

A.7. Solution to exercises in Mathematical Background

SOLUTION A.1.3. Regarding Definition A.1.1(iii), let R and R' be singlevalued. Then, for all $a \in A$, there exists at most one $b \in B$ such that $R(a, b)$ and, if so, there exists at most one $c \in C$ such that $R'(b, c)$. It follows that there exists at most one $c \in C$ such that $\exists b \in B (R(a, b) \wedge R'(b, c))$. Hence $R' \circ R$ is singlevalued. Now assume R and R' are total functions. Then the above argument proves that $R' \circ R$ is a total function by replacing 'at most one' everywhere by 'exactly one'.

Regarding Definition A.1.2(iv), let $f : A \rightarrow B$ be a bijection. Then, for all $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$ as f is surjective, and at most one $a \in A$ such that $f(a) = b$ as f is injective. It follows that there exists exactly one $a \in A$ such that $f(a) = b$. Hence the relation f^{-1} is indeed a total function and we can write $f^{-1}(b) = a$ instead of $(b, a) \in f^{-1}$. Moreover, f^{-1} is surjective since $f^{-1}(f(a)) = a$ for all $a \in A$, and injective since $f^{-1}(b) = a = f^{-1}(b') \Rightarrow (a, b), (a, b') \in f \Rightarrow b = b'$, for all $a \in A$, $b, b' \in B$, so in particular $f(f^{-1}(b)) = f(a) = b$.

Let $R_2 \subseteq A \times B$ and $R_1 \subseteq B \times C$ be relations. We have $(c, a) \in (R_1 \circ R_2)^{-1} \Leftrightarrow (a, c) \in R_1 \circ R_2 \Leftrightarrow \exists b \in B (R_2(a, b) \wedge R_1(b, c)) \Leftrightarrow \exists b \in B (R_1^{-1}(c, b) \wedge R_2^{-1}(b, a)) \Leftrightarrow (c, a) \in R_2^{-1} \circ R_1^{-1}$. Let, moreover, $R_3 \subseteq D \times A$ be a relation. We have $(d, c) \in (R_1 \circ R_2) \circ R_3 \Leftrightarrow \exists a \in A (R_3(d, a) \wedge (a, c) \in R_1 \circ R_2) \Leftrightarrow \exists a \in A (R_3(d, a) \wedge \exists b \in B (R_2(a, b) \wedge R_1(b, c))) \Leftrightarrow \exists a \in A \exists b \in B (R_3(d, a) \wedge R_2(a, b) \wedge R_1(b, c)) \Leftrightarrow \exists b \in B ((\exists a \in A (R_3(d, a) \wedge R_2(a, b))) \wedge R_1(b, c)) \Leftrightarrow \dots \Leftrightarrow (d, c) \in R_1 \circ (R_2 \circ R_3)$.

SOLUTION A.1.9.

(i) Assume R is a strict linear order. R' is reflexive by definition and transitive since R is. R' is antisymmetric since $R(x, y) \wedge R(y, x)$ is impossible and linear since R is. Hence R' is a linear order.

(ii) Assume R is a linear order. R' is irreflexive by definition. If $R'(x, y) \wedge R'(y, z)$, then $R(x, z)$ by the transitivity of R , and $x \neq z$ by the antisymmetry of R , so $R'(x, z)$. So R' is transitive. Since R is linear, $x \neq y$ implies either $R(x, y)$, and hence $R'(x, y)$, or $R(y, x)$, and hence $R'(y, x)$. It follows that R' is a strict linear order.

(iii) Assume R is a quasi-order. R' is irreflexive by definition. Assume $R'(x, y) \wedge R'(y, z)$. Then $R(x, z)$ by the transitivity of R . We get $\neg R(z, x)$ from $R(x, y)$ and $\neg R(z, y)$ using the contraposition of transitivity of R . So $R'(x, z)$ and R' is transitive. Hence R' is a strict partial order.

(iv) Assume R is a quasi-order. \sim is reflexive and transitive since R is. \sim is symmetric by definition. Hence \sim is an equivalence relation. \preceq is well-defined since $x' \sim x$ R $y \sim y'$ implies $x' R y'$ by the transitivity of R . \preceq is reflexive and transitive since R is. Moreover, \preceq is antisymmetric by the definition of \sim : $[x]_{\sim} \preceq [y]_{\sim} \wedge [y]_{\sim} \preceq [x]_{\sim} \Rightarrow R(x, y) \wedge R(y, x) \Rightarrow x \sim y \Rightarrow [x]_{\sim} = [y]_{\sim}$.

(v) R' contains R and is symmetric by definition. Let R'' be a symmetric relation containing R . Assume $R'(x, y)$, then either $R(x, y)$ and hence $R''(x, y)$, or $R(y, x)$, so $R''(y, x)$ and hence $R''(x, y)$ since R'' is symmetric. In both cases we have $R' \subseteq R''$.

SOLUTION A.1.11.

(i) Define $R'(x, z)$ if and only if $x R y_1 R y_2 \cdots y_k R z$ for some $k \geq 0$ and $y_1, \dots, y_k \in S$. By considering the case $k = 0$ we get $R \subseteq R'$. Obviously, R' is transitive. It follows that $R^+ \subseteq R'$. For the converse, assume $x R y_1 R y_2 \cdots y_k R z$ for some $k \geq 0$ and $y_1, \dots, y_k \in S$. By induction on k one proves $R^+(x, z)$, using that R^+ is transitive and contains R .

(ii) Recall the definition of R^n from Definition A.1.1. Now the result follows from (i) and Exercise A.1.9.

(iii) The ‘only if’-part is obvious, so we restrict attention to the ‘if’-part. Assume R is well-founded and assume by contradiction that R^+ is not. Then there exists an infinite R^+ -descending sequence x_0, x_1, \dots . Applying (i) to every $R^+(x_{i+1}, x_i)$ yields an infinite R -descending sequence. Contradiction, so R^+ is well-founded and hence irreflexive. As R^+ is by definition transitive, it follows that R^+ is a well-founded order.

SOLUTION A.1.12.

(i) Let E be a set of equivalence relations. Then $\bigcap E$ is reflexive since every $R \in E$ is. Assume $(x, y) \in \bigcap E$, then $(x, y) \in R$ for every $R \in E$. Since such R are symmetric, we have $(y, x) \in R$ for every $R \in E$, so $(y, x) \in \bigcap E$. Hence $\bigcap E$ is symmetric. Assume $(x, y), (y, z) \in \bigcap E$, then $(x, y), (y, z) \in R$ for every $R \in E$. Since such R are transitive, we have $(x, z) \in R$ for every $R \in E$, so $(x, z) \in \bigcap E$. Hence $\bigcap E$ is transitive. In total, $\bigcap E$ is an equivalence relation.

(ii) Disproof: take $R = \{(a, c), (b, c)\}$, then $a =_R b$ but not $(a, b) \in (R^*)^{\leftrightarrow}$.

(iii) For every relation R , if R is symmetric, then R^* is symmetric. Hence $=_R \subseteq (R^{\leftrightarrow})^*$. For the converse we note the following. For every relation R , if R is symmetric (reflexive and transitive), then $R = R^{\leftrightarrow}$ ($R = R^*$). Moreover, closure operations are monotonic with respect to \subseteq . Now we can calculate: $R \subseteq =_R \Rightarrow R^{\leftrightarrow} \subseteq (=R)^{\leftrightarrow} = =_R \Rightarrow (R^{\leftrightarrow})^* \subseteq (=R)^* = =_R$.

(iv) Define $R'(x, z)$ if and only if $x = y_0 R^{\leftrightarrow} \cdots R^{\leftrightarrow} y_k = z$ for some $k \geq 0$ and $y_0, \dots, y_k \in S$. By considering the case $k = 1$, taking into account that $R \subseteq R^{\leftrightarrow}$, we get $R \subseteq R'$. Obviously, R' is reflexive, symmetric and transitive, and hence an equivalence relation. Hence $=_R \subseteq R'$. The converse is proved by induction on k , using that $R^{\leftrightarrow} \subseteq =_R$ and that $=_R$ is reflexive and transitive.

SOLUTION A.1.19. Irreflexivity of $\prec_{\mathcal{L}}$ is obvious. For transitivity, assume $\langle s_1, \dots, s_n \rangle \prec_{\mathcal{L}} \langle t_1, \dots, t_m \rangle \prec_{\mathcal{L}} \langle s'_1, \dots, s'_{n'} \rangle$. Let $k \leq n, n'$ be maximal such that $\langle s_1, \dots, s_k \rangle = \langle s'_1, \dots, s'_k \rangle$. First we prove $\langle s_1, \dots, s_k \rangle = \langle t_1, \dots, t_k \rangle$. Assume by contradiction $s_i \neq t_i$ for some (least) $1 \leq i \leq k$, then also $s'_i \neq t_i$. It follows by the definition of $\prec_{\mathcal{L}}$ that $s_i \prec t_i \prec s'_i$, which conflicts with the transitivity and irreflexivity of \prec since $s_i = s'_i$. Now we distinguish two cases. The first case is $k = n$. Then $\langle t_1, \dots, t_m \rangle$, and hence $\langle s'_1, \dots, s'_{n'} \rangle$ strictly extends $\langle s_1, \dots, s_n \rangle$ and we are done. The second case is $k < n$. Then we distinguish two subcases, namely $s_{k+1} = t_{k+1}$ and $s_{k+1} \neq t_{k+1}$. In the first subcase we have $t_{k+1} \neq s'_{k+1}$ and hence $s_{k+1} = t_{k+1} \prec s'_{k+1}$, and in the second subcase we have $s_{k+1} \prec t_{k+1} \preceq s'_{k+1}$. In all (sub)cases we can conclude $\langle s_1, \dots, s_n \rangle \prec_{\mathcal{L}} \langle s'_1, \dots, s'_{n'} \rangle$. Hence $\prec_{\mathcal{L}}$ is a strict partial order. If \prec is total, then $\prec_{\mathcal{L}}$ is obviously also total.

For the counterexample, take \mathbb{N} for S and $<$ for \prec , then $<$ is a well-order. However, $\langle 1 \rangle, \langle 0, 1 \rangle, \langle 0, 0, 1 \rangle, \dots$ is an infinite $\prec_{\mathcal{L}}$ -descending sequence.

SOLUTION A.1.8. Assume R is well-founded and let $A \subseteq S$ be non-empty, say $x_0 \in A$. If A does not contain an R -minimal element, then there exists $x_1 \in A$ such that $R(x_1, x_0)$, and also $x_2 \in A$ such that $R(x_2, x_1)$, and so on. This gives an infinite R -descending sequence. Contradiction. For the converse, assume that every non-empty $A \subseteq S$ contains an R -minimal element. Let x_0, x_1, \dots be an arbitrary sequence in S . Then $X = \{x_i \mid i \in \mathbb{N}\}$ is non-empty and contains hence an R -minimal element. It follows that x_0, x_1, \dots is not an infinite R -descending sequence. Hence R is well-founded.

SOLUTION A.3.5. The induction principle for $=_R$ reads as follows: if $R(x, y) \Rightarrow \phi(x, y)$, $\phi(x, x)$, $\phi(x, y) \Rightarrow \phi(y, x)$ and $\phi(x, y) \wedge \phi(y, z) \Rightarrow \phi(x, z)$ for all $x, y, z \in S$, then $x =_R y \Rightarrow \phi(x, y)$ for all $x, y \in S$. Define $R'(x, z)$ if and only if $x = y_0 R^{\leftrightarrow} \dots R^{\leftrightarrow} y_k = z$ for some $k \geq 0$ and $y_0, \dots, y_k \in S$. We prove $\forall x, y \in S (x =_R y \Rightarrow R'(x, y))$ by taking R' for ϕ in the above induction principle. The base cases $R(x, y) \Rightarrow R'(x, y)$ and $R'(x, x)$ are easily verified by taking $k = 1$ and $k = 0$, respectively. The induction cases regarding symmetry and transitivity are also settled in the solution to Exercise A.1.12. For the converse $\forall x, z \in S (R'(x, z) \Rightarrow x =_R z)$ we use induction on k . For $k = 0$ we have $x = z$ and apply the second inductive clause. Now let $k > 0$ and assume we have a proof for all numbers $< k$. Then $x = y_0 R^{\leftrightarrow} y_1 R^{\leftrightarrow} \dots R^{\leftrightarrow} y_k = z$ implies $y_1 =_R z$ by the induction hypothesis. Moreover, $y_0 R^{\leftrightarrow} y_1$ implies either $R(y_0, y_1)$ or $R(y_1, y_0)$, so $y_0 =_R y_1$ by either the first inductive clause or the first and the third inductive clause (symmetry). Now we get $x =_R z$ by applying the fourth inductive clause (transitivity).

SOLUTION A.3.7. The induction principle for R' reads as follows: if $\forall x, y \in S (R(x, y) \Rightarrow \phi(x, y))$ and $\forall x, y, z \in S (R(x, y) \wedge \phi(y, z) \Rightarrow \phi(x, z))$, then $\forall x, y \in S (R'(x, y) \Rightarrow \phi(x, y))$. In order to prove $\forall x, y \in S (R'(x, y) \Rightarrow R^+(x, y))$ we take R^+ for ϕ and verify the base case $R(x, y) \Rightarrow R^+(x, y)$ by the definition of R^+ . The induction case $R(x, y) \wedge R^+(y, z) \Rightarrow R^+(x, z)$ follows immediately from the characterization of R^+ established in Exercise A.1.11. For the converse $\forall x, y \in S (R^+(x, y) \Rightarrow R'(x, y))$ an easy induction on k suffices.

SOLUTION A.5.5.

(i) Disproof: $<$ and \geq partition \mathbb{N}^2 , but there exists no homogeneous set with more than one element.

(ii) Define $X \subseteq [\mathbb{N}]^2$ by $X = \{\{i, j\} \mid i < j \wedge f(i) \leq f(j)\}$, and $Y = [\mathbb{N}]^2 - X$. By Ramsey's Theorem A.5.3 there exists an infinite homogeneous $A \subseteq \mathbb{N}$. We can exclude $[A]^2 \subseteq Y$, since $<$ is well-founded, so $[A]^2 \subseteq X$. Enumerate the elements of A in order of magnitude, $A = \{a_0, a_1, \dots\}$ with $a_0 < a_1 < \dots$. Again since $<$ is well-founded, there exist i, j with $i < j$ (so $a_i < a_j$) and $g(a_i) \leq g(a_j)$. Since $[A]^2 \subseteq X$ we also have $f(a_i) \leq f(a_j)$.

SOLUTION A.5.10. Both (i) and (ii) are easy applications of Lemma A.5.9. Observe that \leq is an almost full relation on \mathbb{N} . Hence the sequence f has a homogeneous subsequence $f(a_0), f(a_1), \dots$. Consider $g(a_0), g(a_1), \dots$. Also this sequence has a homogeneous subsequence $g(a_{i_0}), g(a_{i_1}), \dots$. Finally, consider a homogeneous subsequence of $h(a_{i_0}), h(a_{i_1}), \dots$. The indices of the first three elements of this homogeneous subsequence are the required i, j, k .

SOLUTION A.5.15. Trivial, by putting the elements of a finite set in arbitrary order in a finite sequence and then applying Higman's Lemma A.5.14. The Higman embedding respects the order, which is even more than required for an injection.

SOLUTION A.5.17.

(i) Identity on the finite set $\{0, 1\}$ is an almost full relation. Now observe that R equals $=_H$ and apply Higman's Lemma A.5.14 to conclude that R is almost full. (Note that $=_H$ is not symmetric.)

(ii) The sequence $11, 101, 1001, 10001, 100001, \dots$ shows that R' is not almost full: since the endpoints must be respected due to the fact that they are the only 1's, at least one 0 in between must be deleted.

SOLUTION A.5.20.

(i) Reason: $x \leq y \Rightarrow \sup(\{x, y\}) = y \Rightarrow \sup(\{f(x), f(y)\}) = f(y) \Rightarrow f(x) \leq f(y)$.

(ii) Disproof: isomorphic trees with identical labels are not identified. If we do identify isomorphic trees with identical labels, then indeed R_K is antisymmetric if R is antisymmetric. As it is very natural to take the underlying trees modulo isomorphy, we will tacitly do this.

(iii) Note that $\hat{x} \cap \hat{y}$ is finite, non-empty since it contains $\langle \rangle$, and linearly ordered by \leq . By successively taking the smallest of two elements one sees that $\inf(\hat{x} \cap \hat{y})$ is the minimum of $\hat{x} \cap \hat{y}$. Clearly $\hat{x} \cap \hat{y}$ is the set of upper bounds of $\{x, y\}$ and the minimum of $\hat{x} \cap \hat{y}$ is $\sup(\{x, y\})$.

SOLUTION A.5.22.

(i) This is a trivial weakening of Kruskal's Tree Theorem.

(ii) Disproof: already false in the case of finite sequences, see the disproof (ii) in the solution to Exercise A.5.17 ('Preserving successors' amounts to a subsequence of consecutive elements).

SOLUTION A.6.3.

(i) Define $n \ominus m = n - m$ if $n \geq m$ and 0 otherwise ($n, m \in \mathbb{N}$). Now the result follows from $n = (n \ominus m) + m$ if and only if $n \geq m$, applied to the multiplicities of the elements.

(ii) The 'only if'-part is proved by induction on the number of intermediate steps of the form (A.5) in $M \preceq_{\#} M'$. The base case holds vacuously, as $M = M'$ implies $C = M$. For the induction case, assume $M_1 \preceq_{\#} M_2 \prec_{\#} M_3$, where the result holds for M_1, M_2 and $M_2 \prec_{\#} M_3$ is of the form (A.5). We put $M_3 = M \uplus_{\#} [s]$, $M_2 = M \uplus_{\#} M'$ with $\forall z \in_{\#} M' z \prec s$, $C_{13} = M_1 \cap_{\#} M_3$, $C_{12} = M_1 \cap_{\#} M_2$, $C = M_1 \cap_{\#} M = C_{12} \cap_{\#} M$ (since $M \subseteq_{\#} M_2$) and $C' = C_{12} -_{\#} C \subseteq_{\#} M'$ (so that $C_{12} = C \uplus_{\#} C'$). With this choice of C we have either $C_{13} = C$ or $C_{13} = C \uplus_{\#} [s]$, depending on whether $M_1 -_{\#} C$ contains an occurrence of s . The situation is depicted in Figure A.6. The induction hypothesis is $\forall x \in_{\#} M_1 -_{\#} C_{12} \exists y \in_{\#} M_2 -_{\#} C_{12} x \prec y$ and we have to prove $\forall x \in_{\#} M_1 -_{\#} C_{13} \exists y \in_{\#} M_3 -_{\#} C_{13} x \prec y$. We distinguish two cases: $C_{13} = C$ and $C_{13} = C \uplus_{\#} [s]$.

If $C_{13} = C$, then $M_3 -_{\#} C_{13}$ contains s . Let $x \in_{\#} M_1 -_{\#} C_{13}$. If x occurs in C' , then x occurs in M' , so $x \prec s \in_{\#} M_3 -_{\#} C_{13}$. If x doesn't occur in C' , then x occurs in $M_1 -_{\#} C_{12}$, so we can apply the induction hypothesis to obtain

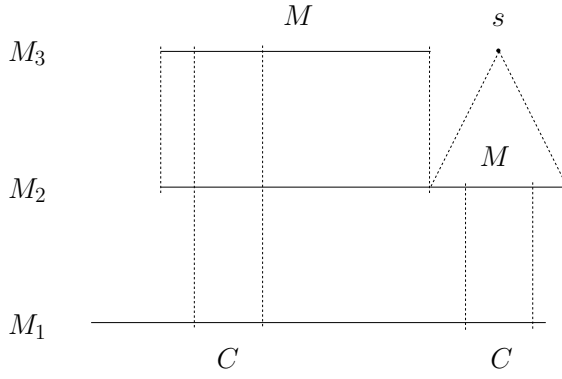


Figure A.6: $M_1 \preceq_{\#} M_2 \prec_{\#} M_3$.

y occurring in $M_2 -_{\#} C_{12}$ with $x \prec y$. Now y occurs either in $M -_{\#} C$, or in $M' -_{\#} C'$. In the former case we have $x \prec y \in_{\#} M_3 -_{\#} C_{13}$. In the latter we have $x \prec y \prec s \in_{\#} M_3 -_{\#} C_{13}$.

If $C_{13} = C \uplus_{\#} [s]$, then $M_1 -_{\#} C$ contains an occurrence of s . We have $C' \subseteq_{\#} M'$ and $\forall z \in_{\#} M' z \prec s$, so s actually occurs in $M_1 -_{\#} C_{12}$. By the induction hypothesis, there exists an occurrence $s' \succ s$ in $M_2 -_{\#} C_{12}$. Since again $\forall z \in_{\#} M' z \prec s$, the occurrence s' is actually in $M -_{\#} C = M_3 -_{\#} C_{13}$. Now the argument from the previous case $C_{13} = C$ can be repeated, using s' instead of s .

The 'if'-part is proved by repeated application of (A.5). Take C for M in (A.5) and replace every y in $M' -_{\#} C$ by a suitable multiset of elements x in $M -_{\#} C$ such that $x \prec y$.

(iii) By the previous result, using $(X \uplus_{\#} Y) \cap_{\#} (X \uplus_{\#} Z) = X \uplus_{\#} (Y \cap_{\#} Z)$.

SOLUTION A.6.7.

(i) By definition, $I_{\#}(S^{\#*}) = S^{\#*} \cup (S^{\#*})^{\#}$, so it suffices to show that $(S^{\#*})^{\#} \subseteq S^{\#*}$. Let $[s_1, \dots, s_n]$ be a multiset over $S^{\#*}$. Then there exists an $m \in \mathbb{N}$ such that $s_i \in I_{\#}^m(S)$ for every $1 \leq i \leq n$. Hence $[s_1, \dots, s_n] \in I_{\#}^{m+1}(S) \subseteq S^{\#*}$.

(ii) Observe that $I_{\#}$ is monotone: $S \subseteq S'$ implies $I_{\#}(S) \subseteq I_{\#}(S')$. If also $I_{\#}(S') = S'$, then it follows by induction that $I_{\#}^n(S) \subseteq S'$. Hence $S^{\#*} \subseteq S'$.

SOLUTION A.6.11. We list initial segments of both orders, then the order isomorphism suggests itself.

$$\begin{array}{lll}
 0, 1, 2, \dots & [], [0], [0, 0], \dots & [1], [0, 1], [0, 0, 1], \dots \\
 [], [], [], \dots & [[[]], [], [[]], [], [], [[]], \dots & [[[]], [[]], [], [[]], [[]], [], [], [[]], \dots
 \end{array}$$

SOLUTION A.6.13.

(i) Every finite, non-empty subset of S contains a maximum since the order is linear (by pairwise comparison of elements). Let M, M' be multisets over S . We prove that $\prec_{\#}$ is linear by induction on $|M|$. The case $|M| = 0$ is obvious. Otherwise, let s be the maximum of the elements occurring in M . If $|M'| = 0$, then we are done, so assume that s' is the maximum of the elements occurring in M' . If $s \prec s'$ ($s' \prec s$), then $M \prec_{\#} M'$ ($M' \prec_{\#} M$). If $s = s'$, then apply the induction hypothesis to $M -_{\#} [s]$ and $M' -_{\#} [s']$, using Exercise A.6.3(iii).

(ii) Let $x, y \in S \cup S^\#$. If both $x, y \in S$, apply the first inductive clause from Definition A.6.8. If both $x, y \in S^\# - S$, apply the second. If $x \in s$ and $y \in S^\# - S$ (or conversely), apply the third clause.

(iii) By induction on n one easily proves that $\prec_{I_\#}^n$ is a strict linear order on $I_\#^n(S)$, for all $n \in \mathbb{N}$. Let $x, y \in S^{\#\ast}$ and let n be the smallest number such that $x, y \in I_\#^n(S)$. Since $\prec_{I_\#}^n$ is a strict linear order we have $x \prec_{I_\#}^n y \vee x = y \vee y \prec_{I_\#}^n x$. Since $\prec_{\#\ast}$ extends $\prec_{I_\#}^n$, it follows that $\prec_{\#\ast}$ is linear.

SOLUTION A.6.14.

(i) Amoebae (as well as the colony) can easily be seen as nested multisets. The activity of amoebae then amounts to going from $[\dots, [A_1, \dots, A_n], \dots]$ ($n \geq 0$) to $[\dots, A_1, \dots, A_1, \dots, A_n, \dots, A_n, \dots]$. This can happen on any arbitrary nesting level, but always the new colony of amoebae is smaller than the old one in the sense of the nested multiset order from Definition A.6.10. By Theorem A.6.12 the nested multiset order is well-founded, and hence the life of a colony of amoebae is finite.

(ii) In this case the previous argument does not work, since the transition from $[\dots, [A_1, \dots, A_n], [B_1, \dots, B_m], \dots]$ to $[\dots, [\dots, A_i, \dots, B_j, \dots], \dots]$ is not necessarily decreasing in the sense of the nested multiset order. A more refined argument is needed, prepared by introducing some auxiliary notions. Let C be a colony of amoebae. We can represent amoebae by a finite tree, essentially their parse tree. The ur-amoeba is represented by just a root, and any other amoeba, as well as a colony, is represented by a root with as subtrees the trees representing the sons of the big amoeba. Now any occurrence of an amoeba in C corresponds with a subtree of the tree T_C representing C . We can define the *depth* of an amoeba with the depth of the corresponding subtree of T_C . Formally, $\text{depth}(\square) = 0$ and $\text{depth}([A_1, \dots, A_n]) = \max\{\text{depth}(A_i) + 1 \mid 1 \leq i \leq n\}$. We also define the *level* of an amoeba occurring in C as the level in T_C on which the root of the corresponding subtree occurs. Formally, $\text{level}(C) = 0$ and $\text{level}(A_i) = 1 + \text{level}([\dots, A_i, \dots])$. Let $d_C = \text{depth}(C)$ and $N_C = \{0, \dots, d_C\}$. Elements of $N_C \times N_C$ can be ordered lexicographically, according to Definition A.1.15: $(l_1, d_1) \prec (l_2, d_2) \Leftrightarrow (l_1 > l_2) \vee (l_1 = l_2 \wedge d_1 < d_2)$. Then \prec is well-founded by Lemma A.1.16, as N_C is finite, so $\prec_\#$ is well-founded on $(N_C \times N_C)^\#$. Let $LD_C \in (N_C \times N_C)^\#$ be the multiset $[\dots, (\text{level}(A), \text{depth}(A)), \dots]$ of pairs $(\text{level}(A), \text{depth}(A))$ for every amoeba A occurring in C . We claim that LD_C decreases in the sense of $\prec_\#$ under both activities of multisets. By a transition under (i), a pair (l, d) corresponding to $[A_1, \dots, A_n]$ is replaced by a number of smaller pairs (l, d') with $d' < d$ (apart from some other innocent changes on higher levels). By a transition under (ii), two pairs $(l, d_1), (l, d_2)$ corresponding to $[A_1, \dots, A_n]$ and $[B_1, \dots, B_m]$ are replaced by one (l, d) with $d \leq d_1, d_2$ (apart from some other innocent changes). In the latter case, if for example $l = 3$, $d_1 = 7$ and $d_2 = 5$, then multiplication of sons with pair $(4, 6)$ is innocent since $(4, 6) \prec (3, 5)$.

There exist alternative solutions. For example, to each amoeba $A = [A_1, \dots, A_n]$ we can assign as *norm* the pair $(\text{depth}(A), [\text{depth}(A_1), \dots, \text{depth}(A_n)])$ consisting of a natural number and a multiset of natural numbers. Norms can be lexicographically ordered, using in the first, and most important, component the usual order $<$ on \mathbb{N} , and in the second component the multiset extension $<_\#$. To a colony of

amoebae we assign the multiset of norms assigned to all occurrences of amoebae in C . As all orders involved are well-founded, it follows again by an analysis of the transitions under (i) and (ii) that the life of a colony of amoebae must be finite.