption is introduced, a box is opened. Discharging assumptions is achieved by closing a box according to the pattern of its particular proof rule. It's useful to make assumptions by opening boxes. *But don't forget you have to close them in the manner prescribed by their proof rule.*

OK, but how do we actually go about constructing a proof? Given a sequent, you write its premises at the top of your page and its conclusion at the bottom. Now, you're trying to fill in the gap, which involves working simultaneously on the premises (to bring them towards the conclusion) and on the conclusion (to massage it towards the premises).

Look first at the conclusion. If it is of the form $\phi \to \psi$, then apply[6] the rule $\to$i. This means drawing a box with ϕ at the top and ψ at the bottom. So your proof, which started out like this:

$$\vdots$$

premises

$$\vdots$$

$$\phi \to \psi$$

now looks like this:

$$\vdots$$

premises

$$\vdots$$

ϕ	assumption
ψ	

$\phi \to \psi$ $\quad\to$i

You still have to find a way of filling in the gap between the ϕ and the ψ. But you now have an extra formula to work with and you have simplified the conclusion you are trying to reach.

[6] Except in situations such as $p \to (q \to \neg r), p \vdash q \to \neg r$ where $\to$e produces a simpler proof.