• Lecture 1: Introduction, Abstract Rewriting
• Lecture 2: Term Rewriting
• Lecture 3: Combinatory Logic
• Lecture 4: Termination
• Lecture 5: Matching, Unification
• Lecture 6: Equational Reasoning, Completion
• Lecture 7: Confluence
• Lecture 8: Modularity
• Lecture 9: Strategies
• Lecture 10: Decidability
• Lecture 11: Infinitary Rewriting
Outline

- Overview
- Strategies
Decidability
Definition

A Turing machine is a tuple \( \langle Q, q_0, F, \Sigma, \square, \delta \rangle \) where:

- \( Q \) is a finite set of states,
- \( q_0 \in Q \) is the initial state,
- \( F \subseteq Q \) is the set of final states,
- \( \Sigma \) is a finite alphabet,
- \( \square \in \Sigma \) is the blank symbol,
- \( \delta : (Q \setminus F) \times \Sigma \rightarrow Q \times \Sigma \times \{L, R\} \) is the transition function.
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Turing machines work on a two-sided infinite tape with one read/write head:
Definition (State Transition)

The current state:

\[ a_{-6} \ a_{-5} \ a_{-4} \ a_{-3} \ a_{-2} \ a_{-1} \ a_0 \ \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ \cdots \]

If \( \delta(q, a_0) = (b, q', L) \), then the following state is:

\[ a_{-6} \ a_{-5} \ a_{-4} \ a_{-3} \ a_{-2} \ a_{-1} \ b \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ \cdots \]

If \( \delta(q, a_0) = (b, q', R) \), then the following state is:

\[ a_{-6} \ a_{-5} \ a_{-4} \ a_{-3} \ a_{-2} \ a_{-1} \ b \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ \cdots \]
A Turing machine $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ is said to halt on $w = a_1 a_2 \ldots a_n \in \Sigma^*$ if it reaches a final state when started on the configuration:

\[ \cdots \square \square \square a_0 a_1 \cdots a_n \cdots \]

\[ \triangle q \]

The tape content upon reaching the final state is called output.
Decidability

**Definition**

A Turing machine $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ is said to halt on $w = a_1 a_2 \ldots a_n \in \Sigma^*$ if it reaches a final state when started on the configuration:

\[
\cdots \square \square \square a_0 a_1 \ldots a_n \cdots
\]

The tape content upon reaching the final state is called output.

**Definition (Decidable Problems)**

A predicate $P \subseteq \mathbb{N}^d$ is **decidable** if there is a Turing machine $M$ such that:

- $M$ terminates on all inputs $s^{n_1}0 \cdots s^{n_d}0$ for $\langle n_1, \ldots, n_d \rangle \in \mathbb{N}^d$, and
- $M$ halts with output 0 if and only if $\langle n_1, \ldots, n_d \rangle \in A$. 
Problem (Halting Problem)

*Input:* Turing machine $M$, word $w \in \Sigma^*$
*Question:* Does $M$ halt on $w$?
Problem (Halting Problem)

Input: Turing machine $M$, word $w \in \Sigma^*$
Question: Does $M$ halt on $w$?

Problem (Empty-Tape Halting Problem)

Input: Turing machine $M$
Question: Does $M$ halt on the empty tape $\epsilon$?
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Problem (Halting Problem)

Input: Turing machine $M$, word $w \in \Sigma^*$
Question: Does $M$ halt on $w$?

Problem (Empty-Tape Halting Problem)

Input: Turing machine $M$
Question: Does $M$ halt on the empty tape $\epsilon$?

Problem (Uniform Halting Problem)

Input: Turing machine $M$
Question: Does $M$ halt on all configurations?

Problem (Totality Problem)

Input: Turing machine $M$
Question: Does $M$ halt on all natural numbers $\{s^n0 \mid n \in \mathbb{N}\}$?
Decidability

Uniform Halting Problem

Totality Problem

Decidable Problems

\[ \Pi^0_0 = \Sigma^0_0 = \Delta^0_0 \]

Recursively Enumerable

Halting Problem

Empty Tape Halting Problem

Term Rewriting Systems - Lecture 10
Definition

A problem $A \subseteq \mathbb{N}^d$ is:

- in $\Sigma^0_1$ if $\bar{x} \in A \iff \exists y. P(\bar{x}, y)$,
- in $\Pi^0_2$ if $\bar{x} \in A \iff \forall y. \exists z. P(\bar{x}, y, z)$,

for some decidable predicate $P$. 

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Decidability

$\Pi^0_0 = \Sigma^0_0 = \Delta^0_0$

Uniform Halting Problem

Recursively Enumerable

Totality Problem

Halting Problem

Decidable Problems

Empty Tape Halting Problem

Term Rewriting Systems - Lecture 10
Decidability

Theorem

- \( SN(\mathcal{R}) \) and \( WN(\mathcal{R}) \) are \( \Pi^0_2 \)-complete.
- \( SN(\mathcal{R}, t) \) and \( WN(\mathcal{R}, t) \) of single terms are \( \Sigma^0_1 \)-complete.
- \( WCR(\mathcal{R}) \) and \( WCR(\mathcal{R}, t) \) are \( \Sigma^0_1 \)-complete.
- \( CR(\mathcal{R}) \) and \( CR(\mathcal{R}, t) \) are \( \Pi^0_2 \)-complete.
Example (From Turing Machine Configurations to Terms)

\[ \ldots \quad \square \quad a_{-2} \quad a_{-1} \quad a_0 \quad a_1 \quad a_2 \quad \square \quad \ldots \]

\[ \triangle q \]

becomes \( q(a_{-1}(a_{-2}(\triangleright)), a_0(a_1(a_2(\triangleright)))) \):

\[ q \]
\[ a_{-1} \quad a_0 \]
\[ a_{-2} \quad a_1 \]
\[ \triangleright \quad a_2 \]
\[ \triangleright \]
Example (From Turing Machine Configurations to Terms)

becomes \( q(a_{-1}(a_{-2}(\triangleright)), a_0(a_1(a_2(\triangleright)))) \):

\[
\begin{align*}
q(x, f(y)) & \rightarrow q'(f'(x), y) & \text{for every } \delta(q, f) = \langle q', f', R \rangle \\
q(g(x), f(y)) & \rightarrow q'(x, g(f'(y))) & \text{for every } \delta(q, f) = \langle q', f', L \rangle
\end{align*}
\]
For a Turing machine $M = \langle Q, q_0, F, \Sigma, \square, \delta \rangle$ we define a TRS $R_M$ as follows:

\[
q(x, f(y)) \rightarrow q'(f'(x), y) \quad \text{for every } \delta(q, f) = \langle q', f', R \rangle
\]
\[
q(g(x), f(y)) \rightarrow q'(x, g(f'(y))) \quad \text{for every } \delta(q, f) = \langle q', f', L \rangle
\]
**Definition (From Turing Machines to Term Rewriting Systems)**

For a Turing machine $M = \langle Q, q_0, F, \Sigma, \sqbox, \delta \rangle$ we define a TRS $R_M$ as follows:

\[
\begin{align*}
q(x, f(y)) &\rightarrow q'(f'(x), y) &\text{for every } \delta(q, f) = \langle q', f', R \rangle \\
q(g(x), f(y)) &\rightarrow q'(x, g(f'(y))) &\text{for every } \delta(q, f) = \langle q', f', L \rangle 
\end{align*}
\]

Together with four rules for ‘extending the tape’:

\[
\begin{align*}
q(\\sqbox, f(y)) &\rightarrow q'(\\sqbox, \sqbox(f'(y))) &\text{for every } \delta(q, f) = \langle q', f', L \rangle \\
q(x, \sqbox) &\rightarrow q'(f'(x), \sqbox) &\text{for every } \delta(q, \sqbox) = \langle q', f', R \rangle \\
q(g(x), \sqbox) &\rightarrow q'(x, g(f'(\sqbox))) &\text{for every } \delta(q, \sqbox) = \langle q', f', L \rangle \\
q(\sqbox, \sqbox) &\rightarrow q'(\sqbox, \sqbox(f'(\sqbox))) &\text{for every } \delta(q, \sqbox) = \langle q', f', L \rangle 
\end{align*}
\]
Decidability

Observations on $\mathcal{R}_M$

**Theorem**

A Turing machine $M$ halts on $w$ if and only if $q_0(\triangleright, w(\triangleright))$ terminates in $\mathcal{R}_M$. 
Decidability

Observations on $\mathcal{R}_M$

**Theorem**

A Turing machine $M$ halts on $w$ if and only if $q_0(\triangleright, w(\triangleright))$ terminates in $\mathcal{R}_M$. 

**Theorem**

A Turing machine $M$ halts on all configurations if and only if $\mathcal{R}_M$ terminates.
Observations on $\mathcal{R}_M$

**Theorem**

A Turing machine $M$ halts on $w$ if and only if $q_0(\triangleright, w(\triangleright))$ terminates in $\mathcal{R}_M$.

**Theorem**

A Turing machine $M$ halts on all configurations if and only if $\mathcal{R}_M$ terminates.

**Lemma**

For ever Turing machine $M$, the TRS $\mathcal{R}_M$ is orthogonal.
As a consequence we obtain:

**Theorem**

\[ SN(R, t) \text{ and } WN(R, t) \text{ of single terms are } \Sigma_1^0\text{-complete.} \]
As a consequence we obtain:

**Theorem**

$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are $\Sigma_1^0$-complete.

**Proof.**

The term $t = q_0(\triangleright, \triangleright)$ is normalizing in $\mathcal{R}_M \iff M$ halts on the blank tape. Moreover $SN(t) \iff WN(t)$ since every reduct contains at most one redex.
As a consequence we obtain:

**Theorem**

$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are $\Sigma_1^0$-complete.

**Proof.**

The term $t = q_0(\triangleright, \triangleright)$ is normalizing in $\mathcal{R}_M \iff M$ halts on the blank tape. Moreover $SN(t) \iff WN(t)$ since every reduct contains at most one redex.

**Theorem**

$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are $\Pi_2^0$-complete.
Decidability

Complexity of Local Confluence

**Definition**

For a Turing machine $M$ we define $H_M$ to be $R_M$ extended with:

- $q(x, f(y)) \rightarrow T$ for every $f \in \Sigma, q \in Q$ with $\delta(q, f)$ is undefined
- $q(x, \triangledown) \rightarrow T$ for every $q \in Q$ with $\delta(q, \square)$ is undefined

**Theorem**

$WCR(R)$ and $WCR(R, q_0(\triangledown, \triangledown))$ are $\Sigma_0^1$-complete.

**Proof.**

We extend $H_M$ with:

$S \rightarrow T S \rightarrow q_0^0(\triangledown, \triangledown)$

Then $\langle T, q_0(\triangledown, \triangledown) \rangle$ and $\langle q_0(\triangledown, \triangledown), T \rangle$ are the only critical pairs.

Hence $WCR(R)$ and $WCR(R, q_0(\triangledown, \triangledown))$ hold

$\iff q_0(\triangledown, \triangledown) \rightarrow^* T$

$\iff M$ halts on the empty tape.
Decidability

Complexity of Local Confluence

Definition

For a Turing machine $M$ we define $H_M$ to be $R_M$ extended with:

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$\iff M$ halts on the empty tape.
Decidability

Complexity of Local Confluence

Definition

For a Turing machine \( M \) we define \( H_M \) to be \( \mathcal{R}_M \) extended with:

\[
q(x, f(y)) \rightarrow T \quad \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined}
\]

\[
q(x, \triangledown) \rightarrow T \quad \text{for every } q \in Q \text{ with } \delta(q, \square) \text{ is undefined}
\]

Theorem

\( \text{WCR}(\mathcal{R}) \) and \( \text{WCR}(\mathcal{R}, t) \) are \( \Sigma_1^0 \)-complete.

Proof.

We extend \( H_M \) with:

\[
S \rightarrow T \quad S \rightarrow q_0(\triangledown, \triangledown)
\]
Decidability

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Definition
For a Turing machine $M$ we define $H_M$ to be $\mathcal{R}_M$ extended with:

$q(x, f(y)) \rightarrow T$ for every $f \in \Sigma$, $q \in Q$ with $\delta(q, f)$ is undefined

$q(x, \triangleright) \rightarrow T$ for every $q \in Q$ with $\delta(q, \square)$ is undefined

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$WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are $\Sigma_1^0$-complete.

Proof.
We extend $H_M$ with:

$S \rightarrow T$

$S \rightarrow q_0(\triangleright, \triangleright)$

Then $\langle T, q_0(\triangleright, \triangleright) \rangle$ and $\langle q_0(\triangleright, \triangleright), T \rangle$ are the only critical pairs.
Decidability

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For a Turing machine $M$ we define $H_M$ to be $R_M$ extended with:

- $q(x, f(y)) \rightarrow T$ for every $f \in \Sigma$, $q \in Q$ with $\delta(q, f)$ is undefined
- $q(x, \triangleright) \rightarrow T$ for every $q \in Q$ with $\delta(q, \Box)$ is undefined

Theorem

$WCR(R)$ and $WCR(R, t)$ are $\Sigma_1^0$-complete.

Proof.

We extend $H_M$ with:

- $S \rightarrow T$
- $S \rightarrow q_0(\triangleright, \triangleright)$

Then $\langle T, q_0(\triangleright, \triangleright) \rangle$ and $\langle q_0(\triangleright, \triangleright), T \rangle$ are the only critical pairs.

Hence $WCR(R)$ and $WCR(R, q_0(\triangleright, \triangleright))$ hold

$\iff q_0(\triangleright, \triangleright) \rightarrow^* T$

$\iff M$ halts on the empty tape.
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For a Turing machine $M$ we define $H_M$ to be $R_M$ extended with:

- $q(x, f(y)) \rightarrow T$ for every $f \in \Sigma$, $q \in Q$ with $\delta(q, f)$ is undefined
- $q(x, \triangleright) \rightarrow T$ for every $q \in Q$ with $\delta(q, \Box)$ is undefined

**Theorem**

$CR(R, T)$ for single terms is $\Pi_{0}^{2}$-complete.

**Proof.**

We extend $H_M$ with:

- $S(x) \rightarrow T$
- $S(x) \rightarrow S(s(x))$
- $S(x) \rightarrow q_0(\triangleright, x)$

Then $T \leftarrow S(0(\triangleright)) \rightarrow^* S(s^n(0(\triangleright))) \rightarrow q_0(\triangleright, s^n(0(\triangleright)))$ for all $n \in \mathbb{N}$.

Hence $CR(R, S(0(\triangleright)))$ holds

$\iff q_0(\triangleright, s^n(0(\triangleright))) \rightarrow^* T$ for all $n \in \mathbb{N}$

$\iff M$ is total.
Decidability

Complexity of Confluence

Definition

For a Turing machine \( M \) we define \( H_M \) to be \( R_M \) extended with:

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q(x, f(y)) \rightarrow T \quad \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined}
\]

\[
q(x, \triangleright) \rightarrow T \quad \text{for every } q \in Q \text{ with } \delta(q, \square) \text{ is undefined}
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Theorem

\( CR(R, t) \) for single terms is \( \Pi^0_2 \)-complete.

Then \( T \leftarrow S(0(\triangleright)) \rightarrow^* S(s^n(0(\triangleright))) \rightarrow q_0(\triangleright, s^n(0(\triangleright))) \) for all \( n \in \mathbb{N} \).

Hence \( CR(R, S(0(\triangleright))) \) holds

\[
\iff q_0(\triangleright, s^n(0(\triangleright))) \rightarrow^* T \text{ for all } n \in \mathbb{N}
\]

\[
\iff M \text{ is total.}
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Decidability

Complexity of Confluence

Definition
For a Turing machine $M$ we define $H_M$ to be $R_M$ extended with:

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We extend $H_M$ with:

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Then $T \leftarrow S(0(\triangleright)) \rightarrow^* S(s^n(0(\triangleright))) \rightarrow q_0(\triangleright, s^n(0(\triangleright)))$ for all $n \in \mathbb{N}$.

Hence $CR(\mathcal{R}, S(0(\triangleright)))$ holds

$\iff q_0(\triangleright, s^n(0(\triangleright))) \rightarrow^* T$ for all $n \in \mathbb{N}$

$\iff M$ is total.
Decidability

Complexity of Confluence

**Theorem**

$CR(\mathcal{R})$ is $\Pi^0_2$-complete.

Then

$$T \leftarrow S(u, \triangleright) \leftrightarrow^* S(\triangleright, v) \rightarrow q_0(\triangleright, v)$$

if and only if $u = v = s^n(\triangleright)$ for some $n \in \mathbb{N}$.

Hence $CR(\mathcal{R})$ holds

$$\iff q_0(\triangleright, s^n(\triangleright)) \rightarrow^* T \text{ for all } n \in \mathbb{N}$$

$$\iff M \text{ is total.}$$
Theorem

$CR(\mathcal{R})$ is $\Pi^0_2$-complete.

Proof.

We extend $H_M$ with:

$$S(x, \triangleright) \rightarrow T$$
$$S(\triangleright, y) \rightarrow q_0(\triangleright, y)$$
$$S(x, s(y)) \rightarrow S(S(x), y)$$
$$S(s(x), y) \rightarrow S(x, s(y))$$

Then $T \leftarrow S(u, \triangleright) \iff S(\triangleright, v) \rightarrow q_0(\triangleright, v)$ if and only if $u = v = s_n(\triangleright)$ for some $n \in \mathbb{N}$.

Hence $CR(\mathcal{R})$ holds $\iff q_0(\triangleright, s_n(\triangleright)) \rightarrow \ast T$ for all $n \in \mathbb{N}$ $\iff M$ is total.
Complexity of Confluence

**Theorem**

$CR(\mathcal{R})$ is $\Pi^0_2$-complete.

**Proof.**

We extend $H_M$ with:

- $S(x, \triangleright) \rightarrow T$
- $S(\triangleright, y) \rightarrow q_0(\triangleright, y)$
- $S(x, s(y)) \rightarrow S(S(x), y)$
- $S(s(x), y) \rightarrow S(x, s(y))$

Then

$$T \leftarrow S(u, \triangleright) \leftrightarrow^* S(\triangleright, v) \rightarrow q_0(\triangleright, v)$$

if and only if $u = v = s^n(\triangleright)$ for some $n \in \mathbb{N}$. 
Decidability

Complexity of Confluence

Theorem

$CR(\mathcal{R})$ is $\Pi^0_2$-complete.

Proof.

We extend $H_M$ with:

\[
\begin{align*}
S(x, \triangleright) & \rightarrow T \\
S(\triangleright, y) & \rightarrow q_0(\triangleright, y) \\
S(x, s(y)) & \rightarrow S(S(x), y) \\
S(s(x), y) & \rightarrow S(x, s(y))
\end{align*}
\]

Then

\[
T \leftarrow S(u, \triangleright) \leftrightarrow^* S(\triangleright, v) \rightarrow q_0(\triangleright, v)
\]

if and only if $u = v = s^n(\triangleright)$ for some $n \in \mathbb{N}$.

Hence $CR(\mathcal{R})$ holds

\[
\begin{align*}
\iff q_0(\triangleright, s^n(\triangleright)) & \rightarrow^* T \text{ for all } n \in \mathbb{N} \\
\iff M \text{ is total.}
\end{align*}
\]
Decidability

\[ SN(\mathcal{R}), WN(\mathcal{R}) \]
\[ CR(\mathcal{R}), CR(\mathcal{R}, t) \]

Decidable Problems

\[ \Pi_0^0 = \Sigma_0^0 = \Delta_0^0 \]

Theorem

- \( SN(\mathcal{R}) \) and \( WN(\mathcal{R}) \) are \( \Pi_2^0 \)-complete.
- \( SN(\mathcal{R}, t) \) and \( WN(\mathcal{R}, t) \) of single terms are \( \Sigma_1^0 \)-complete.
- \( WCR(\mathcal{R}) \) and \( WCR(\mathcal{R}, t) \) are \( \Sigma_1^0 \)-complete.
- \( CR(\mathcal{R}) \) and \( CR(\mathcal{R}, t) \) are \( \Pi_2^0 \)-complete.