Term Rewriting Systems

2010

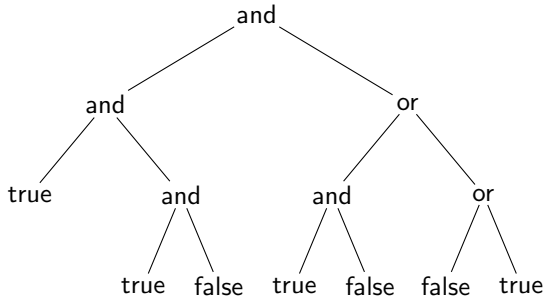
Training Exam

1. This exercise is about ARSs:
   (a) Define the properties CR, UN and WN.
   (b) Give a counterexample for the implication UN $\Rightarrow$ CR.
2. Let $\rightarrow$ defined on A have the property WDP (*Weak Diamond property*), that is
$$\forall a, b, c \in A \ : \ c \leftarrow a \rightarrow b \ \wedge c \neq b \ \Rightarrow \ (\exists d \in A \ : \ c \rightarrow d \leftarrow b)$$
   Prove that if a has a normal form, then
   (a) $a \in \mathrm{SN}$ (i.e. there are no infinte reductions from a).
   (b) All reductions from a to normal form have the same length.
3. Given the following TRS:
$$\mathcal{R} \left\{ \begin{array}{lcl} x + 0 & \longrightarrow & x \\ (x + y) + z & \longrightarrow & x + (y + z) \\ D(x + y) & \longrightarrow & D(x) + D(y) \\ D(0) & \longrightarrow & 0 \end{array} \right.$$
   (a) Find all critical pairs of $\mathcal{R}$ and evaluate WCR.
   (b) In case $\mathcal{R}$ is not WCR, find a completion $\mathcal{R}'$ of $\mathcal{R}$.
   (c) Show that $\mathcal{R}'$ is SN.
   (d) Decide whether $D((x + D(y)) + 0) \ =_{\mathcal{R}'} \ (0 + D(x)) + (D(D(y) + 0))$.
4. Prove or disprove CR for the following two TRSs.
   (a) $f(a, x) \rightarrow f(x, x)$
   $b \rightarrow a$
   (b) $g(h(g(x))) \rightarrow g(h(x))$
5. This exercise is about CL.
   (a) Give a reduction of the term $(IS)(IS)(IS)(IS)$ to normal form.
   (b) Give a complete development of the term $(IS)(IS)(IS)(IS)$.
   (c) Give a CL-term W such that $Wxy \twoheadrightarrow x(yy)$.
   (d) Assume that N is a CL-term such that $N(xy) \twoheadrightarrow y$.
      (i) Show that then $N(K(II)K) \twoheadrightarrow K$ and also $N(K(II)K) \twoheadrightarrow I$. (Which general property of reduction do you use here?)
      (ii) Using (i) argue that such an N can not exist in CL.
   (e) Using a fixed-point combinator Y, give a CL-term F such that $F \twoheadrightarrow FF$.
   (f) Consider the CL-terms
$$s_1 = S \qquad s_2 = SS \qquad s_3 = SSS \qquad s_4 = SSSS \qquad \ldots$$
   That is, $s_1 = S$ and $s_{n+1} = s_n S$ for $n \in \mathbb{N}$. Show that all terms s_n are WN.

6. Consider the TRS

$$\begin{aligned} \mathsf{and}(\mathsf{true}, x) &\to x \\ \mathsf{and}(\mathsf{false}, x) &\to \mathsf{false} \\ \mathsf{or}(\mathsf{true}, x) &\to \mathsf{true} \\ \mathsf{or}(\mathsf{false}, x) &\to x \end{aligned}$$

and the term $t = \mathsf{and}(\mathsf{and}(\mathsf{true}, \mathsf{and}(\mathsf{true}, \mathsf{false})), \mathsf{or}(\mathsf{and}(\mathsf{true}, \mathsf{false}), \mathsf{or}(\mathsf{false}, \mathsf{true})))$.
The tree representation of t is:



Reduce the term t to normal form using the following strategies:

(a) leftmost innermost

(b) leftmost outermost

(c) parallel innermost

(d) parallel outermost

(e) full substitution

7. Consider the TRS

$$\begin{aligned} c(x) &\to f(x, g(c(x))) \\ a &\to b \end{aligned}$$

(a) What is the infinite normal form n of the term $c(a)$?

(b) Give a transfinite reduction from $c(a)$ to n of length $\omega + \omega$.

(c) Give also a transfinite reduction from $c(a)$ to n of length ω.

(d) Give a transfinite reduction from $c(a)$ to n of length $\omega + \omega + 2$.

(e) Indicate a single step p and an infinite reduction ρ, both starting from $c(a)$, such that the set of residuals of p after ρ, i.e. p/ρ is infinite.