

- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: **Infinitary Rewriting**



Outline

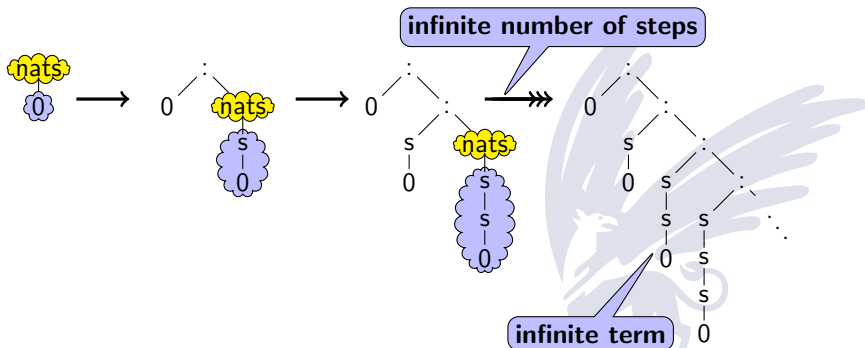
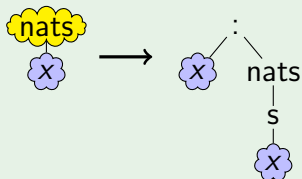
- Overview
- Infinitary Rewriting



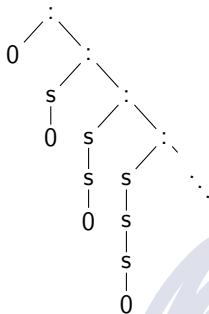
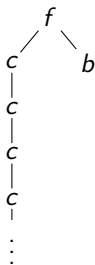
Infinitary Rewriting



Example (The Stream of Natural Numbers)



Infinite Terms

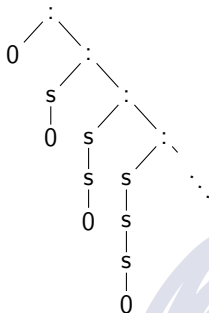
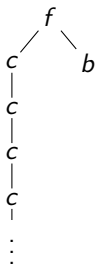


Definition

An **infinite term** is a partial map $t : \mathbb{N}^* \rightarrow \Sigma$ from positions to symbols such that:

- $t(\epsilon) \in \Sigma$, and
- $t(ip) \in \Sigma \iff 1 \leq i \leq \#(t(p))$

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The **set of finite and infinite terms** is denoted by $\mathcal{T}^\infty(\Sigma, \mathcal{X})$.

Infinite Terms as Metric Space

Definition

We define a **metric** d on $\mathcal{T}^\infty(\Sigma, \mathcal{X})$ by:

$$d(s, t) = 2^{-|p|} \text{ where } p \text{ is the highest position such that } s(p) \neq t(p)$$

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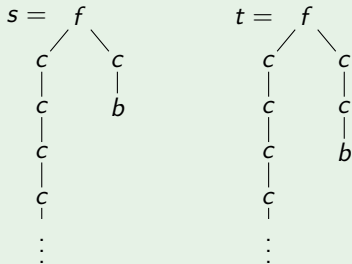
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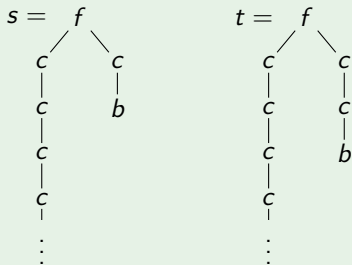
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$$f(x, x) \rightarrow f(a, b)$$

$$a \rightarrow c(a)$$

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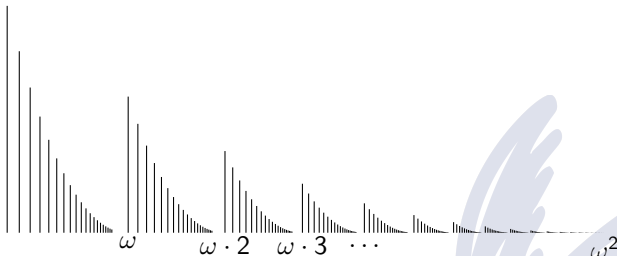
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We need transfinite reductions...



Ordinals

$0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega + \omega = \omega \cdot 2, \dots, \omega \cdot 3, \dots, \omega^2, \dots, \omega^\omega, \dots$



Note that ω is the smallest infinite ordinal.

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A set S is **transitive** if $x \in S$ implies $x \subseteq S$.



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We define $\alpha < \beta \iff \alpha \in \beta$.

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The relation $<$ is a total order on ordinals.

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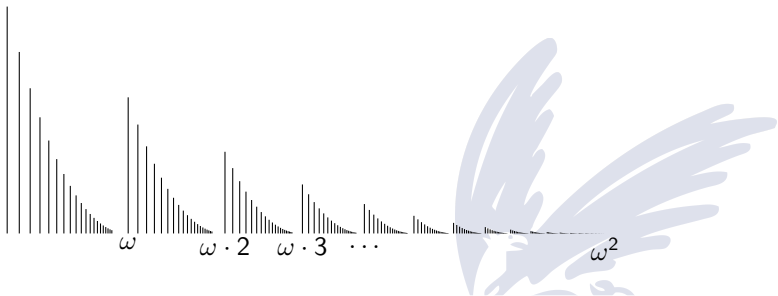
Lemma

For every ordinal β , we have $\beta = \{\alpha \mid \alpha < \beta\}$.

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Definition

For ordinals α , we define $\alpha^+ = \alpha \cup \{\alpha\}$, the **successor of α** .

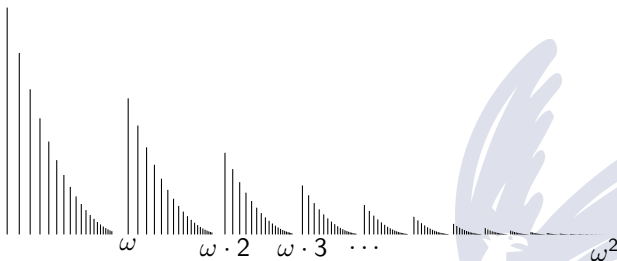


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Successor ordinals: $1, 2, \omega + 1, \omega \cdot 3 + 2, \dots$

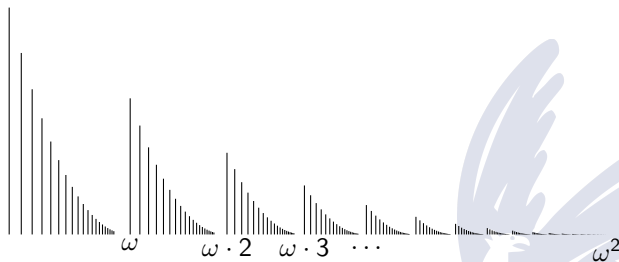
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If $\alpha \neq 0$ and α is not a successor ordinal, then α is called **limit** ordinal.



Example

Successor ordinals: $1, 2, \omega + 1, \omega \cdot 3 + 2, \dots$ Limit ordinals: $\omega, \omega \cdot 2, \omega \cdot 3, \omega^2, \dots$

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A reduction of length $\omega + 1$:

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by alternating $f(a, b) \rightarrow f(c(a), b) \rightarrow f(c(a), c(b)) \rightarrow \dots$



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Then τ is an **infinite reduction** of length α if for every limit ordinal $\lambda < \alpha$:

- 1 the distance $d(t_\beta, t_\lambda)$ tends to 0, $\forall \epsilon > 0. \exists \beta < \lambda. \forall \beta < \gamma < \lambda. d(t_\gamma, t_\lambda) \leq \epsilon$
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Example

Let $R = \{a \rightarrow a, b \rightarrow b\}$. Condition (1) excludes **jumps in the limit**:

$$a \rightarrow a \rightarrow a \rightarrow \dots \underbrace{b}_{t_\omega} \rightarrow b \rightarrow \dots$$

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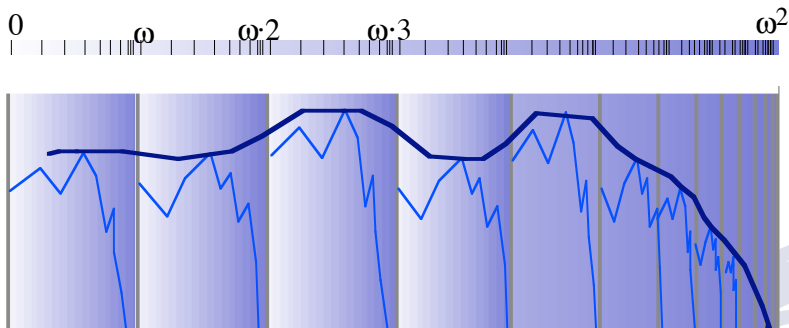
Example (We want more than Cauchy-convergence...)

Let $R = \{f(x) \rightarrow f(c(x))\}$. Condition (2) excludes sequences like:

$$f(a) \rightarrow_\epsilon f(c(a)) \rightarrow_\epsilon f(c(c(a))) \rightarrow_\epsilon \dots \rightarrow_\epsilon^\omega f(c^\omega) \rightarrow \dots$$

where the **activity does not move downwards**.

Transfinite Reductions Visualized



The blue lines indicate the depth of the activity/rewrite steps.

The activity tends to infinity when approaching limit ordinals.

Why more than Cauchy-convergence?

We consider the TRS:

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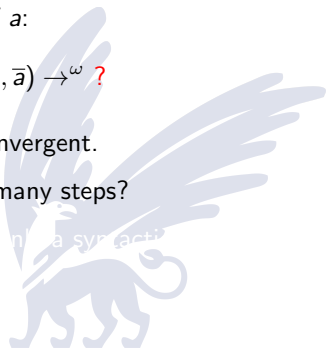
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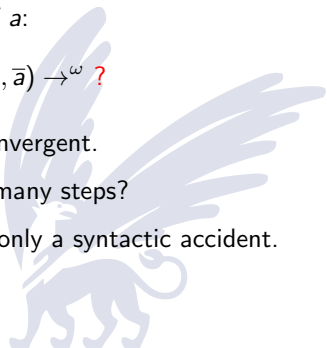
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However, what are the residuals of the left a after ω -many steps?

Although it appears as if the term has a limit, this is only a syntactic accident. The subterms get swapped all the time. . .



Definition

A reduction of length α is **strongly convergent** if for every limit ordinal $\lambda \leq \alpha$ the depth d_β tends to infinity as β approaches λ from below, and **divergent**, otherwise.

... is a divergent rewrite sequence of length ω .

$$\blacksquare R = \{ f(x, x) \rightarrow f(a, b), a \rightarrow c(a), b \rightarrow c(b) \}$$

$$f(a, b) \rightarrow^\omega f(c^\omega, b) \rightarrow^\omega f(c^\omega, c^\omega) \rightarrow f(a, b)$$

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$$\mathbf{2} \quad R = \{ f(x, x) \rightarrow f(a, b), a \rightarrow c(a), b \rightarrow c(b) \}$$

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Lemma

A reduction τ is strongly convergent

\iff for every $n \in \mathbb{N}$ there are only finitely many steps at depth n in τ .

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We write $s \twoheadrightarrow t$ if the rewrite sequence is strongly convergent and with limit t .



But every prefix is convergent!



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... is a divergent rewrite sequence of length ω^2 .

Definition

We write $s \twoheadrightarrow t$ if the rewrite sequence is strongly convergent and with limit t .

Example

$R = \{ a \rightarrow c(a) \}$. Then $a \twoheadrightarrow c^\omega$.

Lemma

Every proper prefix of a (even divergent) rewrite sequence is strongly convergent.

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... is a divergent rewrite sequence of length ω^2 . But every prefix is convergent!

Comparison finitary vs. infinitary rewriting

finitary rewriting	infinitary rewriting
finite reduction	strongly convergent reduction
infinite reduction	divergent reduction

Then R is WN^∞ , SN^∞ and CR^∞ .

- Let $R = \{ a \rightarrow a, a \rightarrow c(a) \}$. Then R is WN^∞ and CR^∞ , but not SN^∞ .

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Definition

Let \mathcal{R} be a TRS and s a term. Then the term s is

- **infinitary strongly normalizing** (SN^∞) if s admits no divergent reductions,
- **infinitary weakly normalizing** (WN^∞) if s admits a reduction to normal form,
- **infinitary confluent** (CR^∞) if $\forall t_1 \leftarrow\leftarrow s \rightarrow\rightarrow t_2. t_1 \rightarrow\rightarrow \cdot \leftarrow\leftarrow t_2$.

Likewise \mathcal{R} has the respective property if all terms from $\mathcal{T}^\infty(\Sigma, \mathcal{X})$ have.

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- Let $R = \{ a \rightarrow c(a) \}$. Then R is WN^∞ , SN^∞ and CR^∞ .
- Let $R = \{ a \rightarrow a, a \rightarrow c(a) \}$. Then R is WN^∞ and CR^∞ , but not SN^∞ .

Remark

- $SN^\infty \not\equiv SN$

$$a \rightarrow c(a)$$

Here, $a \rightarrow^* c^\omega$ which is a normal form.

Even one collapsing rule is sufficient to violate CR^∞ .

Take $R = \{ f(x, y) \rightarrow y \}$. Then

$$f(x, f(x, f(x, \dots))) \leftarrow f(x, f(y, f(x, f(y, \dots)))) \rightarrow f(y, f(y, f(y, \dots)))$$

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$$a \rightarrow b, a \rightarrow c, b \rightarrow d(b), c \rightarrow d(c)$$

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The example $A(x) \rightarrow x, B(x) \rightarrow x$ not every orthogonal TRSs is CR^∞ .

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Remark (The failure of Newmann's Lemma for infinitary rewriting)

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For example:

$$R = \{ a \rightarrow b(a), \\ a \rightarrow c(a), \\ c(b(x)) \rightarrow b(b(x)) \}$$

is WCR and SN^∞ , but not CR^∞ .



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Theorem

Every orthogonal TRS is UN^∞ .

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Weakly orthogonal TRSs are not necessarily UN^∞ :

$$S(P(x)) \rightarrow x$$

$$P(S(x)) \rightarrow x$$

Then

$$S^\omega \leftarrow S^1(P^2(S^3(P^4(\dots)))) \rightarrow P^\omega$$

Compression and Parallel Moves

Theorem (Compression)

Let \mathcal{R} be an left-linear TRS. Then $s \twoheadrightarrow t$ implies $s \rightarrow^{\leq \omega} t$.

That is, every strongly convergent reduction can be compressed to length $\leq \omega$.



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Theorem (Parallel Moves)

Let \mathcal{R} be an orthogonal TRS. Then $t_1 \ll s \twoheadrightarrow t_2 \Rightarrow t_1 \twoheadrightarrow \cdot \ll t_2$.

