Solutions to selected exercises from Terese by Bas Luttik

Exercise 1.3.1 Let $\to_\diamond$ be a reduction relation on A such that $\to\subseteq\to_\diamond\subseteq\twoheadrightarrow$, and suppose that $\to_\diamond$ has the diamond property.

(i) To prove that $\to_\diamond$ is confluent, it suffices by Proposition 1.1.10 to prove for all $a, b, c \in A$ that $c \leftarrow_\diamond a \twoheadrightarrow_\diamond b$ implies the existence of a $d \in A$ such that $c \twoheadrightarrow_\diamond d \leftarrow_\diamond b$.

We proceed by induction on the length of a given reduction $a \twoheadrightarrow_\diamond b$. If $a \equiv b$, then take $d \equiv c$. If $a \twoheadrightarrow_\diamond b' \to_\diamond b$, then by the induction hypothesis there exists a $d' \in A$ such that $c \twoheadrightarrow_\diamond d' \leftarrow_\diamond b'$. Since $\to_\diamond$ has the diamond property, from $d' \leftarrow_\diamond b' \to_\diamond b$ follows the existence of a $d \in A$ such that $d' \to_\diamond d \leftarrow_\diamond b$, so $c \twoheadrightarrow_\diamond d \leftarrow_\diamond b$.

(ii) If $a \twoheadrightarrow b$, then there exist $a_0, \ldots, a_n \in A$ such that $a \equiv a_0 \to \cdots \to a_n \equiv b$, and since $\to\subseteq\to_\diamond$ it follows that $a \equiv a_0 \to_\diamond \cdots \to_\diamond a_n \equiv b$, so $a \twoheadrightarrow_\diamond b$. On the other hand, if $a \twoheadrightarrow_\diamond b$, then there exist $a_0, \ldots, a_n \in A$ such that $a \equiv a_0 \to_\diamond \cdots \to_\diamond a_n \equiv b$. Since $\to_\diamond\subseteq\twoheadrightarrow$, it follows that $a \equiv a_0 \twoheadrightarrow \cdots \twoheadrightarrow a_n \equiv b$, so $a \twoheadrightarrow b$ by transitivity of $\twoheadrightarrow$.

(iii) $\twoheadleftarrow \cdot \twoheadrightarrow = \twoheadleftarrow_\diamond \cdot \twoheadrightarrow_\diamond \subseteq \twoheadrightarrow_\diamond \cdot \twoheadleftarrow_\diamond = \twoheadrightarrow \cdot \twoheadleftarrow$ (the inclusion is by (i); the equalities are by (ii)).

Exercise 2.3.9

(i) Define $\|t\|$ inductively by

$$\begin{aligned}
\|x\| &= 0\\
\|f(t)\| &= \|t\| + 1\\
\|g(t)\| &= \|t\| + 2.
\end{aligned}$$

Note that $\|g(t)\| = \|t\|+2 > \|t\|+1 = \|f(t)\|$ and $\|f(t)\| = \|t\|+1 > \|t\|$. Also note that if $\|t\| > \|u\|$, then $\|f(t)\| > \|f(u)\|$ and $\|g(t)\| > \|g(u)\|$, and hence, by induction, $\|C[t]\| > \|C[u]\|$. It follows that if $t \to u$, then $\|t\| > \|u\|$, so we may conclude SN from the well-foundedness of $<$ on the natural numbers.

(ii) Define $\|t\|$ inductively by

$$\begin{aligned}
\|x\| &= 0\\
\|f(t)\| &= \|t\| + 1\\
\|g(t)\| &= 3\|t\|,
\end{aligned}$$

and verify that $t \to u$ implies $\|t\| > \|u\|$; it follows that the TRS is SN.

(iii) This TRS is not SN: $g(g(f(x))) \to g(f(f(g(g(x))))) \to f(f(g(g(f(g(g(x)))))))$, and the last term contains the first term in the sequence.

(iv) Define $\|t\|$ by

$$\begin{aligned}
\|x\| &= 0\\
\|f(t)\| &= \begin{cases} \|t\| + 1 & t \equiv f(u)\\ \|t\| & \text{otherwise.}\end{cases}\\
\|g(t)\| &= \|t\|.
\end{aligned}$$

Exercise 2.3.11 Suppose there exists an infinite sequence

$$t_0 \rhd t_1 \rhd \cdots \rhd t_k \rhd \cdots \qquad (k \in \omega).$$

Then, since $\subset$ is well-founded and transitive, there also exists an infinite sequence

$$u_0 \supset u'_0 \to^+ u_1 \supset u'_1 \to^+ \cdots \to^+ u_k \supset u'_k \to^+ \qquad (k \in \omega).$$

Note that $u_k \supset u'_k$ means that there exists a (nonempty) context $C_k[]$ such that $u_k = C_k[u'_k]$. Define $v_k \equiv C_0[C_1[\cdots C_k[u'_k]\cdots]]$ for all $k \in \omega$. Then, since $\to^+$ is closed under contexts,

$$v_0 \to^+ v_1 \to^+ \cdots \to^+ v_k \to^+ \cdots \qquad (k \in \omega).$$

We have shown that if $\rhd$ admits an infinite sequence, then so does $\to$. Hence, if $\to$ is SN, then also $\rhd$ is SN.

Exercise 2.7.11 A systematic analysis reveals that there is overlap between the left-hand sides of

1. ρ_1 and ρ_3, resulting in the convergent critical pair $\langle x \cdot z, e \cdot (x \cdot z)\rangle$;
2. ρ_2 and ρ_3, resulting in the critical pair $\langle e \cdot z, I(x) \cdot (x \cdot z)\rangle$, which is not convergent ($I(x) \cdot (x \cdot z)$ is a normal form and, besides $e \cdot z$, z is the only reduct of $e \cdot z$);
3. ρ_3 and ρ_3, resulting in the convergent critical pair $\langle (x \cdot (y \cdot z)) \cdot u, (x \cdot y) \cdot (z \cdot u)\rangle$.

Exercise 2.7.13

2.7.5(i): $\langle L(0), 0\rangle$;

2.7.5(ii): $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$; and

2.7.5(iii): $\langle x, y\rangle$ and $\langle y, x\rangle$.

Exercise 2.7.17 There is overlap between the left-hand sides of

1. the first and the second rule, resulting in the critical pair
$$\langle \varnothing, \varnothing\rangle;$$
2. the first and the sixth rule, resulting in the critical pair
$$\langle x/z, (x/z) \cdot (\varnothing/(z/x))\rangle;$$
3. the first and the seventh rule, resulting in the critical pair
$$\langle z/x, (z/x)/\varnothing\rangle;$$
4. the second and the sixth rule, resulting in the critical pair
$$\langle x/z, (\varnothing/z) \cdot (x/(z/\varnothing))\rangle;$$
5. the second and the seventh rule, resulting in the critical pair
$$\langle z/x, (z/\varnothing)/x\rangle;$$

6. the third and the fourth rule, resulting in the critical pair

   $$\langle \varnothing, \varnothing \rangle;$$

7. the third and the fifth rule, resulting in the critical pair

   $$\langle \varnothing, \varnothing \rangle;$$

8. the third and the sixth rule, resulting in the critical pairs

   $$\langle x \cdot y, (x/\varnothing) \cdot (y/(\varnothing/x)) \rangle \text{ and } \langle (x/\varnothing) \cdot (y/(\varnothing/x)), x \cdot y \rangle;$$

9. the fourth and the fifth rule, resulting in the critical pair

   $$\langle \varnothing, \varnothing \rangle;$$

10. the fourth and the seventh rule, resulting in the critical pairs

    $$\langle \varnothing, (\varnothing/x)/y \rangle \text{ and } \langle (\varnothing/x)/y, \varnothing \rangle;$$

11. the fifth and the sixth rule, resulting in the critical pairs

    $$\langle \varnothing, (x/(x \cdot y)) \cdot (y/((x \cdot y)/x)) \rangle \text{ and } \langle (x/(x \cdot y)) \cdot (y/((x \cdot y)/x)), \varnothing \rangle;$$

12. the fifth and the seventh rule, resulting in the critical pairs

    $$\langle \varnothing, ((x \cdot y)/x)/y \rangle \text{ and } \langle ((x \cdot y)/x)/y, \varnothing \rangle; \text{ and}$$

13. the sixth and the seventh rule, resulting in the critical pairs

$$\langle (x/(x{\cdot}y)){\cdot}(y/((x{\cdot}y)/x)), ((x{\cdot}y)/x)/y \rangle \text{ and } \langle ((x{\cdot}y)/x)/y, (x/(x{\cdot}y)){\cdot}(y/((x{\cdot}y)/x)) \rangle.$$

It is straigthforward to verify that all these critical pairs are convergent, so by the Critical Pair Lemma the TRS is WCR. Since it is also SN (as will be proved in Exercise 6.5.43), it follows by Newman's Lemma that it is CR.

Exercise 2.7.18 There is overlap between the left-hand sides of

1. r_1 and r_2, resulting in the critical pair $\langle \neg(\mathit{false}), \mathit{true} \rangle$ which is convergent according to r_6;
2. r_1 and r_3, resulting in the critical pair $\langle \neg(x), \neg(x) \rangle$ which is trivially convergent; and
3. r_2 and r_5, resulting in the critical pair $\langle \neg(\mathit{true}), \mathit{false} \rangle$ which is convergent according to r_1.

Since all critical pairs are convergent, it follows by the Critical Pair Lemma that the TRS is WCR.

To prove that it is SN, we define a complexity measure $\|\_\|$ on terms, by induction on the structure:

$$\|\mathit{true}\| = \|\mathit{false}\| = 1,$$
$$\|\neg(t)\| = \|t\| + 1,$$
$$\|\mathit{and}(t, u)\| = \|t\| + \|u\| + 1 \text{ and}$$
$$\|\mathit{or}(t, u)\| = \|t\| + \|u\| + 5$$

(so $\|t\|$ is the length of t with every *or* symbol counted five times). It is easy to verify that $t \to u$ implies $\|t\| > \|u\|$, so the TRS is SN. It then follows by Newman's Lemma that the TRS is also CR, so it is complete.

Exercise 2.7.19 By Newman's Lemma, a strongly normalizing TRS is CR if, and only if, it is WCR, and to decide whether a strongly normalizing TRS is WCR it suffices by the Critical Pair Lemma to verify whether all critical pairs are convergent. If a strongly normalizing TRS moreover has finitely many rules, then clearly there is an algorithm that associates with every term t the finite set $\mathrm{Red}(t)$ of all reducts of t. There is also an algorithm that computes the finite set of all critical pairs. To check whether a critical pair $\langle t, u\rangle$ is convergent, it suffices to determine whether $\mathrm{Red}(t) \cap \mathrm{Red}(u)$ is empty, so it is decidable whether all critical pairs of a finite and strongly normalizing TRS are convergent. Hence it is decidable whether a strongly normalizing TRS with finitely many rules is WCR and CR.

Exercise 2.7.20

(i) Straightforward.

(ii) From (i) by the Critical Pair Lemma.

(iii) $\rho : C(A) \to_{\rho_2} D(A, C(A)) \to_{\rho_3} D(C(A), C(A)) \to_{\rho_1} E$ and $C(A) \to_{\rho_3} C(C(A)) \twoheadrightarrow_{\rho} C(E)$.

(iv) We prove that $C(E)$ and E do have a common reduct by assuming, to the contrary, that $C(E)$ and E do have a common reduct, and deriving a contradiction from this assumption. Clearly, if $C(E)$ and E have a common reduct, then, since E is a normal form, $C(E)$ must reduce to E. Consider a reduction $C(E) \twoheadrightarrow E$ of minimal length (i.e., there exists no reduction from $C(E)$ to E consisting of fewer steps); it must be of the form

$$C(E) \to D(E, C(E)) \twoheadrightarrow D(t,t) \to E \qquad \text{for some term } t. \tag{1}$$

Note that $D(E, C(E)) \twoheadrightarrow D(t,t)$ must have subreductions $E \twoheadrightarrow t$ and $C(E) \twoheadrightarrow t$. Hence, since E is a normal form, it has a subreduction $C(E) \twoheadrightarrow E$ contradicting our minimality assumption for the reduction in (1). We conclude that $C(E)$ and E do not have a common reduct.

Exercise 3.3.18

(ii) The terms $([f]D[x]f(xx))F$ and $F(([f]D[x]f(xx))F)$ are convertible according to

$$\begin{aligned} ([f]D[x]f(xx))F &\twoheadrightarrow D[x]F(xx) \\ &\twoheadrightarrow ([x]F(xx))([x]F(xx)) \\ &\twoheadrightarrow F(([x]F(xx))([x]F(xx))) \\ &\twoheadleftarrow F(D[x]F(xx)) \\ &\twoheadleftarrow F(([f]D[x]f(xx))F). \end{aligned}$$

(iii) In a combinatorially complete applicative TRS there are terms s, k and i such that

$$\begin{array}{rcl} sxyz & \twoheadrightarrow & xz(yz) \\ kxy & \twoheadrightarrow & x \\ ix & \twoheadrightarrow & x, \end{array}$$

so in a combinatorially complete applicative TRS there exists for every fixed-point combinator of CL a corresponding fixed-point combinator defined by means of s, k and i.

(iv) Note that $Y(SI)F = SI(Y(SI))F$ since Y is a fixed-point combinator, and that $SI(Y(SI))F \to IF(Y(SI)) \to F(Y(SI)F)$. It follows that $Y(SI)F = F(Y(SI)F)$, so $Y(SI)$ is a fixed-point combinator as well.

Exercise 3.3.21

$$\begin{aligned} \mathit{Pred}' &\equiv Y[y][x_1x_2]\mathit{Cond}(\mathit{Eq}(\mathit{Plus}x_1\mathbf{1})x_2)x_1(y(\mathit{Plus}x_1\mathbf{1})x_2) \\ \mathit{Pred} &\equiv \mathit{Pred}'\mathbf{0} \\ \mathit{Fac} &\equiv Y[y][x]\mathit{Cond}(\mathit{Eq}x\mathbf{0})\mathbf{1}(\mathit{Times}(y\mathit{Pred}x))x) \end{aligned}$$

Exercise 4.2.9

(i) Consider the TRS of Example 4.1.4(i). The steps $E(\infty,\infty) \to \mathit{true}$ and $E(\infty,\infty) \to E(\infty, S(\infty))$ are not orthogonal, for the rule applied in $E(\infty,\infty) \to \mathit{true}$ is not left-linear. Note that $S(\infty)$ has no descendants in true and that $E(\infty, S(\infty))$, the descendant of $E(\infty,\infty)$ over $E(\infty,\infty) \to E(\infty, S(\infty))$ is not a redex.

(ii) Consider the TRS consisting of the rule

$$\begin{aligned} F(G(x)) &\to A, \\ G(x) &\to B. \end{aligned}$$

Then the steps $F(G(x)) \to A$ and $F(G(x)) \to F(B)$ are not orthogonal, for there is overlap. Clearly, $G(x)$ has no descendants over $F(G(x)) \to A$ and the only descendant of $F(G(x))$ over $F(G(x)) \to F(B)$ is $F(B)$, which is not a redex.

Exercise 6.2.7

(i) Define on $\mathbb{N}^+$ unary operations $f_{\mathbb{N}^+}$ and $g_{\mathbb{N}^+}$ for all $n \in \mathbb{N}^+$ by $f_{\mathbb{N}^+}(n) = 2n$ and $g_{\mathbb{N}^+}(n) = n+1$. Clearly, $f_{\mathbb{N}}$ and $g_{\mathbb{N}}$ are strictly monotone in their argument and $f_{\mathbb{N}}(g_{\mathbb{N}}(n)) = 2(n+1) > 2n+1 = g_{\mathbb{N}}(f_{\mathbb{N}}(n))$. It follows that the TRS consisting of the rule $f(g(x)) \to g(f(x))$ is SN.

(ii) Define on $\mathbb{N}^+$ unary operations $f_{\mathbb{N}^+}$ and $g_{\mathbb{N}^+}$ for all $n \in \mathbb{N}^+$ by $f_{\mathbb{N}^+}(n) = 3n$ and $g_{\mathbb{N}^+}(n) = n+1$. Clearly, $f_{\mathbb{N}}$ and $g_{\mathbb{N}}$ are strictly monotone in their argument and $f_{\mathbb{N}^+}(g_{\mathbb{N}^+}(n)) = 3(n+1) > 3n+2 = g_{\mathbb{N}^+}(g_{\mathbb{N}^+}(f_{\mathbb{N}^+}(n)))$. It follows that the TRS consisting of the rule $f(g(x) \to g(g(f(x)))$ is SN.

(iii) Define on $\mathbb{N}^+$ a binary operation $a_{\mathbb{N}^+}$ by $a_{\mathbb{N}^+}(m,n) = m+n$ and unary operations $f_{\mathbb{N}^+}$ and $g_{\mathbb{N}^+}$ by $f_{\mathbb{N}^+}(n) = n^2$ and $g_{\mathbb{N}^+}(n) = n+1$. Clearly, $a_{\mathbb{N}^+}$ is strictly monotone in both its arguments and $f_{\mathbb{N}^+}$ and $g_{\mathbb{N}^+}$ are also strictly monotone in their argument. Moreover, $f_{\mathbb{N}^+}(g_{\mathbb{N}^+}(n)) = (n+1)^2 = n^2+2n+1 > n^2+n+1 = a_{\mathbb{N}^+}(g_{\mathbb{N}^+}(f_{\mathbb{N}^+}(n)), n)$. It follows that the TRS consisting of the rule $f(g(x)) \to a(g(f(x)), x)$ is SN.

Exercise 6.2.15

(i) Let $f_A = 3X$, $g_A = X+1$ and $h_A = 3X+2$; then $3(X+1) - (3X+2) = 1$ and $3X+2-(3X+1) = 1$. It follows that the TRS consisting of the rules $f(g(x)) \to h(x)$ and $h(x) \to g(f(x))$ is SN.

(ii) Let $0_A = 2$, $+_A = X+Y+1$ and $*_A = XY$; then $2+X+1-X > 0$, $X+2+1-X > 0$, $2X-2 > 0$ and $X(Y+Z+1) - (XZ+YX+1) > 0$. It follows that the TRS consisting of the rules $0+x \to x$, $x+0 \to x$, $x*0 \to 0$ and $x*(y+z) \to (x*z)+(y*x)$ is SN.

Exercise 6.2.25

(i) Define on $A = \{0,1\} \times \mathbb{N}^+$ unary operations f_A, g_A and s_A by

$$\begin{aligned} &f_A(0,n) = (0,n) \quad f_A(1,n) = (0,n+1) \\ &g_A(x,n) = (1,n) \\ &s_A(x,n) = (0,n). \end{aligned}$$

Then f_A, g_A and s_A are strictly monotone in their argument, and

$$f_A(g_A(0,n)) = (0,n+1) > (0,n) = f_A(s_A(s_A(g_A(0,n))))$$

and

$$f_A(g_A(1,n)) = (0,n+1) > (0,n) = f_A(s_A(s_A(g_A(1,n)))).$$

It follows that the TRS consisting of the rule $f(g(x)) \to f(s(s(g(x))))$ is SN.

(ii) Let $0_A = (0,1)$, and define on $A = \{0,1\} \times \mathbb{N}^+$ unary operations s_A by $s_A(x,n) = (1,n)$ and a binary operation f_A by

$$\begin{aligned} &f_A((0,m),(x,n)) = (1,m+n+1) \\ &f_A((1,m),(x,n)) = (1,m+n). \end{aligned}$$

Then s_A is strictly monotone in its argument and f_A is strictly monotone in both of its arguments, and

$$f_A((0,1),(x,n)) = (1,n+2) > (1,n+1) = f_A(s_A(0,1),(x,n)).$$

It follows that the TRS consisting of the rule $f(0,x) \to f(s(0),x)$ is SN.