Term Rewriting Systems 2010

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Acknowledgements

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- Femke van Raamsdonk (Vrije Universiteit Amsterdam)

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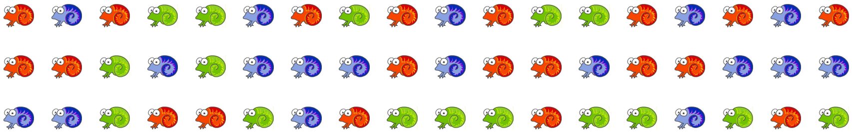
- Roel de Vrijer (Vrije Universiteit Amsterdam)

from the course Term Rewriting Systems in 2008.

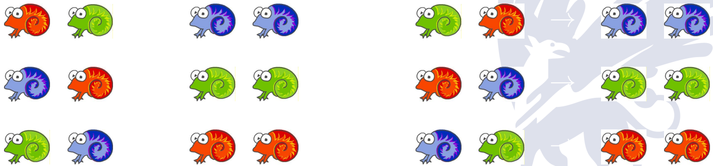
- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: ...
- Lecture 6: ...
- Lecture 7: ...
- Lecture 8: ...
- Lecture 9: ...
- Lecture 10: ...
- Lecture 11: ...

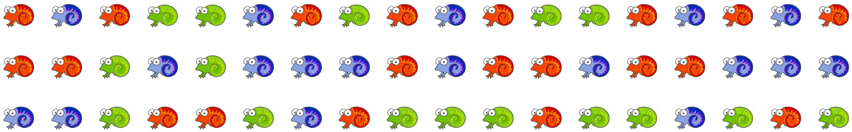
Outline

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?





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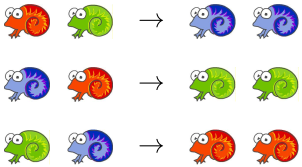

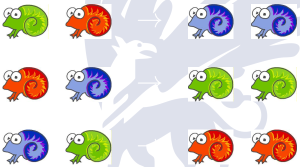

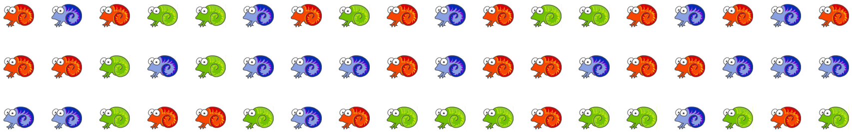

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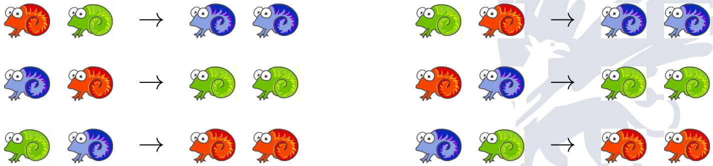

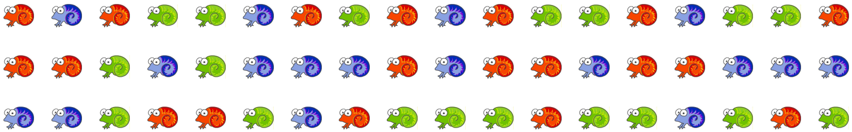

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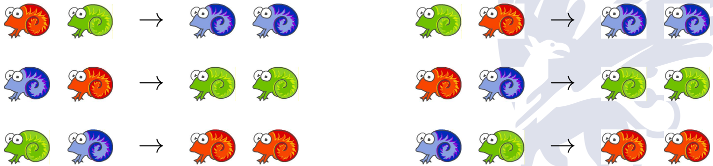

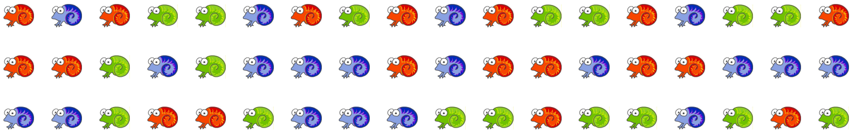

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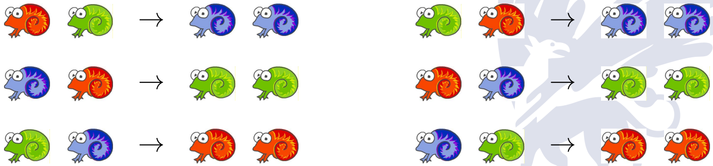

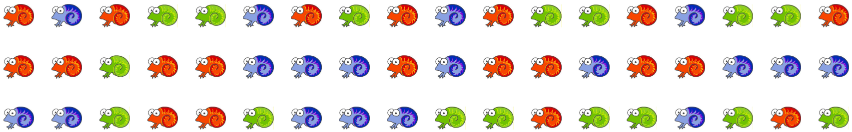

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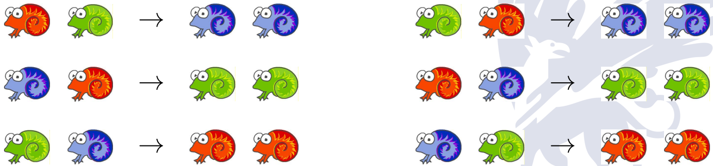

A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

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Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T GAG $\leftrightarrow$ AG CTC $\leftrightarrow$ TC AGTA $\leftrightarrow$ A TAT $\leftrightarrow$ CT

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Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T   GAG $\leftrightarrow$ AG   CTC $\leftrightarrow$ TC   AGTA $\leftrightarrow$ A   TAT $\leftrightarrow$ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature $\quad 0$ (constants) $\quad s$ (unary) $\quad +$ (binary, infix)

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terms $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x) + y$

rewrite rules

$$
\begin{aligned}
0 + y &\to y \\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y)
\end{aligned}
$$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(x) + y$
rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$

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rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\qquad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

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rewrite rules	$0 + y \to y$
	$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \quad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

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terms $\quad \mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(x) + y$

rewrite rules

$$0 + y \to y$$

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rewriting $\quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \qquad y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

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rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \qquad y \mapsto \mathsf{s}(\mathsf{s}(0))$ $\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

Example (Group Theory)

signature e (constant) $^{-}$ (unary, postfix) $\cdot$ (binary, infix)

$\mathcal{E}$

R

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

equations $\mathsf{e} \cdot x \approx x$ $x^- \cdot x \approx \mathsf{e}$ $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$ $\mathcal{E}$

R

Example (Group Theory)

signature	e (constant) $\ ^-$ (unary, postfix) $\cdot$ (binary, infix)	
equations	$\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	$\mathcal{E}$
theorems	$\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$	
		R

Example (Group Theory)

signature: e (constant) $\quad {}^-$ (unary, postfix) $\quad \cdot$ (binary, infix)

equations: $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems: $\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

rewrite rules: $\qquad R$

$$\begin{aligned} \mathsf{e} \cdot x &\to x \\ x^- \cdot x &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) \end{aligned}$$

Example (Group Theory)

signature	e (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)	
equations	$\mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	$\mathcal{E}$
theorems	$\mathsf{e}^{-} \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^{-} \approx_{\mathcal{E}} y^{-} \cdot z^{-}$	
rewrite rules	see below	R

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
x^{-} \cdot x & \to & \mathsf{e} & x \cdot x^{-} & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\
x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y
\end{array}
$$

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signature: e (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

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theorems: $\mathsf{e}^{-} \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^{-} \approx_{\mathcal{E}} y^{-} \cdot z^{-}$

rewrite rules:

$$
\begin{array}{rcl@{\qquad\qquad}rcl@{\qquad}l}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x & R \\
x^{-} \cdot x & \to & \mathsf{e} & x \cdot x^{-} & \to & \mathsf{e} & \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x & \\
\mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} & \\
x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y &
\end{array}
$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

Example (Group Theory)

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equations: $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems: $\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

rewrite rules: R

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\begin{aligned}
\mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\
x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\
(x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\
\mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\
x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y
\end{aligned}
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① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

② R admits no infinite computations

Example (Group Theory)

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equations $\quad \mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

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rewrite rules $\qquad R$

$$
\begin{array}{rclcrcl}
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\end{array}
$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

② R admits no infinite computations

① & ② $\implies$ $\mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature S K I (constants)

inventor (1924)

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

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Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

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Example (Combinatory Logic)

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terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\mathsf{I} \cdot x \to x$$
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$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

inventor (1924)

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rewriting $((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x$

inventor (1924)

Example (Combinatory Logic)

signature	S K I (constants) $\cdot$ (application, binary, infix)
terms	S $\quad ((\mathsf{K}\cdot\mathsf{I})\cdot\mathsf{I})\cdot\mathsf{S}$ $\quad (x\cdot z)\cdot(y\cdot z)$
rewrite rules	$\mathsf{I}\cdot x \to x$ $(\mathsf{K}\cdot x)\cdot y \to x$ $((\mathsf{S}\cdot x)\cdot y)\cdot z \to (x\cdot z)\cdot(y\cdot z)$
rewriting	$((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x \;\to\; (\mathsf{K}\cdot x)\cdot(\mathsf{K}\cdot x)$
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rewriting

$$((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x \to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \to x$$

inventor   (1924)

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rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

inventor Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature λ (binds variables)

$$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$$

$\lambda x.\, x \cdot y$ $\lambda z.\, z \cdot y$

$(\lambda x.\, M) \cdot N$ M

inventor (1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

$$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$$

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$$(\lambda x.\, M) \cdot N \qquad M$$

inventor (1936)

Example (Lambda Calculus)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

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inventor (1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

$(\lambda x.\, M) \cdot N \qquad M$

inventor (1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \rightarrow_\beta M[x := N]$

inventor (1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \rightarrow_\beta M[x := N]$

replace free occurrences of x in M by N

inventor (1936)

Example (Lambda Calculus)

signature	λ (binds variables) $\cdot$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$
β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$ replace free occurrences of x in M by N
rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$
inventor	(1936)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

inventor (1936)

Example (Lambda Calculus)

signature — λ (binds variables) $\cdot$ (application, binary, infix)

terms — $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion — $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction — $(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting — $(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

inventor — Alonzo Church (1936)

Outline

- Overview
- Examples
- Abstract Rewrite Systems
  - Definitions
  - Properties
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

- string rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting

Motivation

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- graph rewriting

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abstract rewriting

- no structure on objects that are rewritten

Motivation

concrete rewrite formalisms

- string rewriting
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abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

Definitions

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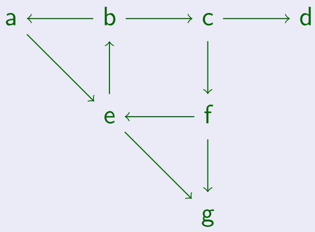

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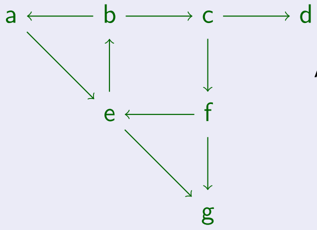

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to = \left\{ \begin{matrix} (a, e), (b, a), (b, c), (c, d), (c, f) \\ (e, b), (e, g), (f, e), (f, g) \end{matrix} \right\}$

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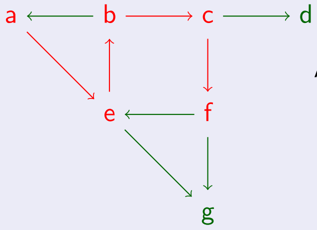

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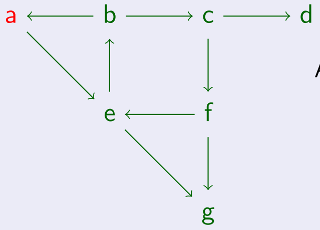

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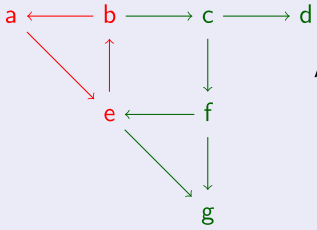

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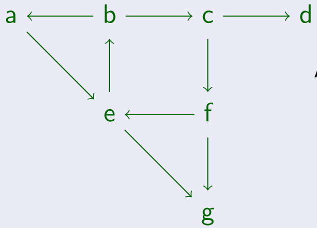

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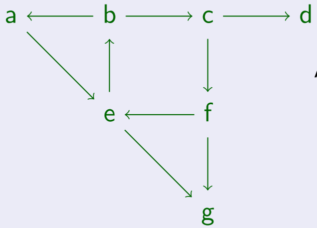

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- rewrite sequence

  rewrite step

  - finite $\quad a \to e \to b \to c \to f \quad$ length 4
  - empty $\quad a \quad$ length 0
  - infinite $\quad a \to e \to b \to a \to e \to b \to \cdots \quad$ length ω

The length of a rewrite sequence is the number of rewrite steps.

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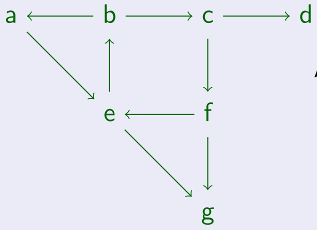

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- rewrite sequence

  rewrite step

  - finite $\mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$ length 4 $a \to^* f$
  - empty a length 0 $a \to^* a$
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The length of a rewrite sequence is the number of rewrite steps.
We write $x \to^* y$ if x rewrites to y in 0 or more steps.

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- $\leftarrow$ or $\to^{-1}$ inverse of $\to$

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- $\downarrow$ — joinability $\downarrow \, = \, \rightarrow^{*} \cdot {}^{*}\!\leftarrow$

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- if $x \to^* y$ then x rewrites to y and y is reduct of x

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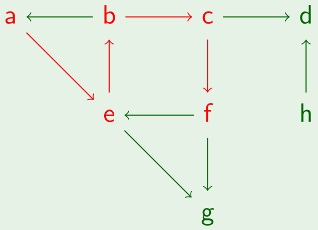
Example



- a $\to^*$ f

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\leftarrow y$ then z is common reduct of x and y

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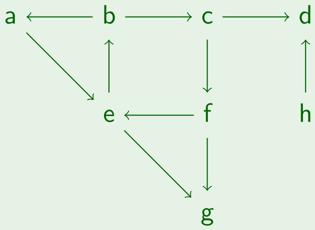

- $\mathsf{a} \to^* \mathsf{f}$

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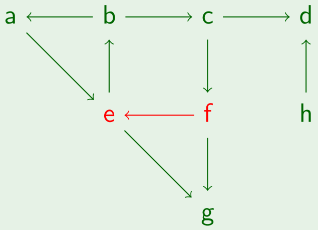

- $a \to^* f$
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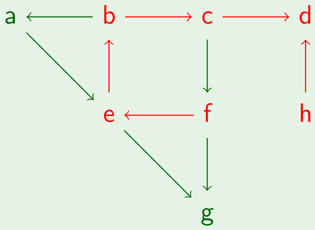

- a $\to^*$ f
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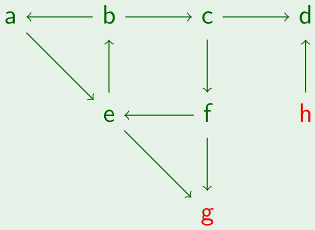

- $a \to^* f$
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- if $x \leftrightarrow^* y$ then x and y are convertible

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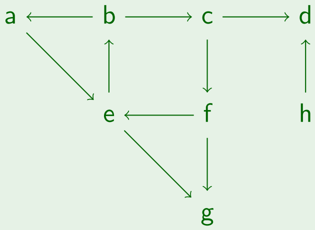

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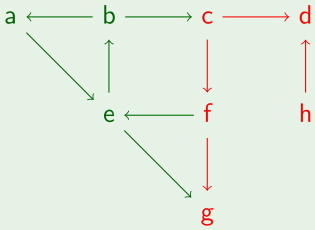

- a $\to^*$ f
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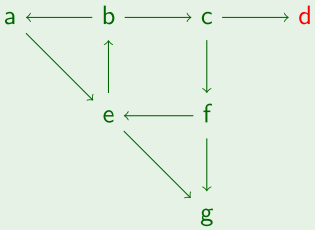

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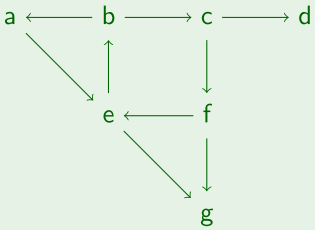

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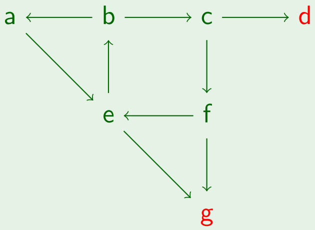

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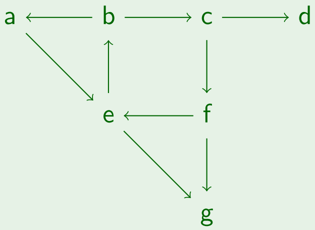

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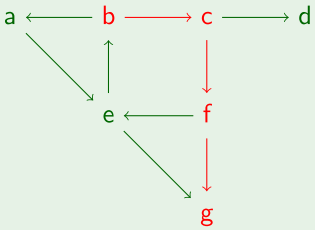

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Outline

Definitions

- SN strong normalization or termination
  - no infinite rewrite sequences

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  - $\forall a \; \exists b \quad a \to^! b$

Lemmata

1. SN $\implies$ WN

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1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN

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  - if an element a is convertible with a normal form b, then a rewrites to b
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- UN unique normal forms
  - convertible normal forms are equal

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- $\mathrm{UN}^{\to}$ unique normal forms with respect to reduction
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  - ${}^!{\leftarrow} \cdot \to^! \subseteq =$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowleft \mathsf{a} \longrightarrow \mathsf{b}$
3. $\mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN

3. NF $\Longrightarrow$ UN $\Longrightarrow$ $\mathrm{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$

Lemmata

1. SN $\Longrightarrow$ WN
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5. NF $\not\Longleftarrow$ UN

Lemmata

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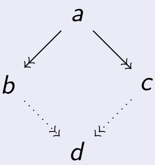
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$

Definition

- CR confluence or Church-Rosser property
  - $\uparrow \subseteq \downarrow$

Definition

- CR   confluence   or   Church-Rosser property
  - $\uparrow \subseteq \downarrow$
  - $\forall a, b, c$ $\exists d$



in diagrams: $\twoheadrightarrow$ for $\to^*$

Definition

- CR confluence or Church-Rosser property
  - $\uparrow \subseteq \downarrow$
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    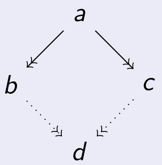

    in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

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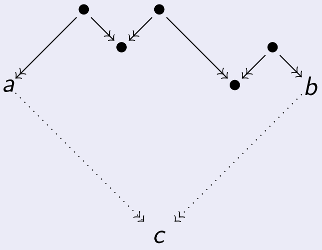

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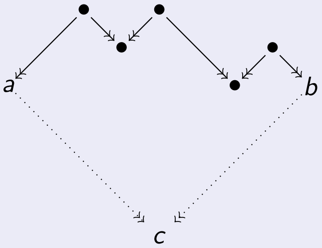

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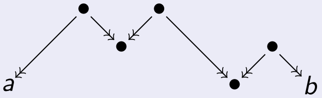

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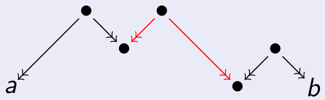

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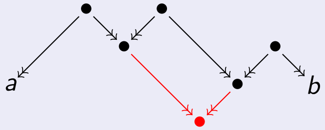

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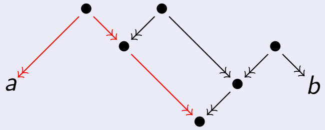

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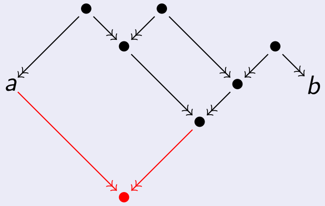

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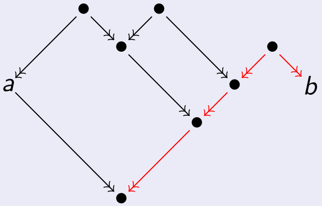

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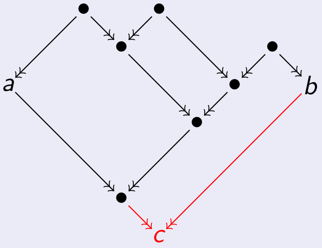

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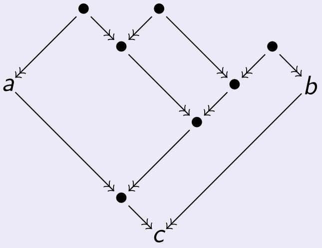

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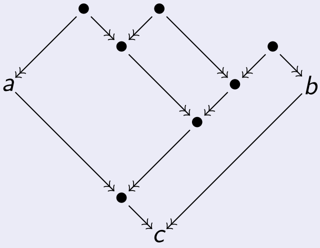

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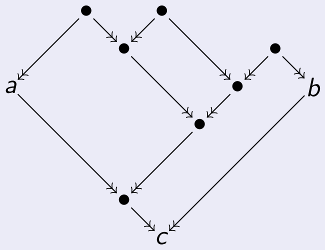

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Induction

Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

1. *The base case:*
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2. *The inductive step:*
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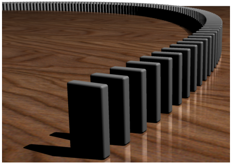

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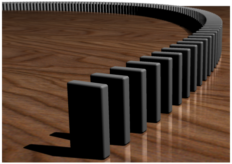

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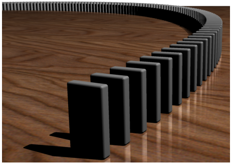

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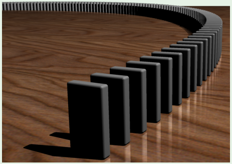

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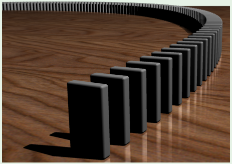

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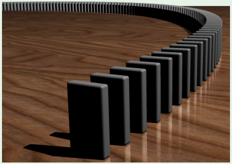

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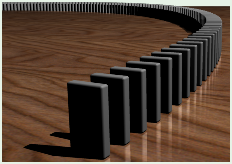

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- Base case: proof that the first domino falls
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Then you have proven that all dominoes will fall.

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We use induction to prove that:

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

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Hence the formula holds for all $n \in \mathbb{N}$.

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Second proof: induction.

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We proof by induction on n that $a\,(\leftarrow^* \cdot \rightarrow^*)^n\, b$ implies $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$.

- Base case $n = 0$: $a\,(\leftarrow^* \cdot \rightarrow^*)^0\, b$. Then $a = b$ and hence $a \rightarrow^* \cdot {}^*\!\!\leftarrow b$.
- Induction step $n + 1$: (assume it holds for n, show it for $n + 1$)
  Let $a\,(\leftarrow^* \cdot \rightarrow^*)^{n+1}\, b$. Then $a \leftarrow^* \cdot \rightarrow^* d\,(\leftarrow^* \cdot \rightarrow^*)^n\, b$.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

$\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

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  - Hence $a \rightarrow^* e \; {}^*\!\!\leftarrow d$ since by assumption $\uparrow \subseteq \downarrow$.

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$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

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  - Hence $a \rightarrow^* e\ {}^*\!\!\leftarrow d$ since by assumption $\uparrow \subseteq \downarrow$.
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  We conclude $a \rightarrow^* \cdot {}^*\!\leftarrow b$, that is, $a \downarrow b$.

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  - Thus $e \rightarrow^* \cdot {}^*\!\leftarrow b$ by induction hypothesis.

  We conclude $a \rightarrow^* \cdot {}^*\!\leftarrow b$, that is, $a \downarrow b$.

Hence we have shown $\uparrow^* \subseteq \downarrow$. It follows $\leftrightarrow^* \subseteq \downarrow$ since $\leftrightarrow^* \subseteq \uparrow^*$. ∎

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Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$

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- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
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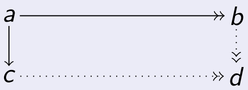
  $\exists d$



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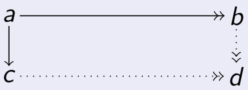

- $\forall a, b, c.\quad a \rightarrow^* b \;\wedge\; a \rightarrow c \quad\Rightarrow\quad \exists d.\quad b \rightarrow^* d \;\wedge\; c \rightarrow^* d$

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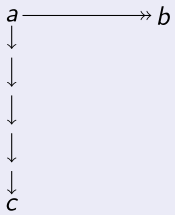

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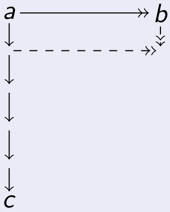

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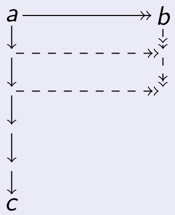

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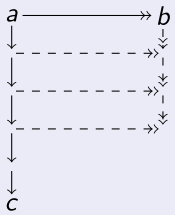

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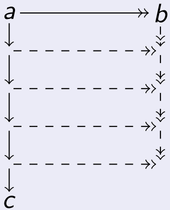

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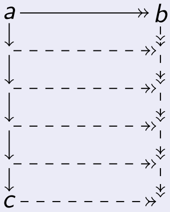

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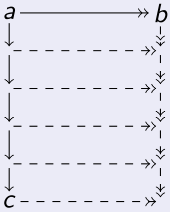
By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.



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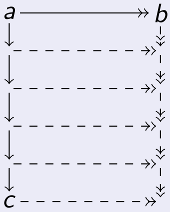

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.

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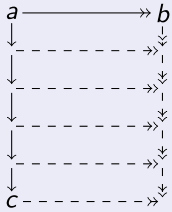

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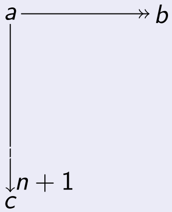

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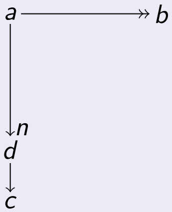

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  Then $c \leftarrow d \ {}^{n}\!\leftarrow a \rightarrow^* b$ for some d, and:

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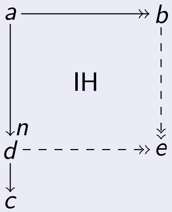

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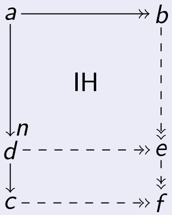

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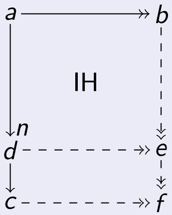

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  - Then $c \rightarrow^* f \ {}^{*}\!\leftarrow e$ since $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

  Hence $c \rightarrow^* \cdot {}^{*}\!\leftarrow b$.

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$

3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$

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Lemmata

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2. SN $\not\Longleftarrow$ WN

3. CR $\Longrightarrow$ NF $\Longrightarrow$ UN $\Longrightarrow$ $\mathrm{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$

5. NF $\not\Longleftarrow$ UN

6. CR $\not\Longleftarrow$ NF

Lemmata

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4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow c^{\circlearrowleft} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow$

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6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow$
8. $\mathsf{WN}\ \&\ \mathsf{UN} \implies \mathsf{CR}$

Definitions

- WCR local confluence or weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$

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  - $\leftarrow \cdot \rightarrow \ \subseteq \ \downarrow$
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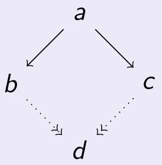

$\exists d$

Lemmata

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4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$ 

5. NF $\not\Longleftarrow$ UN 

6. CR $\not\Longleftarrow$ NF 

7. CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$
8. WN & UN $\Longrightarrow$ CR
9. CR $\Longrightarrow$ WCR

Lemmata

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6. CR $\not\Longleftarrow$ NF $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$
8. WN & UN $\Longrightarrow$ CR
9. CR $\Longrightarrow$ WCR
10. CR $\not\Longleftarrow$ WCR

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
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7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow$
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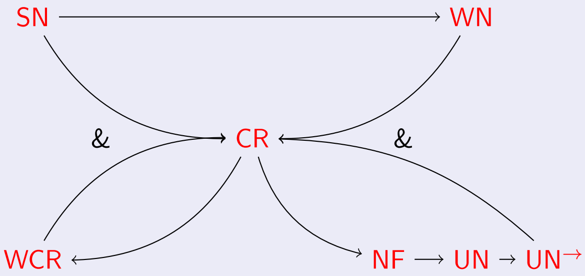
11. SN & WCR $\Longrightarrow$ CR — Newman's Lemma

Summary



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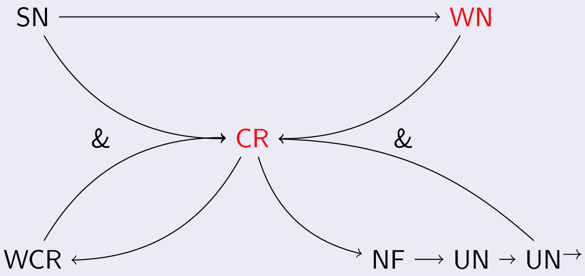

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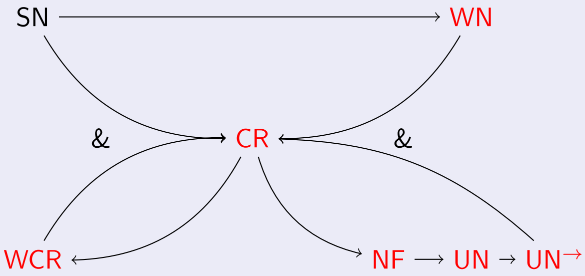

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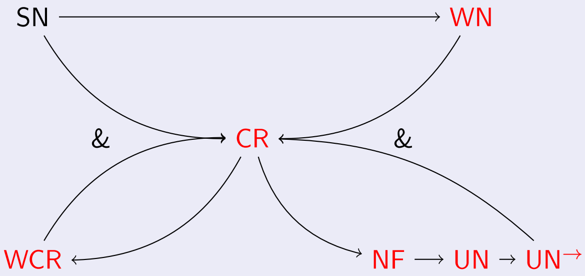

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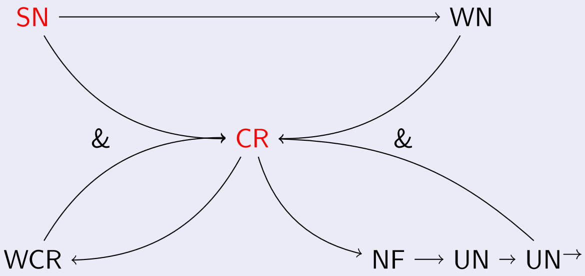

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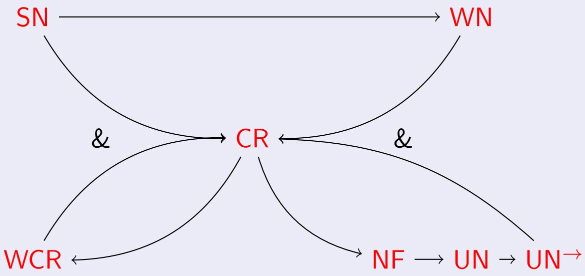

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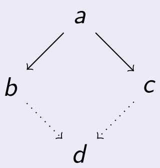

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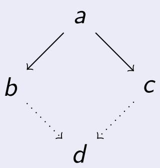

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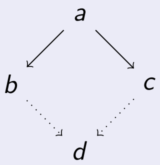

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    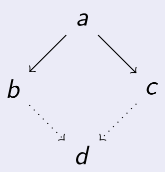

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Proof.

Exercise. ∎

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ARS $\mathcal{A} = \langle A, \to_1 \rangle$ *is confluent if*

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Outline

Well-Founded Induction

given

- strongly normalizing ARS $\mathcal{A} = \langle A, \to \rangle$

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$$\Big(\forall a \colon \big(\forall b \colon a \to b \implies \mathsf{P}(b)\big) \implies \mathsf{P}(a)\Big) \quad \implies \quad \forall a \colon \mathsf{P}(a)$$

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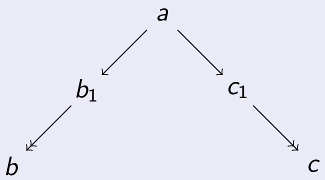
Proof.



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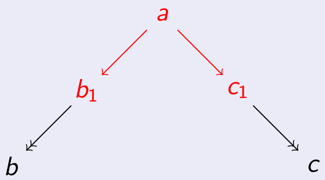

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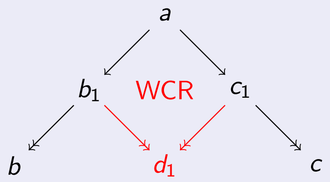

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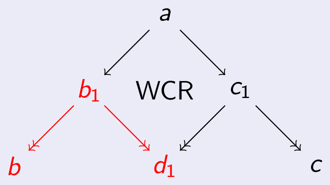

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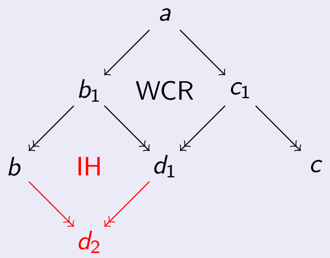

induction hypothesis $\mathrm{CR}(b_1)$

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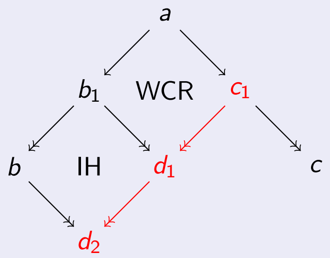

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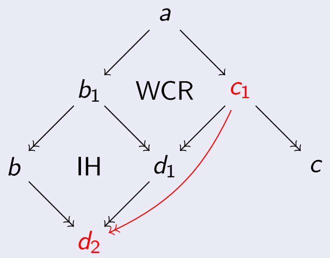

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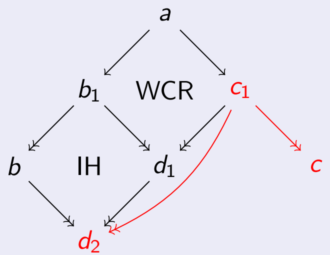

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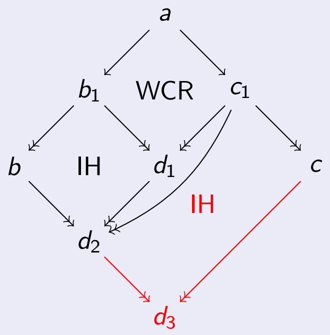

induction hypothesis $\mathrm{CR}(c_1)$

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An ARS has the property if all its elements have the respective property.

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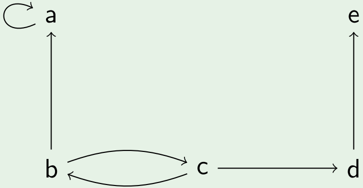

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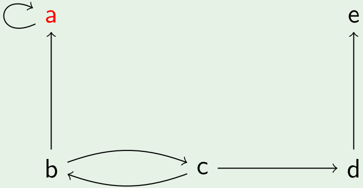

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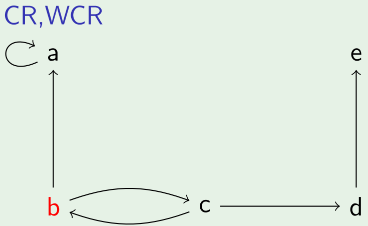

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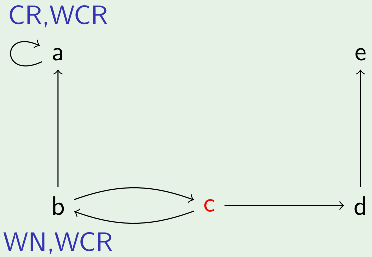

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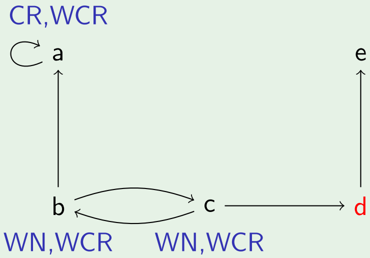

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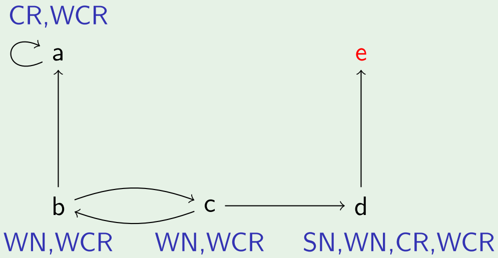

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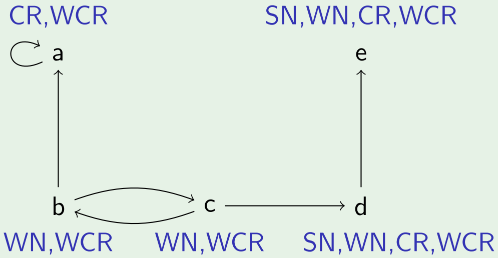

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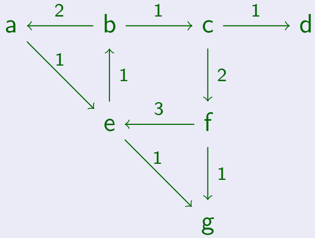

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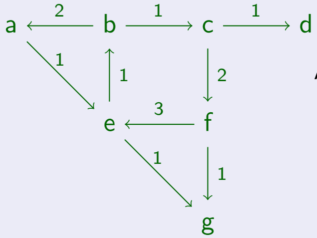

ARS $\mathcal{A} = \langle A, \to_1, \to_2, \to_3 \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to_1 = \left\{ \begin{matrix} (a, e), (b, c), (c, d), \\ (e, b), (e, g), (f, g) \end{matrix} \right\}$
- $\to_2 = \{(b, a), (c, f)\}$
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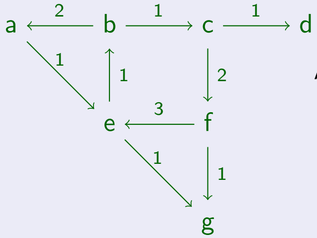

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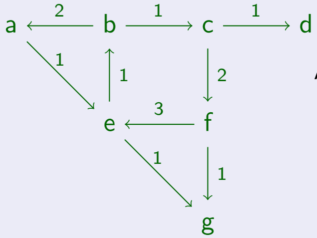

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- $\to_{12} = \to_1 \cup \to_2, \quad \to_{13} = \to_1 \cup \to_3, \quad \dots$

Definition

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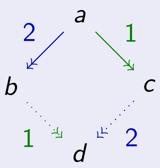

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    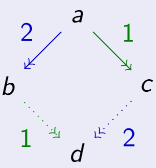

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    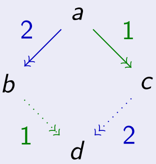

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    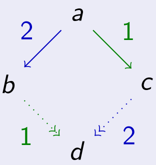

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