- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Strategies

Strategies

Definition

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Lemma

For terminating TRSs every strategy is normalizing and perpetual.

Definition

Let $\rho = t_0 \to t_1 \to \ldots$ be a finite or infinite rewrite sequence.

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Definition

A strategy $\mathcal{S}$ is fair if every every maximal $\mathcal{S}$ rewrite sequence is fair.

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A one-step strategy maps every reducible term to a set of one-step reductions.

Example

There exists no fair one-step strategy for $\mathcal{R} = \{I(x) \to I(x)\}$.

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- $\mathcal{S}(t) = \{I(I(x)) \to_\varepsilon I(I(x)),\ I(I(x)) \to_1 I(I(x))\}$.

None of these is fair as we can always continue to reduce the same occurrence of I.

Example

- rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{cc}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

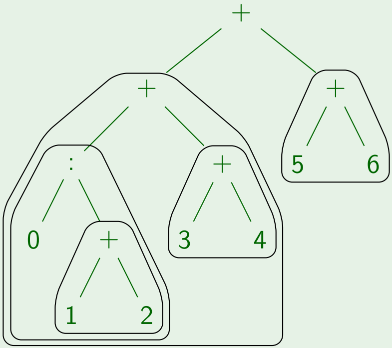
- term

$$((0:(1+2))+(3+4))+(5+6)$$

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1 + 2}} + \boxed{3 + 4}} + \boxed{5 + 6}$

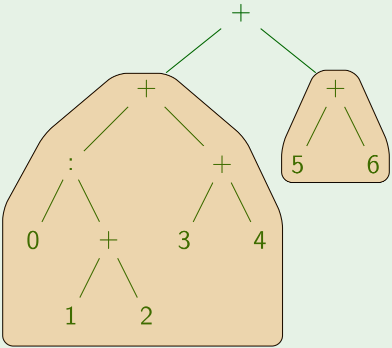
tree representation



Example (cont'd)

term $0 : 1+2 + 3+4 + 5+6$

tree representation

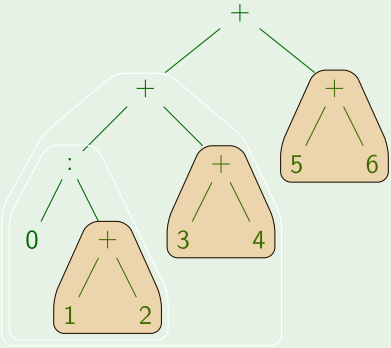


outermost redexes

Example (cont'd)

term $0 : 1 + 2 + 3 + 4 + 5 + 6$

tree representation



innermost redexes

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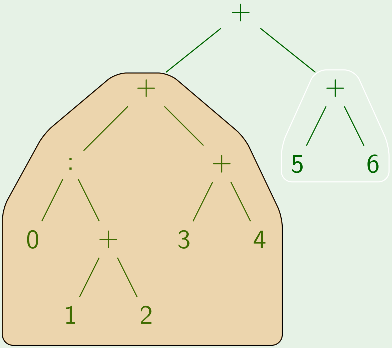
term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

tree representation

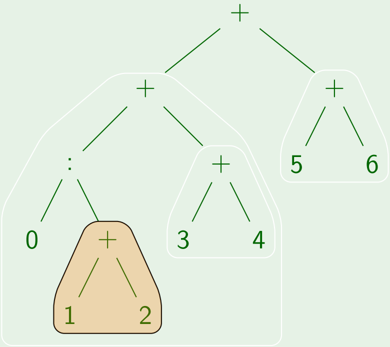


leftmost outermost strategy

Example (cont'd)

term $0 : 1 + 2 + 3 + 4 + 5 + 6$

tree representation

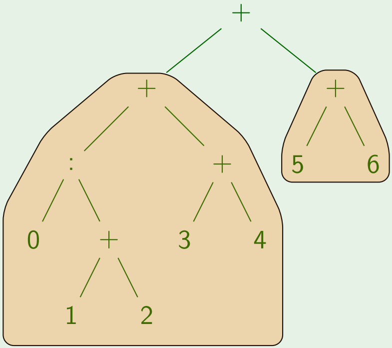


leftmost innermost strategy

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

tree representation

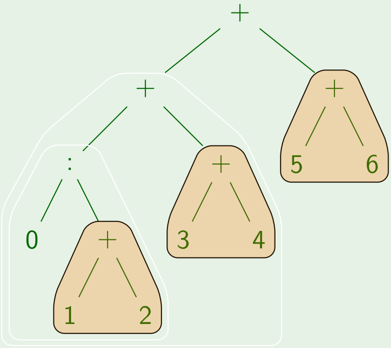


parallel outermost strategy

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

tree representation



parallel innermost strategy

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

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Example (cont'd)

$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$

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Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \to (0 : ((1 + 2) + (3 + 4))) + (5 + 6)$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \to (0 : ((1 + 2) + (3 + 4))) + (5 + 6)$$
$$\to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$\begin{aligned}((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \\ &\to 0 : (((1 + 2) + (3 + 4)) + (5 + 6)) \\ &\to ((1 + 2) + (3 + 4)) + (5 + 6)\end{aligned}$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\to (0 : ((1+2) + (3+4))) + (5+6) \\
&\to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\to ((1+2) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6)
\end{aligned}
$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$\begin{aligned}((0:(1+2))+(3+4))+(5+6) &\to (0:((1+2)+(3+4)))+(5+6)\\ &\to 0:(((1+2)+(3+4))+(5+6))\\ &\to ((1+2)+(3+4))+(5+6)\\ &\to (3+(3+4))+(5+6)\\ &\to (3+7)+(5+6)\end{aligned}$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6)\\
&\to 0 : (((1 + 2) + (3 + 4)) + (5 + 6))\\
&\to ((1 + 2) + (3 + 4)) + (5 + 6)\\
&\to (3 + (3 + 4)) + (5 + 6)\\
&\to (3 + 7) + (5 + 6)\\
&\to (1 : 0) + (5 + 6)
\end{aligned}
$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\to (0:((1+2)+(3+4)))+(5+6) \\
&\to 0:(((1+2)+(3+4))+(5+6)) \\
&\to ((1+2)+(3+4))+(5+6) \\
&\to (3+(3+4))+(5+6) \\
&\to (3+7)+(5+6) \\
&\to (1:0)+(5+6) \\
&\to 1:(0+(5+6))
\end{aligned}
$$

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Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\to (0 : ((1+2) + (3+4))) + (5+6) \\
&\to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\to ((1+2) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to 1 : (0 + (5+6)) \\
&\to 1 : (0 + (1:1))
\end{aligned}
$$

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Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\to (0 : ((1+2) + (3+4))) + (5+6) \\
&\to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\to ((1+2) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1 : 0) + (5+6) \\
&\to 1 : (0 + (5+6)) \\
&\to 1 : (0 + (1 : 1)) \\
&\to 1 : (1 : (0+1))
\end{aligned}
$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \\
&\to 0 : (((1 + 2) + (3 + 4)) + (5 + 6)) \\
&\to ((1 + 2) + (3 + 4)) + (5 + 6) \\
&\to (3 + (3 + 4)) + (5 + 6) \\
&\to (3 + 7) + (5 + 6) \\
&\to (1 : 0) + (5 + 6) \\
&\to 1 : (0 + (5 + 6)) \\
&\to 1 : (0 + (1 : 1)) \\
&\to 1 : (1 : (0 + 1)) \\
&\to (1 + 1) : (0 + 1)
\end{aligned}
$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\to (0 : ((1+2) + (3+4))) + (5+6) \\
&\to 0 : (((1+2) + (3+4)) + (5+6)) \\
&\to ((1+2) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to 1 : (0 + (5+6)) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to (1+1) : (0+1) \\
&\to 2 : (0+1)
\end{aligned}
$$

Definition

Leftmost outermost strategy always reduces the leftmost or the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\to (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \\
&\to 0 : (((1 + 2) + (3 + 4)) + (5 + 6)) \\
&\to ((1 + 2) + (3 + 4)) + (5 + 6) \\
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&\to (3 + 7) + (5 + 6) \\
&\to (1 : 0) + (5 + 6) \\
&\to 1 : (0 + (5 + 6)) \\
&\to 1 : (0 + (1 : 1)) \\
&\to 1 : (1 : (0 + 1)) \\
&\to (1 + 1) : (0 + 1) \\
&\to 2 : (0 + 1) \\
&\to 2 : 1
\end{aligned}
$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

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Example (cont'd)

$$(0 : (1 + 2)) + (3 + 4) + (5 + 6)$$

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Example (cont'd)

$$(0 : (1 + 2)) + (3 + 4) + (5 + 6) \to ((0 : 3) + (3 + 4)) + (5 + 6)$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$(0 : (1 + 2)) + (3 + 4) + (5 + 6) \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6)$$
$$\rightarrow (3 + (3 + 4)) + (5 + 6)$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6)
\end{aligned}
$$

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Example (cont'd)

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\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6)\\
&\to (3 + (3+4)) + (5+6)\\
&\to (3+7) + (5+6)\\
&\to (1:0) + (5+6)
\end{aligned}
$$

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Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1)
\end{aligned}
$$

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Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1))
\end{aligned}
$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
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&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1))
\end{aligned}
$$

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Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to 1 : (1:1)
\end{aligned}
$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1 + 2)) + (3 + 4) + (5 + 6) &\to ((0 : 3) + (3 + 4)) + (5 + 6) \\
&\to (3 + (3 + 4)) + (5 + 6) \\
&\to (3 + 7) + (5 + 6) \\
&\to (1 : 0) + (5 + 6) \\
&\to (1 : 0) + (1 : 1) \\
&\to 1 : (0 + (1 : 1)) \\
&\to 1 : (1 : (0 + 1)) \\
&\to 1 : (1 : 1) \\
&\to (1 + 1) : 1
\end{aligned}
$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to 1 : (1:1) \\
&\to (1+1) : 1 \\
&\to 2 : 1
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

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Example (cont'd)

$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$((0:(1+2))+(3+4))+(5+6) \twoheadrightarrow\!\!\!\!\!\shortparallel\; (0:((1+2)+(3+4)))+(1:1)$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\Vdash\!\!\rightarrow (0:((1+2)+(3+4)))+(1:1) \\
&\rightarrow 0:(((1+2)+(3+4))+(1:1))
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\mathrel{+\!\!\!\!\twoheadrightarrow} (0:((1+2)+(3+4)))+(1:1)\\
&\to 0:(((1+2)+(3+4))+(1:1))\\
&\to ((1+2)+(3+4))+(1:1)
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\twoheadrightarrow\!\!\!\!\!\!\| \;\; (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
&\to 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
&\to ((1 + 2) + (3 + 4)) + (1 : 1) \\
&\to 1 : (((1 + 2) + (3 + 4)) + 1)
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\mathrel{+\!\!\!\!\twoheadrightarrow} (0:((1+2)+(3+4)))+(1:1)\\
&\to 0:(((1+2)+(3+4))+(1:1))\\
&\to ((1+2)+(3+4))+(1:1)\\
&\to 1:(((1+2)+(3+4))+1)\\
&\mathrel{+\!\!\!\!\twoheadrightarrow} 1:((3+7)+1)
\end{aligned}$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\twoheadrightarrow\!\!\!\!\!\!\shortparallel\;\; (0 : ((1+2)+(3+4))) + (1:1) \\
&\rightarrow 0 : (((1+2)+(3+4)) + (1:1)) \\
&\rightarrow ((1+2)+(3+4)) + (1:1) \\
&\rightarrow 1 : (((1+2)+(3+4)) + 1) \\
&\twoheadrightarrow\!\!\!\!\!\!\shortparallel\;\; 1 : ((3+7)+1) \\
&\rightarrow 1 : ((1:0)+1)
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{+\!\!\!\!\!\twoheadrightarrow} (0 : ((1+2) + (3+4))) + (1 : 1) \\
&\to 0 : (((1+2) + (3+4)) + (1 : 1)) \\
&\to ((1+2) + (3+4)) + (1 : 1) \\
&\to 1 : (((1+2) + (3+4)) + 1) \\
&\mathrel{+\!\!\!\!\!\twoheadrightarrow} 1 : ((3+7) + 1) \\
&\to 1 : ((1 : 0) + 1) \\
&\to 1 : (1 : (0+1))
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\twoheadrightarrow\!\!\!\!\!\shortmid\; (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
&\to 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
&\to ((1 + 2) + (3 + 4)) + (1 : 1) \\
&\to 1 : (((1 + 2) + (3 + 4)) + 1) \\
&\twoheadrightarrow\!\!\!\!\!\shortmid\; 1 : ((3 + 7) + 1) \\
&\to 1 : ((1 : 0) + 1) \\
&\to 1 : (1 : (0 + 1)) \\
&\to (1 + 1) : (0 + 1)
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\twoheadrightarrow\!\!\!\!\!\!\!+\ (0:((1+2)+(3+4)))+(1:1)\\
&\to 0:(((1+2)+(3+4))+(1:1))\\
&\to ((1+2)+(3+4))+(1:1)\\
&\to 1:(((1+2)+(3+4))+1)\\
&\twoheadrightarrow\!\!\!\!\!\!\!+\ 1:((3+7)+1)\\
&\to 1:((1:0)+1)\\
&\to 1:(1:(0+1))\\
&\to (1+1):(0+1)\\
&\twoheadrightarrow\!\!\!\!\!\!\!+\ 2:1
\end{aligned}
$$

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Example (cont'd)

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

Definition

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Example (cont'd)

$$((0 : (1+2)) + (3+4)) + (5+6) \mathrel{+\!\!\!\!\Vdash\!\!\!\to} ((0:3) + 7) + (1:1)$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1 + 2)) + (3 + 4)) + (5 + 6) &\mathrel{+\!\!\!\!\!\!\twoheadrightarrow} ((0 : 3) + 7) + (1 : 1) \\
&\to (3 + 7) + (1 : 1)
\end{aligned}
$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\twoheadrightarrow\!\!\!\!\!\!\parallel\; ((0:3) + 7) + (1:1) \\
&\to (3+7) + (1:1) \\
&\to (1:0) + (1:1)
\end{aligned}
$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{+\!\!\!\!\!\rightarrow} ((0:3) + 7) + (1:1) \\
&\to (3+7) + (1:1) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1))
\end{aligned}
$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{\rightarrow\!\!\!\!\!\!\parallel}\; ((0:3)+7)+(1:1) \\
&\rightarrow (3+7)+(1:1) \\
&\rightarrow (1:0)+(1:1) \\
&\rightarrow 1:(0+(1:1)) \\
&\rightarrow 1:(1:(0+1))
\end{aligned}
$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{\|\!\!\!\to} ((0:3) + 7) + (1:1) \\
&\to (3+7) + (1:1) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to 1 : (1 : 1)
\end{aligned}
$$

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Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{\|\!\!\!\rightarrow} ((0 : 3) + 7) + (1 : 1) \\
&\rightarrow (3+7) + (1 : 1) \\
&\rightarrow (1 : 0) + (1 : 1) \\
&\rightarrow 1 : (0 + (1 : 1)) \\
&\rightarrow 1 : (1 : (0 + 1)) \\
&\rightarrow 1 : (1 : 1) \\
&\rightarrow (1+1) : 1
\end{aligned}
$$

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Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{+\!\!\!\!\!\!\rightarrow} ((0:3) + 7) + (1:1) \\
&\to (3+7) + (1:1) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to 1 : (1:1) \\
&\to (1+1) : 1 \\
&\to 2 : 1
\end{aligned}
$$

Definition

A development of set of redex positions Q in term t is rewrite sequence starting from t in which all contracted redex positions descend from position in Q.

Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0)$$ ☺

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$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0 \quad ☺$$

Definition (Overlining)

For a TRS $\mathcal{R} = \langle \Sigma, R \rangle$ we define the overlined TRS $\overline{\mathcal{R}} = \langle \overline{\Sigma}, \overline{R} \rangle$:

- $\overline{\Sigma} = \Sigma \cup \{\overline{f} \mid f \in \Sigma\}$,
- $\overline{R} = \{\overline{\rho} \mid \rho \in R\}$

where $\overline{\rho}$ is obtained from ρ by overlining the head symbol of the left-hand side.

$$\rho : f(s_1, \ldots, s_n) \to r \quad \text{yields} \quad \overline{\rho} : \overline{f}(s_1, \ldots, s_n) \to r$$

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Example

The overlined version of Combinatory Logic:

$$\begin{array}{lcl} \overline{Ap}(Ap(Ap(S,x),y),z) & \to & Ap(Ap(x,z),Ap(y,z)) \\ \overline{Ap}(Ap(K,x),y) & \to & x \\ \overline{Ap}(I,x) & \to & x \end{array}$$

We write $t \geq s$ if t can be obtained from s by overlining some redex positions.

Definition (Lifting)

A $\mathcal{R}$ rewrite sequence $A : s_1 \to s_2 \to \ldots \to s_n$ can be lifted if:

$$\begin{array}{ccccccc}
t_1 & \xrightarrow{\langle \overline{\rho}_1,\ p_1 \rangle} & t_2 & \xrightarrow{\langle \overline{\rho}_2,\ p_2 \rangle} & \cdots & \xrightarrow{\langle \overline{\rho}_{n-1},\ p_{n-1} \rangle} & t_n \\
\vee\!| & & \vee\!| & & & & \vee\!| \\
s_1 & \xrightarrow{\langle \rho_1,\ p_1 \rangle} & s_2 & \xrightarrow{\langle \rho_2,\ p_2 \rangle} & \cdots & \xrightarrow{\langle \rho_{n-1},\ p_{n-1} \rangle} & s_n
\end{array}$$

for some $\overline{\mathcal{R}}$ rewrite sequence $B : t_1 \to \ldots t_n$.

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for some $\overline{\mathcal{R}}$ rewrite sequence $B : t_1 \to \ldots t_n$.

Lemma

For orthogonal TRSs: a reduction is a development $\iff$ *it can be lifted.*

Theorem

Properties of $\overline{\mathcal{R}}$ for orthogonal TRSs $\mathcal{R}$:

- *$\overline{\mathcal{R}}$ is orthogonal.*
- *$\overline{\mathcal{R}}$ is SN.*
- *$\overline{\mathcal{R}}$ is CR.*

For SN we show that $\overline{\mathcal{R}}$ is ILPO terminating where $\overline{f} > g$ for every $f, g \in \Sigma$.

Let $\ell = \overline{f}(\ell_1, \ldots, \ell_n)$ and $r \in \mathcal{T}(\Sigma, \mathcal{X})$ with $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}\ell$.

Then $\ell^* \to_{ilpo}^* r$ by induction on r:

- If $r \in \mathcal{V}\mathrm{ar}(\ell)$, then we use $\to_{put}$ and $\to_{select}$.
- If $r = g(r_1, \ldots, r_m)$, then we use $\ell^* \to_{copy} g(\ell^*, \ldots, \ell^*)$. Moreover by the induction hypothesis $\ell^* \to_{ilpo}^* r_i$ for every i.

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Theorem

Developments are finite.

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Definition

A development $A\colon s \to^* t$ of $Q \subseteq \mathcal{P}\mathrm{os}(s)$ is complete if $Q/A = \varnothing$.

We write $s \;\circ\!\!\!\!\to\; t$ (called multi-step) if there is a complete development $s \to^* t$.

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Example

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☹$$

$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0 \qquad ☺$$

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Theorem

All complete developments of Q are permutation equivalent.

Definition

For orthogonal TRSs the full substitution strategy performs complete development of all redexes.

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- rewrite rules

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- full substitution strategy

$$\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)))$$

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- full substitution strategy

$$\begin{aligned} &\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\ &\quad \xrightarrow{\circ}\!\!\!\!\to (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \end{aligned}$$

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$$\begin{aligned} &\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\ &\quad \mathrel{\circ\!\!\!\to} (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \\ &\quad \mathrel{\circ\!\!\!\to} ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \end{aligned}$$

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&\quad \mathrel{\circ\!\!\!\!\rightarrow} ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\quad \to (0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))
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$$\begin{aligned}
&\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\
&\qquad \mathrel{\circ\!\!\!\!\longrightarrow} (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \\
&\qquad \mathrel{\circ\!\!\!\!\longrightarrow} ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\qquad \to (0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\qquad \to \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))
\end{aligned}$$

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For orthogonal TRSs the full substitution strategy performs complete development of all redexes.

Example

- rewrite rules

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- full substitution strategy

$$\begin{aligned}
&\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\
&\qquad \twoheadrightarrow\!\!\!\!\circ\; (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \\
&\qquad \twoheadrightarrow\!\!\!\!\circ\; ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\qquad \to (0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\qquad \to \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\qquad \to \ldots \to \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))))
\end{aligned}$$

Outline

Theorem

For orthogonal TRSs

- *full substitution and parallel outermost strategies are normalizing*

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- *full substitution and parallel outermost strategies are normalizing*
- *innermost strategies are perpetual*

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- *innermost strategies are perpetual*
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Theorem

For orthogonal TRSs

- *full substitution and parallel outermost strategies are normalizing*
- *innermost strategies are perpetual*
- *leftmost outermost strategy is not normalizing*
- *full substitution is fair*

Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\mathsf{f}(\mathsf{c}, \mathsf{a})$
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Theorem

For orthogonal TRSs

- *full substitution* *and* *parallel outermost* *strategies are* *normalizing*
- *innermost* *strategies are* *perpetual*
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$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

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Definition

A reduction $\rho = t_0 \to t_1 \to \dots$ is cofinal if for every $t_0 \to^* s$ there exists t_n in ρ such that $s \to^* t_n$.

Let t_0 be a term that has a normal form u ($t_0 \to^* u$).

Consider a maximal $\mathcal{S}$ rewrite sequence $\rho : t_0 \to t_1 \to \dots$ starting from t_0.

By cofinality there must be t_n in ρ such that $u \to^* t_n$.

Hence $t_n = u$ since u is a normal form. ∎

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A strategy $\mathcal{S}$ is cofinal if every every maximal $\mathcal{S}$ rewrite sequence is cofinal.

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Theorem

For orthogonal TRSs, every fair strategy is cofinal.

Let $\tau : t_0 \to u_0$. We show that $u_0 \to^* t_n$ for some t_n in ρ.

$$\begin{array}{ccccccccc}
t_0 & \longrightarrow & t_1 & \longrightarrow & t_2 & \longrightarrow \cdots \longrightarrow & t_n & \longrightarrow & \cdots \\
\downarrow \tau & & \downarrow \tau/\rho_1 & & \downarrow \tau/\rho_2 & & \mid \varnothing & & \\
u_0 & \twoheadrightarrow & u_1 & \twoheadrightarrow & u_2 & \twoheadrightarrow \cdots \twoheadrightarrow & u_n & &
\end{array}$$

Here ρ_i consists of the first i steps of ρ.

By fairness of ρ there exists n such that $\tau/\rho_n = \varnothing$. Hence $u_n = t_n$ and $u_0 \to^* t_n$.

The reduction $u_0 \to u_1 \to \ldots \to u_n \to t_{n+1} \to t_{n+2}$ is fair again.
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Leftmost outermost strategy is normalizing for orthogonal left-normal TRSs.

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Theorem

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Remark

Combinatory Logic is left-normal

$$\mathsf{I}\,x \to x \qquad \mathsf{K}\,x\,y \to x \qquad \mathsf{S}\,x\,y\,z \to x\,z\,(y\,z)$$