1.4. Solutions to exercises on ARSs

SOLUTION 1.3.1.

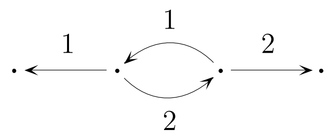
(i) DP and TP imply both $\mathrm{CR}^{\leq 1}$, which implies CR, see Figure 1.8.

(ii) By monotonicity of reflexive–transitive closure.

(iii) By (i) and (ii).

SOLUTION 1.3.2. Analogous to the multiset argument for Newman's Lemma 1.2.1, using that $\rightarrow_{12}$ is SN.

The counterexample is given in the following picture, which is analogous to Figure 1.2.



SOLUTION 1.3.3. Since $\rightarrow_1$ is subcommutative, it follows by Theorem 1.2.2(iii) that $\rightarrow_1$ is confluent. By Proposition 1.1.10 this is equivalent to $\twoheadrightarrow_1$ being subcommutative. If $\twoheadrightarrow_1 = \twoheadrightarrow_2$, then also $\twoheadrightarrow_2$ is subcommutative. Hence again by Proposition 1.1.10, $\rightarrow_2$ is confluent.

SOLUTION 1.3.4. Use Exercise 1.3.1 with $\mathrel{\diamond\!\!\twoheadrightarrow} = \bigcup\{\twoheadrightarrow_\alpha \mid \alpha \in I\}$.

SOLUTION 1.3.5.

(i) Follows immediately from $=_\alpha = \twoheadrightarrow_{\alpha\alpha^{-1}}$ and Proposition 1.1.10(vi).

(ii) The assumption that $\rightarrow_\alpha$ commutes with $\leftarrow_\beta$ can be rephrased as $\twoheadrightarrow_\beta \cdot \twoheadrightarrow_\alpha \subseteq \twoheadrightarrow_\alpha \cdot \twoheadrightarrow_\beta$. Repeated application to $\twoheadrightarrow_{\alpha\beta}$, being equal to the transitive closure of $\twoheadrightarrow_\beta \cdot \twoheadrightarrow_\alpha$, gives the desired postponement of β-steps after α-steps.

SOLUTION 1.3.6. First we prove $\forall a, b, c \in A\ (b \leftarrow_1 a \twoheadrightarrow_2 c \Rightarrow \exists d \in A\ b \twoheadrightarrow_2 d \leftarrow_1^{\equiv} c)$. This is proved by induction on the length of the reduction sequence for $a \twoheadrightarrow_2 c$. Assume $b \twoheadleftarrow_1 a \twoheadrightarrow_2 c$ for some $a, b, c \in A$. Now one proves $\exists d \in A\ (b \twoheadrightarrow_2 d \twoheadleftarrow_1 c)$ by induction on the length of the reduction sequence for $a \twoheadrightarrow_1 b$, using the previous result for the induction case.

SOLUTION 1.3.7. Assume $b \twoheadleftarrow_1 a \twoheadrightarrow_2 c$ for some $a, b, c \in A$. One easily proves $\exists d \in A\ (b \twoheadrightarrow_2 d \twoheadleftarrow_1 c)$ by induction on the length of the reduction sequence for $a \twoheadrightarrow_1 b$, using the assumption $\forall a, b, c \in A\ (b \leftarrow_1 a \twoheadrightarrow_2 c \Rightarrow \exists d \in A\ b \twoheadrightarrow_2 d \twoheadleftarrow_1 c)$ for the induction case.

SOLUTION 1.3.8. Pictures are given in Figure 14.35.

(i) First prove by induction on the length of the reduction sequence for $a \twoheadrightarrow_2 c$ that $\forall a, b, c \in A\ (b \twoheadleftarrow_1 a \twoheadrightarrow_2 c \Rightarrow \exists e \in A\ b =_2 e \twoheadleftarrow_1 c)$. Here $=_2$ is the convertibility relation corresponding to $\rightarrow_2$. Then apply the confluence of $\rightarrow_2$ in the formulation of Proposition 1.1.10(vi).

(ii) First note that $\twoheadrightarrow_3$ equals $\twoheadrightarrow_{12}$ (as every reduction sequence of $\rightarrow_{12}$ can be taken to start with zero or more steps $\rightarrow_1$ and to end with zero or more steps $\rightarrow_2$). We will prove that $\twoheadrightarrow_3$ satisfies $\mathrm{CR}^{\leq 1}$. It follows by Proposition 1.1.10(iv) that $\rightarrow_{12}$ is confluent. For a proper understanding of the following reasoning, making a picture can be very helpful. Assume $b \leftarrow_3 a \rightarrow_3 c$ for some $a, b, c \in A$. Then there exist $d, e \in A$ such that $b \twoheadleftarrow_2 d \twoheadleftarrow_1 a \twoheadrightarrow_1 e \twoheadrightarrow_2 c$. Since $d \twoheadleftarrow_1 a \twoheadrightarrow_1 e$ there exists $f \in A$ such that $d \rightarrow_3 f \leftarrow_3 e$, so $d \twoheadrightarrow_1 g \twoheadrightarrow_2 f \twoheadleftarrow_2 h \twoheadleftarrow_1 e$ for some $g, h \in A$. To $b \twoheadleftarrow_2 d \twoheadrightarrow_1 g$ and $h \twoheadleftarrow_1 e \twoheadrightarrow_2 c$ we apply that $\rightarrow_1$ requests $\rightarrow_2$. It follows that there exist $i, j, k, l \in A$ such that $b \twoheadrightarrow_1 i \twoheadrightarrow_2 j \twoheadleftarrow_2 g$ and $h \twoheadrightarrow_2 k \twoheadleftarrow_2 l \twoheadleftarrow_1 c$. We have $i =_2 l$, so by the confluence of $\rightarrow_2$ there exists a common $\rightarrow_2$-reduct, say m, of i and l. In total we have $b \twoheadrightarrow_1 i \twoheadrightarrow_2 m \twoheadleftarrow_2 l \twoheadleftarrow_1 c$, i.e. $b \rightarrow_3 m \leftarrow_3 c$.

(iii) Included in the previous case, since the confluence of $\rightarrow_1$ implies the condition given in (ii).

Solution 1.3.9. Note that $\rightarrow_1 \subseteq \twoheadrightarrow_2$ implies $\twoheadrightarrow_1 \subseteq \twoheadrightarrow_2$.

(i) The 'only if'-part is trivial: use $\twoheadrightarrow_1 \subseteq \twoheadrightarrow_2$. For the 'if'-part, assume $a \twoheadrightarrow_2 b$. We will prove $\exists c \in A\ a \twoheadrightarrow_1 c \twoheadleftarrow_1 b$ by induction on the length of the reduction sequence for $a \twoheadrightarrow_2 b$. In the base case there is nothing to prove. For the induction case the induction hypothesis is applied to the end of the reduction sequence: assume $a \rightarrow_2 a' \rightarrow_2 \cdots \rightarrow_2 b$, then there exists a $c' \in A$ such that $a' \twoheadrightarrow_1 c' \twoheadleftarrow_1 b$. Consider $a \rightarrow_2 a' \twoheadrightarrow_1 c'$. By assumption there exists $c \in A$ such that $a \twoheadrightarrow_1 c \twoheadleftarrow_1 c'$, so $a \twoheadrightarrow_1 c \twoheadleftarrow_1 b$.

(ii) Assume $\rightarrow_1$ is confluent. For the confluence of $\rightarrow_2$, assume $b \twoheadleftarrow_2 a \twoheadrightarrow_2 c$ for some $a, b, c \in A$. Since $\rightarrow_2$ is a compatible refinement of $\rightarrow_1$, there exist $d, e \in A$ such that $b \twoheadrightarrow_1 d \twoheadleftarrow_1 a \twoheadrightarrow_1 e \twoheadleftarrow_1 c$, so $b =_1 c$. By the confluence of $\rightarrow_1$, there exists a common $\rightarrow_1$-reduct of b and c, say f. Now use $\twoheadrightarrow_1 \subseteq \twoheadrightarrow_2$ to conclude $b \twoheadrightarrow_2 f \twoheadleftarrow_2 c$. For the converse, assume that $\rightarrow_2$ is confluent and consider $b \twoheadleftarrow_1 a \twoheadrightarrow_1 c$ for some $a, b, c \in A$. By $\twoheadrightarrow_1 \subseteq \twoheadrightarrow_2$ and the confluence of $\rightarrow_2$ there exists a $d \in A$ such that $b \twoheadrightarrow_2 d \twoheadleftarrow_2 c$. To $b \twoheadrightarrow_2 d$ we apply that $\rightarrow_2$ is a compatible refinement of $\rightarrow_1$ to find e such that $b \twoheadrightarrow_1 e \twoheadleftarrow_1 d$. Again we use $\twoheadrightarrow_1 \subseteq \twoheadrightarrow_2$ and get $c \twoheadrightarrow_2 e$, to which we apply that $\rightarrow_2$ is a compatible refinement of $\rightarrow_1$ in order to get f such that $c \twoheadrightarrow_1 f \twoheadleftarrow_1 e$. In total: $b \twoheadrightarrow_1 e \twoheadrightarrow_1 f \twoheadleftarrow_1 c$. This proves the confluence of $\rightarrow_1$.

Solution 1.3.10. Consider the following formulas, extending conditions (i)–(iii) from Exercise 1.3.10. Making a picture for each formula will be very helpful.

(i) $\leftarrow_1 \cdot \twoheadrightarrow_2 \subseteq =_2 \cdot \leftarrow_1$

(ii) $\leftarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadleftarrow_2 \cdot \leftarrow_1$

(iii) $\twoheadleftarrow_2 \cdot \leftarrow_1 \cdot \rightarrow_1 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2 \cdot \leftarrow_1$

(iv) $\twoheadleftarrow_2 \cdot \leftarrow_1 \cdot \rightarrow_2 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2 \cdot \leftarrow_1$

(v) $\leftarrow_1 \cdot \twoheadrightarrow_{12} \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2 \cdot \leftarrow_1 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_{12}$

(vi) $\twoheadleftarrow_2 \cdot \rightarrow_1 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2$

(vii) $\twoheadleftarrow_2 \cdot \rightarrow_2 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2$

(viii) $\leftarrow_2 \cdot \twoheadrightarrow_{12} \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_2 \subseteq \twoheadrightarrow_{12} \cdot \twoheadleftarrow_{12}$

Formula (iv) follows by iterated application of (iii), and yields (v) by applying (ii), the confluence of $\rightarrow_2$. Formula (vi) follows from condition (i), DP for $\rightarrow_1$, and the mirror image of (v). Formula (vii) follows from condition (iii) and again (ii), using $\twoheadrightarrow_2 \subseteq \twoheadrightarrow_{12}$. Formulas (vi) and (vii) combine by iteration into (viii), using that $\twoheadrightarrow_2$ is reflexive. Formula (ix) is obtained from (the mirror image of) formula (v), and formula (x) follows directly from condition (ii), using $\rightarrow_2 \subseteq \twoheadrightarrow_2 \subseteq \twoheadrightarrow_{12}$. Formulas

(ix) and (x) combine into (xi), using that $\twoheadrightarrow_2$ contains $\rightarrow_2$. Finally, formulas (viii) and (xi) yield the confluence of $\rightarrow_{12}$ by applying Proposition 1.1.10(v).

SOLUTION 1.3.11. We prove the confluence of $\rightarrow$ in the formulation of Proposition 1.1.10(v). Assume $b \leftarrow a \twoheadrightarrow c$ for some $a, b, c \in A$. By a simple induction on the length of the reduction sequence for $a \twoheadrightarrow c$ one gets a common reduct of b and c, taking care that the strong confluence is applied with $\leftarrow^{\equiv}$ parallel to the reduction step $b \leftarrow a$.

SOLUTION 1.3.12. The following is essentially a variation of the multiset argument for Newman's Lemma 1.2.1. Here we associate the multiset $[f(a_0), \ldots, f(a_n)]$ to the landscape $a_0 \leftrightarrow \cdots \leftrightarrow a_n$.

(i) We eliminate the peak $a_{i-1} \leftarrow a_i \rightarrow a_{i+1}$ by replacing it in the landscape by $d_0 \leftrightarrow \cdots \leftrightarrow d_k$, where $a_{i-1} \equiv d_0 \leftrightarrow \cdots \leftrightarrow d_k \equiv a_{i+1}$ such that $f(d_j) \prec f(a_i)$ for all $0 < j < k$. Termination is ensured since the multiset order $\prec_{\#}$ is well-founded on $S^{\#}$.

(ii) The argument is similar in case a_{i-1} and a_{i+1} are connected below a_i. Otherwise, there exists a $d \in A$ with $a_{i-1} \rightarrow d \leftarrow a_{i+1}$ and $f(d) \preceq f(a_i)$. The latter case requires an additional argument. First, observe that the multiset obtained by the replacement of a_i by d in the landscape is smaller than or equal to the multiset before the replacement. Second, there are at most finitely many successive replacements of the latter kind possible. This is most easily seen by a geometrical counting argument. There is no such replacement possible if the landscape is of the form $a_0 \rightarrow \cdots \rightarrow a_j \leftarrow \cdots \leftarrow a_n$, but in that case we have already a CR-landscape and we are done. The number of tiles in between the landscape $a_0 \leftrightarrow \cdots \leftrightarrow a_n$ and any resulting CR-landscape is a natural number which decreases under the above replacements. So after finitely many replacements under which the multiset may be invariant, there must come a step by which the multiset becomes strictly smaller. Again termination is ensured since the multiset order $\prec_{\#}$ is well-founded on $S^{\#}$.

SOLUTION 1.3.13.

(i) Assume CP and let $b \twoheadleftarrow a \twoheadrightarrow c$ for some $a, b, c \in A$. Let $a_0 \rightarrow a_1 \cdots$ be a reduction sequence such that $\{a_n \mid n \geq 0\}$ is cofinal in $\mathcal{G}(a)$. We have $b, c \in \mathcal{G}(a)$, so by the cofinality there exists $i, j \geq 0$ such that $b \twoheadrightarrow a_i$ and $c \twoheadrightarrow a_j$. If $i < j$, then a_j is the desired common reduct, otherwise a_i.

(ii) Assume CR, A countable, and let $a \equiv a_0, a_1, \ldots$ be an enumeration of $\mathcal{G}(a)$. Define recursively $b_0 \equiv a$ and b_{n+1} as a common reduct of b_n and a_{n+1}. Then $b_0 \twoheadrightarrow b_1 \cdots$ yields a reduction sequence and $\{b_n \mid n \geq 0\}$ is cofinal in $\mathcal{G}(a)$.

The following ARS $\mathcal{A} = (A, \rightarrow)$ shows that countability is necessary for the implication. Take one starting point, $a_0 \in A$. Take uncountably many a_1^{α} such that $a_0 \rightarrow a_1^{\alpha}$. For every (unordered) pair of distinct $a_1^{\alpha}, a_1^{\alpha'}$, take a unique a_2 such that $a_1^{\alpha} \rightarrow a_2$ and $a_1^{\alpha'} \rightarrow a_2$. We continue this layered construction infinitely many times. By construction we have confluence and $\mathcal{G}(a) = A$. Moreover, every reduction sequence contains exactly one element of each layer. Let $a_0 \rightarrow a_1 \cdots$ be a reduction sequence.For every $x \in A$ there exist at most two elements $y \in A$ such that $y \rightarrow x$. It follows that $\{x \in A \mid \exists i \in \mathbb{N}\ x \twoheadrightarrow a_i\}$ is countable, hence not equal to A, and the cofinality fails.

As an alternative solution using ordinal and cardinal numbers, see [1977], the uncountable ARS $\mathcal{A} = (\aleph_1, <)$ can be shown to satisfy CR, whereas CP fails. Here $\aleph_1$ is (by definition) the set of all countable ordinals and $<$ is the usual total order on ordinals taken as reduction relation. Clearly, $\mathcal{A}$ is CR: if $\gamma > \alpha < \beta$, then the maximum of γ and β is their common reduct. However, CP fails, as there cannot be a sequence $\alpha_0 < \alpha_1 < \cdots$ in $\mathcal{G}(0) = \aleph_1$ such that $\{\alpha_n \mid n \geq 0\}$ is cofinal in $\aleph_1$. More generally, any ARS $\mathcal{A} = (\aleph, <)$ such that $\aleph$ is an ordinal in which ω cannot be cofinally embedded (a so-called regular ordinal) provides such a counterexample.

(iii) $\mathrm{CP}^=$ trivially implies CP since any cofinal reduction sequence $a \equiv a_0 \to a_1 \to \cdots$ in $\mathcal{C}(a)$ is also a cofinal reduction sequence in $\mathcal{G}(a)$. For the converse, assume CP. By (i) we have CR. Let $a \in A$ and $a \equiv a_0 \to a_1 \to \cdots$ be cofinal in $\mathcal{G}(a)$. If $a' \in \mathcal{C}(a)$, then $a' = a$, so by CR we have $a' \twoheadrightarrow a''$ for some $a'' \in \mathcal{G}(a)$, hence $a' \twoheadrightarrow a_i$ for suitable $i \geq 0$. It follows that $a \equiv a_0 \to a_1 \to \cdots$ is cofinal in $\mathcal{C}(a)$.

Solution 1.3.14. Assume $a' \twoheadleftarrow_\alpha a \twoheadrightarrow_\alpha a''$ for some $a, a', a'' \in A$. By (ii), both a' and a'' have a normal form in $\mathcal{A}$, say n' and n'', respectively. Since $n' \twoheadleftarrow_\alpha a \twoheadrightarrow_\alpha n''$ we have $\varphi(n') =_\beta \varphi(a) =_\beta \varphi(n'')$ by (iii). Since $\varphi(n')$ and $\varphi(n'')$ are normal forms in $\mathcal{B}$ by (iv), we must have $\varphi(n') \equiv \varphi(n'')$ by (i). Now $n' \equiv n''$ by (v), and we are done.

Solution 1.3.15. Since $\to_2$ is SN, the inverse relation $\leftarrow_2$ is WF, and hence also $R = \leftarrow_2^+$. We will prove $\forall c, a, b \in A\ (c \twoheadleftarrow_1 a \twoheadrightarrow_2 b \Rightarrow \exists d \in A\ c \twoheadrightarrow_2 d \twoheadleftarrow_1 b)$ by well-founded induction on R. Let $c \in A$ and assume $\exists d \in A\ (c' \twoheadleftarrow_1 a \twoheadrightarrow_2 b \Rightarrow c' \twoheadrightarrow_2 d \twoheadleftarrow_\beta b)$ for all $c', a, b \in A$ with $c'\ R\ c$. Assume $c \equiv a_n \leftarrow_1 \cdots \leftarrow_1 a_1 \leftarrow_1 a \twoheadrightarrow_2 b$ and let $a \to_2 b_0$ be the first step of the reduction sequence for $a \twoheadrightarrow_2 b$. Applying $\leftarrow_1 \cdot \to_2 \subseteq \to_2^+ \cdot \twoheadleftarrow_1$ to $a_1 \leftarrow_1 a \to_2 b_0$ yields $d_1 \in A$ such that $a_1 \to_2^+ d_1 \twoheadleftarrow_1 b_0$. Let $a_1 \to_2 b_1$ be the first step in the reduction sequence for $a_1 \to_2^+ d_1$. Applying $\leftarrow_1 \cdot \to_2 \subseteq \to_2^+ \cdot \twoheadleftarrow_1$ to $a_2 \leftarrow_1 a_1 \to b_1$ yields $d_2 \in A$ such that $a_2 \to_2^+ d_2 \twoheadleftarrow_1 b_1$. Let $a_2 \to_2 b_2$ be the first step in the reduction sequence for $a_2 \to_2^+ d_2$. Iterating this construction yields $d_1, d_2, \ldots$ and $b_1, b_2, \ldots$ such that $a \twoheadrightarrow_1 a_i \to_2 b_i \twoheadrightarrow_2 d_i \twoheadleftarrow_1 b_{i-1}$ for all $1 \leq i \leq n$. Making a picture can be very helpful here. The induction hypothesis applies to d_n, since $d_n\ R\ c$. We apply the induction hypothesis to $d_n \twoheadleftarrow_1 b_{n-1} \twoheadrightarrow_2 d_{n-1}$ to find e_n such that $d_n \twoheadrightarrow_2 e_n \twoheadleftarrow_1 d_{n-1}$, then to $e_n \twoheadleftarrow_1 d_{n-1} \twoheadleftarrow_1 b_{n-2} \twoheadrightarrow_2 d_{n-2}$ to find e_{n-1} such that $e_n \twoheadrightarrow_2 e_{n-1} \twoheadleftarrow_1 d_{n-2}$, and so on. After $n-1$ steps we find e such that $a_n \to_2^+ e \twoheadleftarrow_1 b_0$. Applying the induction hypothesis once more to $e \twoheadleftarrow_1 b_0 \twoheadrightarrow_2 b$ yields d such that $a_n \twoheadrightarrow_2 d \twoheadleftarrow_1 b$.

Solution 1.3.16. By SN we can apply well-founded induction, Theorem A.1.7, with respect to $\leftarrow$. Let $\phi(a)$ express that the reduction graph of a is finite. It suffices to show that ϕ is $\leftarrow$-inductive. Let $a \in A$ and assume $\phi(b)$ for all b with $b \leftarrow a$. We have to prove $\phi(a)$. Consider the reduction graph $\mathcal{G}(a)$ of a. By FB we have finitely many b with $a \to b$, say $b_1, \ldots, b_n$. The reduction graphs $\mathcal{G}(b_i)$ are all finite by the induction hypothesis. As $\mathcal{G}(a)$ is built up from a and $\mathcal{G}(b_i)$ for every $1 \leq i \leq n$, it follows that $\mathcal{G}(a)$ is finite.

Solution 1.3.17. Under the conditions of the exercise, any sequence $a_0 \leftrightarrow a_1 \leftrightarrow \cdots$ can only contain finitely many different a_i's, since $f(a_0) = f(a_1) = \ldots$ and the inverse image of $f(a_0)$ is finite. It follows that every infinite $\to$-ascending sequence yields an infinite $\rightarrow$-descending sequence by using a (cyclic) subsequence $a_i \rightarrow a_{i+1} \rightarrow \cdots \rightarrow a_j$ with $a_i = a_j$, and conversely.

SOLUTION 1.3.18. Let $a \in A$, then a reduces to a normal form by WN. Assume there is a reduction sequence of length n to this normal form. We will prove by induction on n that every reduction sequence starting with a has length at most n. The base case $n = 0$ is trivial, since then a is a normal form itself. For the induction case, let $a \rightarrow a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n$ be a reduction sequence with a_n a normal form. Assume we have a proof for all lengths $\leq n$. Consider a reduction step $a \rightarrow b$. If b equals one of the a_i's $(0 \leq i \leq n)$, then we are done. Otherwise, there exists $c \in a$ such that $b \rightarrow c \leftarrow a_0$. Now we can apply the induction hypothesis first to a_0, then to c and finally to b to conclude that every reduction sequence starting with b has length at most n. Since b was arbitrary, it follows that every reduction sequence starting with a has length at most $n+1$. This proves SN. The counterexample to $\mathrm{CR}^{\leq 1} \wedge \mathrm{WN} \Rightarrow \mathrm{SN}$ is $a \rightarrow a$ and $a \rightarrow b$.

SOLUTION 1.3.19. The 'only if'-part is obvious since $\beta \subseteq \beta/\alpha$. Now we prove the 'if'-part. By an easy induction on the length of the reduction sequence for $a \twoheadrightarrow_\alpha b$ one proves $\forall a, b, c \in A\ (a \twoheadrightarrow_\alpha b \rightarrow_\beta c \Rightarrow \exists d \in A\ a \rightarrow_\beta d \twoheadrightarrow_{\alpha\beta} c)$. Assume there exists $a \in A$ and an infinite β/α-reduction sequence starting from a. Let $a \twoheadrightarrow_\alpha b \rightarrow_\beta c \twoheadrightarrow_\alpha d \rightarrow_\beta e$ be an initial segment of the infinite β/α-reduction sequence starting from a. Then there exists $a' \in A$ such that $a \rightarrow_\beta a' \twoheadrightarrow_{\alpha\beta} c$. Also, $a' \twoheadrightarrow_{\alpha\beta} c \twoheadrightarrow_\alpha d \rightarrow_\beta e$ is an initial segment of an infinite β/α-reduction sequence starting from a'. Repeated application of this argument gives an infinite β-reduction sequence, and hence proves the 'if'-part.

SOLUTION 1.3.20. The 'only if'-part is trivial. For the 'if'-part, assume that α and β are SN. We have to prove that $\rightarrow_{\alpha\beta}$ is SN. Assume by contradiction that $a_0 \rightarrow_{\alpha\beta} a_1 \rightarrow_{\alpha\beta} \cdots$ is an infinite reduction sequence. Define $X = \{\{i,j\} \mid i < j \wedge a_i \rightarrow_\alpha a_j\}$ and $Y = \{\{i,j\} \mid i < j \wedge a_i \rightarrow_\beta a_j\}$. Since $\rightarrow_{\alpha\beta}$ is transitive, we have $X \cup Y = [\mathbb{N}]^2$. By Ramsey's Theorem A.5.3 there exists an infinite homogeneous set $I \subseteq \mathbb{N}$, that is, either $[I]^2 \subseteq X$ or $[I]^2 \subseteq Y$. Ordering the elements of I yields an infinite reduction sequence with respect to either $\rightarrow_\alpha$ or $\rightarrow_\beta$. Contradiction.

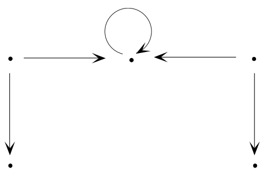
SOLUTION 1.3.21. If $\mathcal{B}$ is an extension of $\mathcal{A}$, then $\forall a, a' \in A\ (a =_\alpha a' \Rightarrow a =_\beta a')$ can be proved by induction on the generation of $=_\alpha$. Moreover, by the clauses (ii) and (iii) of Definition 1.1.6, we have $\forall a \in A\ \forall b \in B\ (a \twoheadrightarrow_\beta b \Rightarrow b \in A \wedge a \twoheadrightarrow_\alpha b)$.

(i) If $\mathcal{B}$ is conservative over $\mathcal{A}$, then $a \neq_\alpha a' \Rightarrow a \neq_\beta a'$, so $\mathcal{B}$ is consistent if $\mathcal{A}$ is consistent.

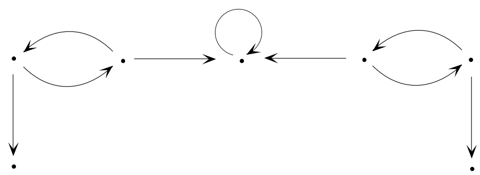
(ii) If $\mathcal{B}$ is a confluent extension of $\mathcal{A}$, then, for all $a, a' \in A$, $a =_\alpha a' \Rightarrow a =_\beta a' \Rightarrow a \twoheadrightarrow_\beta b \twoheadleftarrow_\beta a'$ for some $b \in B$. Hence $b \in A$ and $a \twoheadrightarrow_\alpha b \twoheadleftarrow_\alpha a'$, so $\mathcal{A}$ is confluent.

(iii) Now let $\mathcal{B}$ be a confluent extension of $\mathcal{A}$. Then $\forall a, a' \in A\ (a =_\alpha a' \Leftrightarrow a =_\beta a')$ follows from the above results. Hence $\mathcal{B}$ is conservative over $\mathcal{A}$.

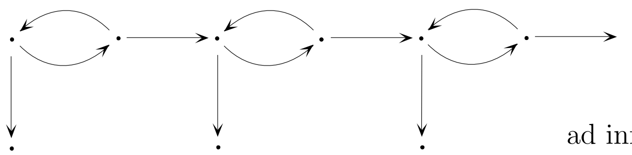
SOLUTION 1.3.22. UN $\Rightarrow$ $\mathrm{UN}^{\rightarrow}$ is trivial, since $b \twoheadleftarrow a \twoheadrightarrow c$ implies $b = c$. The other answers can be found in the following picture.



(i)



(ii)



(iii)

SOLUTION 1.3.23.

(i) Note that $\twoheadrightarrow$ is semi-decidable since $\rightarrow$ is semi-decidable. Let $a, a' \in A$ be given. We have the following decision procedure for testing $a = a'$. Enumerate all $b, b' \in A$ such that $a \twoheadrightarrow b$ and $a' \twoheadrightarrow b'$ and test each time whether b, b' are normal or not. (This actually involves the simultaneous execution of countably many computations, a technique which is called 'dove-tailing'.) By WN such b and b' exist, so after finitely many time they are found. By UN we have $a = a' \Leftrightarrow b \equiv b'$, so testing $a = a'$ reduces to testing $b \equiv b'$.

(ii) Take $n \rightarrow m$ if and only if $T(n, n, m)$. Here $T(x, y, z)$ is Kleene's predicate expressing that the program with code x applied to input y halts after an execution with code z. Then $\rightarrow$ is decidable (even primitive recursive), but n is a normal form if and only if the program with code n does not halt on input n. This latter problem is known to be undecidable.