AN INVITATION TO $\beta\omega$

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CONTENTS

1. NOTATION

If A and B are sets then A^B denotes the set of all functions $B \to A$.

If X is a set then $\mathcal{P}(X)$ denotes the power set of X, i.e. the set consisting of all subsets of X.

$\mathbb{R}$ and $\mathbb{I}$ denote the set of reals and its subinterval $[0,1]$, respectively. $\mathbb{N}$ denotes the set of natural numbers.

If X is a set and κ is a cardinal number then

$$\begin{aligned}
[X]^{<\kappa} &= \{A \subseteq X : |A| < \kappa\},\\
[X]^{\leq\kappa} &= \{A \subseteq X : |A| \leq \kappa\},\\
[X]^{\kappa} &= \{A \subseteq X : |A| = \kappa\}.
\end{aligned}$$

A collection sets $\mathcal{A}$ has the *finite intersection property* provided that for any $\mathcal{B} \in [\mathcal{A}]^{<\omega}$ we have $\bigcap \mathcal{B} \neq \emptyset$.

Observe that a topological space is compact if and only if every family consisting of closed sets with the finite intersection property has non-empty intersection.

If X and Y are topological spaces then $C(X,Y)$ denotes the set of all continuous functions $X \to Y$.

The symbol $X \approx Y$ means that X and Y are homeomorphic spaces.

If X is a space and $A \subseteq X$ then $\overline{A}$ and $\operatorname{int}(A)$ denote respectively the closure of A and the interior of A.

Unless stated otherwise, all functions between topological spaces are continuous.

2. SET THEORY (REVIEW)

2.1. **Cardinal numbers.** If X is a set then $|X|$ denotes its cardinality. $\aleph_0$ is the cardinality of $\mathbb{N}$. If $\kappa = |X|$ and $\lambda = |Y|$ are cardinal numbers then $\kappa < \lambda$ if and only if there exists an injection $f\colon X \to Y$. The famous theorem of Cantor-Bernstein says that if $\kappa \leq \lambda$ and $\lambda \leq \kappa$ then $\kappa = \lambda$. The class of all cardinal numbers is well-ordered by $<$, i.e. $<$ is a linear order and every non-empty subclass has a minimal element. Addition, multiplication and exponentiation of cardinal numbers is defined as follows:

$$\begin{aligned}
\sum_{i\in I} \kappa_i &= |\bigcup_{i\in I}\{i\} \times X_i|, \text{where } \kappa_i = |X_i| \text{ for every } i,\\
\prod_{i\in I} \kappa_i &= |\prod_{i\in I} X_i|, \text{where } \kappa_i = |X_i| \text{ for every } i,\\
\kappa^{\lambda} &= |A^B|, \text{ where } \kappa = |A| \text{ and } \mu = |B|.
\end{aligned}$$

It is easy to prove that the cardinality of $\mathcal{P}(X)$ is equal to $2^{|X|}$.

$\aleph_1$ denotes the first uncountable cardinal, and $\mathfrak{c}$ denotes $2^{\aleph_0}$. Observe that $\mathfrak{c} = |\mathbb{R}|$. Since clearly $\mathfrak{c}$ is uncountable, it follows that $\aleph_1 \leq \mathfrak{c}$. The *Continuum Hypothesis* states that $\aleph_1 = \mathfrak{c}$, i.e. $\mathfrak{c}$ is the first uncountable cardinal number.

2.2. **Ordinal numbers.** Let $(X,<)$ and $(Y,\prec)$ be linearly ordered sets. A function $f\colon X \to Y$ is called *order-preserving* provided that for all $x_1, x_2 \in X$ with $x_1 < x_2$ we have $f(x_1) \prec f(x_2)$. In addition, $(X,<)$ and $(Y,\prec)$ are called *order-isomorphic* if there exists an order-preserving bijection $f\colon X \to Y$. Order-isomorphism is an equivalence relation on the class of all linearly ordered sets. The equivalence class $[X,<]$ of $(X,<)$ is called the *order type* of $(X,<)$.

An ordinal number is the order type of a well-ordered set. The ordinal number of $\mathbb{N}$ with its natural order is denoted by ω_0, or ω. If $\alpha = [X,<]$ and $\beta = [Y,\prec]$ are ordinal numbers then $\alpha \leq \beta$ if there exists an order preserving injection $f\colon X \to Y$ such that $f[X]$ is an initial segment of Y. This defines a well-order on the class of all ordinal numbers. As a consequence one can prove the following

Theorem 2.1. *Let α and β be ordinal numbers, say $\alpha = [X,<], \beta = [Y,<]$.*

(a) *If $\alpha \leq \beta$ then $|X| \leq |Y|$.*

(b) *If $|Y| < |X|$ then $\beta < \alpha$.*

- *Exercise* 1. Prove Theorem 2.1.

There is a smallest uncountable ordinal number, denoted by ω_1. If $\omega_1 = [X,<]$ then $|X| = \aleph_1$.

If α is an ordinal number then $\mathrm{W}(\alpha) = \{\beta : \beta \text{ is an ordinal number and } \beta < \alpha\}$. The set $W(\alpha)$ is called the set of *predecessors* of α.

Since the class of ordinal numbers is well-ordered by $<$ it follows that if A is a set of ordinal numbers then $[A,<]$ is again an ordinal number.

Theorem 2.2. $[W(\alpha),<] = \alpha$.

This transforms $\mathrm{W}(\alpha)$ into a canonical representant of α. Sometimes α and $\mathrm{W}(\alpha)$ are identified. So then every ordinal number is a set.

To every ordinal number α there corresponds a cardinal number, the cardinality of any well-ordered set of type α. It is denoted by $|\alpha|$.

An infinite ordinal number λ is an *initial ordinal number* if λ is the smallest among all ordinal numbers α satisfying $|\alpha| = |\lambda|$ (i.e., if $|\xi| < |\lambda|$ for every $\xi < \lambda$). An initial ordinal number λ is *regular* if there is no $\alpha < \lambda$ which is cofinal with λ. This means that if $\alpha = [X,<]$ and $\lambda = [Y,\prec]$ are such that $\alpha < \lambda$ then there does not exist an injective order-preserving function $f\colon X \to Y$ such that for every $y \in Y$ there exists an $x \in X$ with $y \prec f(x)$.

For every cardinal number κ there exists an initial ordinal number λ such that $|\lambda| = \kappa$ and this λ is unique. A cardinal number κ is *regular* if the initial ordinal number λ is regular. A cardinal number which is not regular is called *singular*. It is easy to show that for every cardinal number κ its successor κ^+ is regular. We sometimes identity κ and its corresponding initial ordinal number λ. So cardinal numbers can be thought of as initial ordinal numbers. Since ordinal numbers can be thought of as sets, the same applies to cardinal numbers. Observe that if κ is a cardinal number, thought of as a set, then its cardinality is again κ.

2.3. **The Axiom of Choice.**

We assume the *Axiom of Choice* (abbreviated: AC):

If $\{X_i\}_{i\in I}$ is a non-empty family of non-empty sets, then $\prod_{i\in I} X_i$ is non-empty as well.

Definition 2.3. Let $(X,\leq)$ be a partially ordered set.

(a) A *chain* in X is a subset of X which is linearly ordered by $\leq$.
(b) A *maximal element* in X is an $x\in X$ such that for all $y\in X$ with $x\leq y$ we have $x=y$.

Definition 2.4. Let $(X,\leq)$ be a linearly ordered set. $(X,\leq)$ is well-ordered (and $\leq$ is a well-ordering) if every non-empty subset of X has a minimal element.

Theorem 2.5 (Zorn's Lemma)**.** *AC is equivalent to the statement that every partially ordered set in which every chain has an upperbound, has a maximal element.*

Theorem 2.6 (Zermelo's Theorem)**.** *AC is equivalent to the statement that every set can be well-ordered.*

3. TYCHONOFF SPACES

A T_1-space X is called *Tychonoff* (or $T_{3\frac{1}{2}}$, or completely regular) if for every $x\in X$ and every neighborhood V of x there exists a continuous function $f\colon X\to\mathbb{I}$ such that $f(x)=0$ and $f(y)=1$ for every $y\in X\setminus V$.

By Urysohn's Lemma, normal spaces are Tychonoff.

Lemma 3.1. *Let X be Tychonoff. If $Y\subseteq X$ then Y is Tychonoff.*

Proof. Let $y\in Y$ and let $F\subseteq Y$ be a relatively closed set not containing y. There is a closed set $F'\subseteq X$ such that $F'\cap Y=F$. By assumption there exists a continuous function $f\colon X\to\mathbb{I}$ such that $f(y)=0$ and $f[X\setminus F']\subseteq\{1\}$. Now define $g\colon Y\to\mathbb{I}$ by $g=f\restriction Y$. Then g is clearly as required. ☐

Theorem 3.2. *Let $\{X_i : i\in I\}$ be a family of Tychonoff spaces. Then $\Pi=\prod_{i\in I}X_i$ is Tychonoff.*

Proof. Let $x\in\Pi$ and let V be a neighborhood of x. We may assume without loss of generality that V is a basic open neighborhood of x. That is, there is a *finite* subset $J\subseteq I$ and for every $j\in J$ an open set $V_j\subseteq X_j$ such that

$$V=\bigcap_{j\in J}\pi_j^{-1}[V_j]$$

(here $\pi_j\colon\Pi\to X_j$ denotes the standard projection). There exists by assumption for every $j\in J$ a continuous function $f_j\colon X_j\to\mathbb{I}$ such that $f_j(x_j)=0$ and $f_j[X_j\setminus V_j]\subseteq\{1\}$. Let

$$\pi\colon\prod_{i\in I}X_i\to\prod_{j\in J}X_j$$

denote the projection, and define $f\colon\prod_{j\in J}X_j\to\mathbb{I}$ by

$$f(y)=\max\{f_j(y_j) : j\in J\}. \tag{3.1}$$

Then f is continuous. Let $F\colon \Pi \to \mathbb{I}$ denote the composition $f \circ \pi$. Then F is continuous. It is clear that $F(x) = 0$. Moreover, if $y \notin V$ then there is an index $j \in J$ such that $y_j \notin V_j$. But then $f_j(y_j) = 1$ which implies that $F(y) = 1$. □

• *Exercise* 2. Prove that the function f in (3.1) is continuous.

A (topological) product of copies of $\mathbb{I}$ is called a *Tychonoff cube*. So a Tychonoff cube is a product of the form

$$\prod_{i \in I} X_i, \tag{3.2}$$

where I is an arbitrary index set, and $X_i = \mathbb{I}$ for every $i \in I$. The product in (3.2) is denoted by $\mathbb{I}^I$.

• *Exercise* 3. Let I and J be index sets having the same cardinality. That is, there exists a bijection $\varphi\colon I \to J$. Suppose that for every $i \in I$ there exist spaces X_i and $Y_{\varphi(i)}$ and a homeomorphism $f_i\colon X_i \to Y_{\varphi(i)}$. Prove that the products

$$\prod_{i \in X} X_i \text{ and } \prod_{j \in J} Y_j$$

are homeomorphic.

So Tychonoff cubes $\mathbb{I}^I$ and $\mathbb{I}^J$ are homeomorphic if I and J have the same cardinality. The converse to this statement also holds. This is surprising but basically trivial if the index sets are infinite, and not surprising but very complicated if the index sets are finite.

Let X be a space. Let

$$\kappa = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}.$$

Since every set of cardinal numbers is well-ordered by $<$, the cardinal number κ is well-defined. It is called the *weight* of X and is denoted by $\mathrm{w}(X)$.

• *Exercise* 4. Let X and Y be homeomorphic spaces. Prove that $w(X) = w(Y)$. Give an example of two non-homeomorphic spaces X and Y having the same weight.

• *Exercise* 5. Let I be an index set. Prove that

$$\mathrm{w}(\mathbb{I}^I) = |I|$$

if I is infinite, and

$$\mathrm{w}(\mathbb{I}^I) = \aleph_0$$

if I is finite. Conclude that if I and J are infinite such that $|I| \neq |J|$ then $\mathbb{I}^I$ is not homeomorphic to $\mathbb{I}^J$.

• *Exercise* 6. Prove that if I is a set containing more than one point then $\mathbb{I}$ and $\mathbb{I}^I$ are not homeomorphic.

Theorem 3.3. *Let X be a space. The following conditions are equivalent:*

1. *X is Tychonoff.*
2. *There is a cardinal number κ such that X can be embedded in $\mathbb{I}^\kappa$.*

Proof. By Theorem 3.2 it follows that $\mathbb{I}^\kappa$ is Tychonoff for every cardinal number κ. This means that all subspaces of Tychonoff cubes are Tychonoff by Lemma 3.1. This proves (2) $\Rightarrow$ (1).

For (1) $\Rightarrow$ (2), let X be Tychonoff, and put

$$\mathcal{A} = C(X, \mathbb{I}),$$

the set consisting of all continuous function $X \to \mathbb{I}$. We will prove that X can be embedded in the Tychonoff cube $\mathbb{I}^{\mathcal{A}}$. Indeed, define a function

$$e\colon X \to \mathbb{I}^{\mathcal{A}}$$

by the formula

$$e(x)_f = f(x) \qquad (x \in X, f \in \mathcal{A}).$$

Claim 3.3.1. e is continuous.

This is easy. For every $f \in \mathcal{A}$ let $\pi_f\colon \mathbb{I}^{\mathcal{A}} \to \mathbb{I}$ denote the projection onto its f-th factor. Then

$$\pi_f \circ e = f$$

for every $f \in \mathcal{A}$, since if $x \in X$ is arbitrary then

$$(\pi_f \circ e)(x) = \pi_f(e(x)) = e(x)_f = f(x).$$

This means that all compositions of the form $\pi_f \circ e$ are continuous. But then e is continuous, by the definition of the product topology on $\mathbb{I}^{\mathcal{A}}$.

Claim 3.3.2. e is injective.

If x and y are distinct points in X then there exists $f \in \mathcal{A}$ such that $f(x) = 0$ and $f(y) = 1$. But then $0 = e(x)_f \neq e(x)_f = 1$, which proves that $e(x) \neq e(y)$.

Claim 3.3.3. $e\colon X \to e[X]$ is closed.

Let $F \subseteq X$ be an arbitrary closed set, and pick an arbitrary point $e(x) \notin e[F]$. Then $x \notin F$, and so there exists an element $f \in \mathcal{A}$ such that $f(x) = 0$ and $f[F] \subseteq \{1\}$. Again let $\pi_f\colon \mathbb{I}^{\mathcal{A}} \to \mathbb{I}$ denote the projection onto the f-th factor. Then

$$U = \pi_f^{-1}\big[[0,1)\big]$$

is open in $\mathbb{I}^{\mathcal{A}}$ and contains $e(x)$ since $e(x)_f = 0$. Now assume that $e(p)$ is an arbitrary point in $e[X] \cap U$. Then $e(p)_f < 1$, which implies that $f(p) < 1$. We conclude that $p \notin F$ and also, by Claim 2, that $e(p) \notin e[F]$. It therefore follows that $U \cap e[X]$ is a neighborhood of $e(x)$ in $e[X]$ which misses $e[F]$. Since $e(x)$ was chosen arbitrarily, it consequently follows that $e[F]$ is closed.

We conclude that $e\colon X \to e[X]$ is a homeomorphism, whence e is an embedding. ☐

So Tychonoff spaces can be thought of as subspaces of Tychonoff cubes.

From now on, all spaces under discussion are Tychonoff.

4. The Čech-Stone compactification of a Tychonoff space

Let X be a Tychonoff space, and let $\mathcal{A} = C(X, \mathbb{I})$. By the proof of Theorem 3.3 it follows that the function $e\colon X \to \mathbb{I}^{\mathcal{A}}$ defined by by the formula

$$e(x)_f = f(x) \qquad (x \in X, f \in \mathcal{A})$$

is an embedding. It will sometimes be convenient to identify X and $e[X]$.

Since $\mathbb{I}^{\mathcal{A}}$ is compact, the closure of $e[X]$ in $\mathbb{I}^{\mathcal{A}}$ is compact as well. It contains $e[X]$ which is a dense homeomorph of X. So it is a *compactification* of X. It is called the *Čech-Stone compactification* of X, and is denoted by βX.

We think of X as a subspace of βX, i.e. when discussing βX we identify X and $e[X]$.

Lemma 4.1. *Let $f\colon X \to \mathbb{I}$ be continuous. Then f can be extended to a unique continuous function $\beta f\colon \beta X \to \mathbb{I}$.*

Proof. As before, let $\pi_f\colon \mathbb{I}^{\mathcal{A}} \to \mathbb{I}$ be the projection onto the f-th factor. Observe that if $x \in X$ then

$$\pi_f \circ e(x) = \pi_f(e(x)) = e(x)_f = f(x).$$

So $\beta f = \pi_f \restriction \beta X$ is a continuous function extending f.

If g and h are continuous functions $\beta X \to \mathbb{I}$ extending f then the set

$$\{x \in \beta X : h(x) = g(x)\}$$

is closed and contains the dense set X. So it is equal to all of βX, i.e. $h = g$. This proves that βf is unique. ☐

This lemma can be improved.

Theorem 4.2. *Let $f\colon X \to K$ be continuous, where K is compact. Then f can be extended to a unique continuous function $\beta f\colon \beta X \to K$.*

Proof. By Theorem 3.3, we may assume that for some cardinal κ we have that K is a subspace of the Tychonoff cube $\mathbb{I}^\kappa$. For every $\alpha \in \kappa$ let $\pi_\alpha\colon \mathbb{I}^\kappa \to \mathbb{I}$ denote the projection onto the α-th factor. In addition, for every $\alpha \in \kappa$, let

$$f_\alpha = (\pi_\alpha \restriction K) \circ f\colon X \to \mathbb{I}.$$

Then f_α is continuous, being the composition of two continuous functions. By Lemma 4.1, there exists for every $\alpha \in \kappa$ a continuous function $F_\alpha\colon \beta X \to \mathbb{I}$ such that $F_\alpha \restriction X = f_\alpha$. Define $F\colon \beta X \to \mathbb{I}^\kappa$ coordinatewise as follows:

$$F(x)_\alpha = F_\alpha(x) \qquad (x \in \beta X, \alpha \in \kappa).$$

Then F is continuous since for every $\alpha \in \kappa$ we have that the composition

$$\pi_\alpha \circ F = F_\alpha$$

is continuous. In addition, if $x \in X$ and $\alpha \in \kappa$ then

$$F(x)_\alpha = F_\alpha(x) = f_\alpha(x) = (\pi_\alpha \restriction K) \circ f(x) = f(x)_\alpha.$$

We conclude that $F \restriction X = f$, and hence that $F[X] \subseteq K$. By observing that X is dense in βX this means that

$$F[\beta X] = F[\overline{X}] \subseteq \overline{F[X]} \subseteq \overline{K} = K$$

since K is closed in $\mathbb{I}^\kappa$. So F is the required extension of f. That F is unique follows by again observing that X is dense in βX. ☐

Corollary 4.3. *Let $f\colon X \to Y$ be continuous. Then f can be extended to a continuous function $\beta f\colon \beta X \to \beta Y$.*

Proof. This follows from Theorem 4.2 since βY is compact and the function f considered to be a function from X into βY is continuous. ☐

• *Exercise* 7. Let X and Y be spaces, and let $\varphi\colon X \to Y$ be a homeomorphism. Prove that the function $\beta\varphi$ from the previous exercise is also a homeomorphism.

Compactifications aX and bX of X are said to be *equivalent* provided there is a homeomorphism $\varphi\colon aX \to bX$ such that $\varphi \restriction X$ is the identity on X. Observe that $\equiv$ is an equivalence relation on the set of all compactifications of X. It is sometimes convenient to identify equivalent compactifications. If aX and bX are compactifications of X then we say that $aX \leq bX$ provided that there is a continuous function $f\colon bX \to aX$ such that $f \restriction X$ is the identity on X.

Proposition 4.4. *Let X be a space and let aX and bX be compactifications of X such that $aX \leq bX$ and $bX \leq aX$. Then $aX \equiv bX$.*

Proof. If $f\colon bX \to aX$ and $g\colon aX \to bX$ are the functions that we get from our assumptions then the composition

$$g \circ f\colon bX \to bX$$

is continuous and has the property that it restricts to the identity on X. So it must be the identity on all of bX. A similar argument shows that the composition

$$f \circ g\colon aX \to aX$$

is the identity on aX. It therefore follows that f is a homeomorphism $bX \to aX$ which is the identity on X. So $aX \equiv bX$. ☐

• *Exercise* 8. Give an example of a space X having homeomorphic compactifications aX and bX which are not equivalent.

Observe that βX is the largest compactification of X. For if γX is a compactification of X then the identity on X can be extended by Theorem 4.2 to a continuous function $\beta X \to \gamma X$. But this means that $\gamma X \leq \beta X$.

The question remains whether βX is unique, that is, whether up to equivalence X has only one largest compactification. But this question has a trivial answer. For if γX is another largest compactification then $\gamma X \leq \beta X$ and $\beta X \leq \gamma X$. But this means that $\gamma X \equiv \beta X$.

So every Tychonoff space X has a largest compactification βX which is characterized by the property stated in Theorem 4.2. It is this compactification that we wish to study, for the simplest infinite space, namely, the natural numbers $\mathbb{N}$ endowed with the discrete topology.

We will finish this section by proving another interesting characterization of βX.

Lemma 4.5. *Let Y be a dense subspace of X, let Z be compact, and let $f\colon Y \to Z$ be continuous. The following statements are equivalent:*

(a) *f can be extended to a continuous function $\bar{f}\colon X \to Z$.*
(b) *For all closed sets $A, B \subseteq Z$ we have*

$$A \cap B = \emptyset \Rightarrow \overline{f^{-1}[A]} \cap \overline{f^{-1}[B]} = \emptyset.$$

(Here closure means closure in X.)

Proof. For (a) $\Rightarrow$ (b), pick disjoint closed sets $A, B \subseteq Z$. Since $\bar{f}$ extends f we have

$$f^{-1}[A] \subseteq \bar{f}^{-1}[A] \text{ and } f^{-1}[B] \subseteq \bar{f}^{-1}[B]. \tag{4.1}$$

Observe that by continuity of $\bar{f}$, $\bar{f}^{-1}[A]$ and $\bar{f}^{-1}[B]$ are disjoint closed sets of X. So we get what we want from (4.1).

The proof of (b) $\Rightarrow$ (a) is more complicated. Pick an arbitrary point $x \in X$ and let $\mathcal{B}(x)$ be the collection of all open neighborhoods of x in X. Put

$$\mathcal{F}(x) = \{\overline{f[U \cap Y]} : U \in \mathcal{B}(x)\}.$$

(Here closure means closure in X.) Since for a finite subcollection $\mathcal{U} \subseteq \mathcal{B}(x)$ we have $\bigcap \mathcal{U} \in \mathcal{B}(x)$ and

$$\overline{f\bigl[\bigcap \mathcal{U} \cap Y\bigr]} \subseteq \bigcap_{U \in \mathcal{U}} \overline{f[U \cap Y]}, \tag{4.2}$$

the family $\mathcal{F}(x)$ has the finite intersection property and consequently by compactness has non-empty intersection.

We shall prove that $\bigcap \mathcal{F}(x)$ consists of a single point. Striving for a contradiction, suppose that $y_1, y_2 \in \bigcap \mathcal{F}(x)$ are distinct. There exist closed neighborhoods V_1 and V_2 of y_1 and y_2 respectively, such that $V_1 \cap V_2 = \emptyset$. From (b) it follows that $\overline{f^{-1}[V_1]} \cap \overline{f^{-1}[V_2]} = \emptyset$. We may assume without loss of generality that $x \notin \overline{f^{-1}[V_1]}$. Thus $X \setminus \overline{f^{-1}[V_1]} \in \mathcal{B}(x)$ and $y_1 \in \bigcap \mathcal{F}(x) \subseteq \overline{f[Y \setminus f^{-1}[V_1]]}$. However, since $f[Y \setminus f^{-1}[V_1]]$ misses V_1, its closure misses the interior of V_1. But y_1 belongs to the interior of V_1, a contradiction.

Observe that if $x \in X$ then $f(x) \in \bigcap \mathcal{F}(x)$ so that, by what we just proved, the unique point in $\bigcap \mathcal{F}(x)$ is $f(x)$.

Assigning to $x \in X$ the unique point in $\bigcap \mathcal{F}(x)$ we define a function $\bar{f}\colon X \to Z$, which extends f. It remains to check that $\bar{f}$ is continuous.

Let V be a neighborhood of $\bar{f}(x)$ in the space Z. Then

$$\{\bar{f}(x)\} = \bigcap \mathcal{F}(x) \subseteq V,$$

and so by compactness of Z there is a finite subfamily $\mathcal{U} \subseteq \mathcal{B}(x)$ such that

$$\bigcap_{U \in \mathcal{U}} \overline{f[Y \cap U]} \subseteq V.$$

It is clear that $\bigcap \mathcal{U}$ is a neighborhood of x for which $\bar{f}\left[\bigcap \mathcal{U}\right] \subseteq V$. $\square$

Let X be a space. A *zero-set* of X is a set of the form $f^{-1}[\{0\}]$, where $f\colon X \to \mathbb{I}$ is continuous. Let $\mathcal{Z}(X)$ denote the collection of zero-sets of X. A *cozero-set* of X is the complement of a zero-set.

• *Exercise* 9. Prove that $\mathcal{Z}(X)$ is closed under finite unions and countable intersections. Prove that if $A, B \in \mathcal{Z}(X)$ are disjoint then there is a continuous function $f\colon X \to \mathbb{I}$ such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$.

• *Exercise* 10. Prove that if X is metrizable (= the topology on X is generated by a metric) then every closed subset of X belongs to $\mathcal{Z}(X)$.

Corollary 4.6. *Let X be a space and let γX be a compactification of X. The following statements are equivalent:*

(a) $\gamma X = \beta X$.
(b) *Disjoint elements in $\mathcal{Z}(X)$ have disjoint closures in γX.*

Proof. For (a) $\Rightarrow$ (b), let A and B be disjoint zero-sets in X. There is a continuous function $f\colon X \to \mathbb{I}$ such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$ (Exercise 9). By Lemma 4.1 this function can be extended to a continuous function $\bar{f}\colon \beta X \to \mathbb{I}$. So $A \subseteq \bar{f}^{-1}[\{0\}]$ and $B \subseteq \bar{f}^{-1}[\{1\}]$ which obviously implies that A and B have disjoint closures in βX.

For (b) $\Rightarrow$ (a), let $f\colon X \to \mathbb{I}$ be continuous. If A and B are disjoint closed subsets of $\mathbb{I}$ then evidently $f^{-1}[A]$ and $f^{-1}[B]$ are disjoint zero-sets of X (Exercise 10), which by assumption have disjoint closures. This shows that f can be extended to a continuous function $\bar{f}\colon \gamma X \to \mathbb{I}$ (Lemma 4.5). So $\gamma X = \beta X$. $\square$

For normal spaces we can do a little better.

Corollary 4.7. *Let X be a normal space and let γX be a compactification of X. The following statements are equivalent:*

(a) $\gamma X \equiv \beta X$.
(b) *Disjoint closed sets in X have disjoint closures in γX.*

Proof. If A and B are disjoint closed sets in X then by normality there is a Urysohn function $f\colon X \to \mathbb{I}$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. So clearly A and B are contained in the disjoint zero-sets $f^{-1}[\{0\}]$ and $f^{-1}[\{1\}]$. But then A and B clearly have disjoint closures in βX (Corollary 4.6).

Now if γX has the property stated in (b) then since zero-sets are closed, disjoint zero-sets have disjoint closures in γX. Hence $\gamma X \equiv \beta X$ again by Corollary 4.6. $\square$

Corollary 4.8. *Let X be a space and let $Y \subseteq X$ be closed. Then the closure of Y in βX is (equivalent to) βY.*

Proof. The closure of Y in βX is certainly a compactification of Y, which we denote by γY. If A and B are disjoint closed sets in Y, then A and B are disjoint closed sets in X, and consequently have disjoint closures in βX, whence in the closure of Y. But then this closure is βY by the previous corollary. $\square$

Let $X = \omega$ with the discrete topology. The aim of this course is to study $\beta\omega$. We can already paint a clear picture of $\beta\omega$. It is a compactification of ω which has the property that disjoint subsets of ω have disjoint closures in $\beta\omega$. Consider an arbitrary point $x \in \beta\omega \setminus \omega$, and consider the collection

$$\mathcal{F}_x = \{A \subseteq \omega : x \in \overline{A}\}.$$

- *Exercise* 11. Prove that $\mathcal{F}_x$ has the following properties:
  1. Every element of $\mathcal{F}_x$ is infinite.
  2. $\mathcal{F}_x$ is closed under finite intersections.
  3. $\mathcal{F}_x$ is closed under supersets (i.e. if $A \in \mathcal{F}_x$ and $B \subseteq \omega$ is such that $A \subseteq B$ then $B \in \mathcal{F}_x$).
  4. If $A \subseteq \omega$ then either $A \in \mathcal{F}_x$ or $\omega \setminus A \in \mathcal{F}_x$.

A collection of sets such as in this exercise will be called a *free ultrafilter* later. So to each "new" point $x \in \beta\omega \setminus \omega$ we can associate a collection of subsets of ω with an interesting combinatorial structure. In fact, as will turn out later, the points in $\beta\omega \setminus \omega$ are in one-to-one correspondence with ultrafilters, and this allows one to study $\beta\omega$ with success.

- *Exercise* 12. Let X be a space and let $X \subseteq Y \subseteq \beta X$. Prove that $\beta Y = \beta X$.

5. Boolean Algebra's

A *Boolean algebra*, abbreviated Ba, is a set A together with two distinguished elements 0 and 1, two binary operations $\wedge$ and $\vee$, a unary operation $'$, satisfying the following identities:

$$\begin{array}{llr}
(1) & 0' = 1 & 1' = 0 \\
(2) & p \wedge 0 = 0 & p \vee 1 = 1 \\
(3) & p \wedge 1 = p & p \vee 0 = p \\
(4) & p \wedge p' = 0 & p \vee p' = 1 \\
(5) & \multicolumn{2}{c}{p'' = p} \\
(6) & p \wedge p = p & p \vee p = p \\
(7) & (p \wedge q)' = p' \vee q' & (p \vee q)' = p' \wedge q' \\
(8) & p \wedge q = q \wedge p & p \vee q = q \vee p \\
(9) & p \wedge (q \wedge r) = (p \wedge q) \wedge r & p \vee (q \vee r) = (p \vee q) \vee r \\
(10) & p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) & p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)
\end{array} \tag{5.1}$$

Let X be a set. Then $\mathcal{P}(X)$ is a Ba with the following operations:

$$\begin{array}{rcl}
0 & = & \emptyset \\
1 & = & X \\
\vee & = & \cup \\
\wedge & = & \cap \\
A' & = & X \setminus A
\end{array} \tag{5.2}$$

A more general way is to consider an arbitrary non-empty subset $\mathcal{A}$ of $\mathcal{P}(X)$ such that if $P, Q \in \mathcal{A}$, then $P \cap Q$, $P \cup Q$ and $X \setminus P$ are also in $\mathcal{A}$. So a σ-algebra of subsets of X is a Ba. Another example is the collection $\{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}$.

- *Exercise* 13. Prove that $\mathcal{A}$ contains $\emptyset$ and X.

It follows that $\mathcal{A}$ is a Ba with operations such as in (5.2). Every Ba obtained in this way is called a *field of sets.*

A *subalgebra* of a Ba A is a subset B of A containing 0 and 1 such that if $p, q \in B$ then so are $p \vee q$, $p \wedge q$ and p'. Since the identities (5.1) also hold when restricted to the elements of B, the operations on A make B into a Ba.

So a field of sets is simply a subalgebra of some Ba of the form $\mathcal{P}(X)$.

Each Ba B can be endowed with a natural partial order. Indeed, if $p, q \in B$ then define

$$p \leq q \Leftrightarrow p \wedge q = p.$$

- *Exercise* 14. Prove that $\leq$ is a partial order on B.

If A and B are Ba's then they a *Boolean homomorphism* is a function $f\colon A \to B$ such that

$$\begin{array}{rcl} f(p \vee q) & = & f(p) \vee f(q) \\ f(p \wedge q) & = & f(p) \wedge f(q) \\ f(p') & = & (f(p))' \end{array}$$

whenever $p, q \in A$.

- *Exercise* 15. Prove that if $f\colon A \to B$ is a homomorphism then $f(0) = 0$ and $f(1) = 1$. Prove that $f[A]$ is a subalgebra of B.

An *isomorphism* between Ba's B and E is a bijection $f\colon B \to E$ which is simultaneously a homomorphism.

- *Exercise* 16. Let $f\colon B \to E$ be an isomorphism between Ba's. Prove that $f^{-1}\colon E \to B$ is an isomorphism.

A *Boolean ideal* in a Ba B is a subset $M \subseteq B$ such that

$$\begin{array}{c} 0 \in M, \\ p, q \in M \Rightarrow p \vee q \in M, \\ p \in M, q \leq p \Rightarrow q \in M. \end{array}$$

Observe that if $f\colon B \to A$ is a homomorphism then its *kernel*

$$\ker(f) = \{x \in B : f(x) = 0\}$$

is an ideal.

$\{0\}$ is the *trivial ideal*, all other ideals will be called *non-trivial.* Every Ba B has an *improper* ideal, namely B itself; all other ideals will be called *proper.*

- *Exercise* 17. Prove that an ideal is proper if and only if it does not contain 1.

If $E \subseteq B$ then the intersection of all ideals containing E is called *the ideal generated by* E.

- *Exercise* 18. Prove that if $E = \emptyset$, then the ideal generated by E is $\{0\}$.

A *Boolean filter* in a Ba B is a subset $N \subseteq B$ such that

$$\begin{gathered} 1 \in N, \\ p, q \in N \Rightarrow p \wedge q \in N, \\ p \in N, p \leq q \Rightarrow q \in N. \end{gathered}$$

If M is an ideal then $N = \{p' : p \in M\}$ is a filter, and vice versa. So filters and ideals come in dual pairs. If $E \subseteq B$ then the *filter generated by* E is the intersection of all filters that contain E.

$\{1\}$ is the *trivial filter*, all other filters will be called *non-trivial.* Every Ba B has an *improper* filter, namely B itself; all other filters will be called *proper.*

- *Exercise* 19. Prove that a filter is proper if and only if it does not contain 0.

- *Exercise* 20. Let $E \subseteq B$ be such that there exist $p, q \in E$ with $p \wedge q = 0$. Prove that the filter generated by E is B.

Let p and q be elements in the Ba B. Define

$$p \triangle q = (p \wedge q') \vee (q \wedge p').$$

Observe that $p \triangle q = q \triangle p$. Let $M \subseteq B$ be a proper ideal. Define an equivalence relation $\equiv$ on B by

$$p \equiv q \Leftrightarrow p \triangle q \in M.$$

- *Exercise* 21. Prove that $\equiv$ is an equivalence relation.

Let A denote the set of all $\equiv$-equivalence classes. If $[p]$ and $[q]$ are such classes then define

$$\begin{aligned} [p] \wedge [q] &= [p \wedge q], \\ [p] \vee [q] &= [p \vee q], \\ [p]' &= [p']. \end{aligned}$$

In addition, put $0 = [0]$ and $1 = [1]$.

- *Exercise* 22. Prove that these operations are well-defined and that A with these operations is a Ba.

The Ba A is denoted by B/M and is called the *quotient of* B *modulo* M.

- *Exercise* 23. Prove that the function $f\colon B \to B/M$ defined by $f(x) = [x]$ is a surjective homomorphism with kernel M. Conclude that every proper ideal is the kernel of some surjective homomorphism of Ba's. (Where did we use that M is proper?)

- *Exercise* 24. Let $f\colon B \to E$ be a surjective homomorphism of Ba's. Prove that E is isomorphic to $B/\ker(f)$.

6. Stone duality

Let X be a zero-dimensional space. That is, a space with a base consisting of clopen (= open and closed) sets. It is clear that

$$\mathcal{B} = \{B \subseteq X : B \text{ is clopen}\}$$

is a subalgebra of $\mathcal{P}(X)$. This Boolean algebra is denoted by $CO(X)$.

So to every zero-dimensional space X one can assign a natural Ba. Our aim now is to prove the converse, namely, to every Ba one can assign a natural zero-dimensional space.

Let B be a Ba. A non-empty subset $A \subseteq B$ has the *finite intersection property*, abbreviated fip, provided that for every *finite* $F = \{x_1, \ldots, x_n\} \subseteq A$ we have

$$\bigwedge F = x_1 \wedge \cdots \wedge x_n \neq 0.$$

Lemma 6.1. *If $A \subseteq B$ has the fip then the set*

$$\hat{A} = \{x \in B : (\exists F \in [A]^{<\omega})(\bigwedge F \leq x)\}$$

is the filter generated by A and is proper.

Proof. That $\hat{A}$ is a filter is easy, and it is proper since A has the fip. The filter generated by A contains $\bigwedge F$ for every $F \in [A]^{<\omega}$ and hence contains $\hat{A}$, hence is equal to $\hat{A}$. ☐

Let B be a Ba. A subset $N \subseteq B$ is called an *ultrafilter* if it is a proper filter while moreover for any proper filter $N' \subseteq B$ with $N \subseteq N'$ we have $N = N'$. So, in other words, an ultrafilter is a *maximal* proper filter.

Lemma 6.2. *Every proper filter is contained in an ultrafilter.*

Proof. Let N be a proper filter in a Ba B. Put

$$\mathcal{E} = \{A \subseteq B : A \text{ is a proper filter containing } N\}.$$

Partial order $\mathcal{E}$ by inclusion. Now let $\mathcal{L} \subseteq \mathcal{E}$ be a non-empty chain. We first claim that $P = \bigcup \mathcal{L}$ is a filter. That $1 \in P$ is clear since $N \subseteq P$. If $p, q \in P$ then there exist $A_0, A_1 \in \mathcal{L}$ such that $p \in A_0$ and $q \in A_1$. Since $\mathcal{L}$ is a chain, $A_0 \subseteq A_1$ or $A_1 \subseteq A_0$. But this means that there exists an $A \in \mathcal{L}$ containing p as well as q. But then $p \wedge q \in A \subseteq P$. The proof that the third property in the definition of a filter holds is similar.

We next claim that P is proper. But this is trivial. By Exercise 19 we only need to show that $0 \notin P$. Striving for a contradiction, assume that $0 \in P$. Then there exists $L \in \mathcal{L}$ such that $0 \in L$, showing that L is improper, which is a contradiction. So by Zorn's Lemma, $\mathcal{E}$ contains a maximal element, i.e. an ultrafilter. ☐

Lemma 6.3. *Let B be a Ba. For a subset $N \subseteq B$ the following conditions are equivalent:*

1. *N is an ultrafilter.*
2. *N has the following properties:*
   - (a) $0 \notin N$.
   - (b) $p, q \in N \Rightarrow p \wedge q \in N$.
   - (c) *If $p \in B$ then either $p \in N$ or $p' \in N$.*

Proof. Suppose first that N is an ultrafilter. Then N is a proper filter, so (a) and (b) are trivially satisfied. Now pick an arbitrary $p \in B$. If both p and p' are in N then so is $0 = p \wedge p'$. But this contradicts the fact that N is proper. Suppose that $p \notin N$. Suppose first that $p \wedge n \neq 0$ for every $n \in N$. Then the set $\{p\} \cup N$ has the fip, hence is contained in a proper filter (Lemma 6.1), hence must be equal to N since N is an ultrafilter, hence $p \in N$ which is a contradiction. We conclude that there exists $n \in N$ such that $p \wedge n = 0$. But then $n \leq p'$ which implies that $p' \in N$ since N is a filter. This proves (c).

Now assume that $N \subseteq B$ satisfies (a), (b) and (c). Take an arbitrary $n \in N$ and let $n \leq p$. Then $p' \wedge n = 0$, so if $p' \in N$ then $0 \in N$ by (b) which contradicts (a). By (c) it therefore follows that $p \in N$. We conclude that N is a filter, and it is proper by (a).

Now let $\hat{N}$ be a proper filter containing N. If $\hat{N} \neq N$ then there exists an element $x \in \hat{N} \setminus N$. Then by (c), $x' \in N \subseteq \hat{N}$. But then $x \wedge x' = 0 \in \hat{N}$, which contradicts the fact that $\hat{N}$ is proper. □

For a Ba B let $\mathrm{st}(B)$ denote the set of all ultrafilters of B. Our aim is to define a natural topology on $\mathrm{st}(B)$. Indeed, for every $b \in B$ define

$$b^* = \{N \in \mathrm{st}(B) : b \in N\}.$$

Lemma 6.4. *If $b, e \in B$ then the following equalities hold:*

(a) $b^* \cap e^* = (b \wedge e)^*$,
(b) $b^* \cup e^* = (b \vee e)^*$,
(c) $\mathrm{st}(B) \setminus b^* = (b')^*$.

Proof. We will only prove (a). First observe that if $p \leq q$ then $p^* \subseteq q^*$. This is trivial since if $N \in p^*$ then $p \in N$ and so $q \in N$. This proves that

$$(b \wedge e)^* \subseteq b^* \cap e^*.$$

Now assume that $N \in b^* \cap e^*$. Then $b, e \in N$ and hence $b \wedge e \in N$ since N is a filter. This implies that $N \in (b \wedge e)^*$, as required. □

- *Exercise* 25. Prove (b) and (c) of the previous lemma.

Lemma 6.4 shows that the collection

$$B^* = \{b^* : b \in B\}$$

is a base for a topology on $\mathrm{st}(B)$. This topology is called the *Stone topology* on $\mathrm{st}(B)$, and $\mathrm{st}(B)$ is called the *Stone space* of B.

Observe that by Lemma 6.4(c) it follows that B^* is closed under complements. This means that the elements of B^* are open and closed, i.e. clopen. So $\mathrm{st}(B)$ is zero-dimensional.

Lemma 6.5. $\mathrm{st}(B)$ *is compact and Hausdorff (hence normal, hence Tychonoff).*

Proof. Let $\mathcal{U}$ be an open cover of $\mathrm{st}(B)$. We will prove that it has a finite subcover. We may assume without loss of generality that $\mathcal{U}$ is a subcollection of the base B^*. Let $U \subseteq B$ be such that

$$\mathcal{U} = \{u^* : u \in U\}.$$

Striving for a contradiction, assume that $\mathcal{U}$ does not have a finite subcover. Consider the collection

$$\mathcal{V} = \{\mathrm{st}(B) \setminus u^* : u \in U\} = \{(u')^* : u \in U\}.$$

Then $\mathcal{V}$ has the finite intersection property. We claim that

$$U' = \{u' : u \in U\}$$

has the fip. This is easy since if $u_1, \dots, u_n \in U$ then there exists $N \in (u_1')^* \cap \dots \cap (u_n')^*$, or, equivalently, $u_1' \wedge \dots \wedge u_n' \in N$, which is as required since $0 \notin N$.

By Lemmas 6.1 and 6.2 it follows that there exists an ultrafilter $N \supseteq U'$. Since $\mathcal{U}$ covers, there exists $u \in U$ such that $N \in u^*$. But now both u and u' belong to N which contradicts Lemma 6.3(c).

We will now prove that $\mathrm{st}(B)$ is T_2. To this end, take distinct elements $N_0, N_1 \in \mathrm{st}(B)$. Since $N_0 \neq N_1$ we may assume without loss of generality that there exists an element $p \in N_0 \setminus N_1$. Then $p' \in N_1$ and

$$p^* \cap (p')^* = (p \wedge p')^* = 0^* = \emptyset.$$

We conclude that p^* and $(p')^*$ are disjoint open neighborhoods of N_0 and N_1, respectively. □

So to every Ba B we can assign the zero-dimensional compact space $\mathrm{st}(B)$. As we saw above, to every zero-dimensional space X we can assign the Ba $CO(X)$. So we have the following situation:

$$B \to \mathrm{st}(B) \to CO(\mathrm{st}(B)) \to \mathrm{st}(CO(\mathrm{st}(B))).$$

We will now show that there is in fact duality: the Ba's B and $CO(\mathrm{st}(B))$ are isomorphic and the spaces $\mathrm{st}(B)$ and $\mathrm{st}(CO(\mathrm{st}(B)))$ are homeomorphic. Moreover, the isomorphism and the homeomorphism are canonical. This is the so-called *Stone duality* between Ba's and compact zero-dimensional spaces.

Lemma 6.6. *If $C \subseteq \mathrm{st}(B)$ is clopen then there exists a $b \in B$ such that $C = b^*$.*

Proof. For every $x \in C$ pick an element $b_x \in B$ such that $x \in b_x^* \subseteq C$. Here we use that C is open. Since C is compact, the cover

$$\{b_x^* : x \in C\}$$

has a finite subcover. So there exists a finite subset $F \subseteq C$ such that

$$C = \bigcup_{x \in F} b_x^*.$$

By Lemma 6.4(b) it follows that

$$\bigcup_{x \in F} b_x^* = (\bigvee_{x \in F} b_x)^*.$$

So we are done. □

The previous lemma shows that $CO(\mathrm{st}(B))$ is equal to the collection $\{b^* : b \in B\}$. From Lemma 6.4 it now clearly follows that the function

$$b \mapsto b^*$$

is a natural isomorphism between B and $CO(\mathrm{st}(B))$.

Corollary 6.7. *Every Ba is isomorphic to a field of sets.*

Now start with a compact zero-dimensional space X, and let $Y = \mathrm{st}(CO(X))$. We claim that X and Y are homeomorphic. To see this, let $\mathcal{N} \in \mathrm{st}(CO(X))$. Then $\mathcal{N}$ is a collection of clopen subsets of X with the finite intersection property. By compactness it therefore follows that $\bigcap \mathcal{N} \neq \emptyset$. We claim that this intersection contains at most one point. Striving for a contradiction, assume that it contains at least two distinct points, say x and y. Since X is zero-dimensional, there is an element $C \in CO(X)$ such that $x \in C$ but $y \notin C$. Since $\mathcal{N}$ is an ultrafilter, either C is in $\mathcal{N}$ or $X \setminus C$ is in $\mathcal{N}$. If e.g. $C \in \mathcal{N}$ then y must belong to C which is a contradiction. So the function

$$f\colon \mathrm{st}(CO(X)) \to X$$

defined by $\bigcap \mathcal{N} = \{f(\mathcal{N})\}$ is well-defined. A moments reflection shows that it is surjective and open since for $C \in CO(X)$ the equality

$$f[C^*] = C$$

holds. This proves that f is a homeomorphism.

If X and Y are zero-dimensional spaces and $f\colon X \to Y$ is a homeomorphism then the function

$$F\colon CO(X) \to CO(Y)$$

defined by $F(A) = f[A]$ is easily seen to be a natural isomorphism. So homeomorphic spaces have isomorphic clopen algebra's.

If B and E are Ba's and $f\colon B \to E$ is an isomorphism then a subset $N \subseteq B$ is an ultrafilter if and only if $f[N]$ is an ultrafilter. This defines a function

$$F\colon \mathrm{st}(B) \to \mathrm{st}(E).$$

• *Exercise* 26. Prove that F is a homeomorphism.

Corollary 6.8. *Let X and Y be compact zero-dimensional spaces. The following statements are equivalent:*

(a) *X and Y are homeomorphic.*
(b) *$CO(X)$ and $CO(Y)$ are isomorphic.*

Proof. We only need to prove that (b) implies (a). If $CO(X)$ and $CO(Y)$ are isomorphic then the Stone spaces $\mathrm{st}(CO(X))$ and $\mathrm{st}(CO(Y))$ are homeomorphic. But $X \approx \mathrm{st}(CO(X))$ and $Y \approx \mathrm{st}(CO(Y))$, so we are done. ☐

7. Subalgebra's and homomorphic images

Stone duality says that zero-dimensional compact spaces are being characterized by their algebra's of clopen sets. In this section we will think about the question whether continuous images and subspaces can be characterized in a similar way.

Theorem 7.1. *Let X and Y be zero-dimensional compact spaces. The following statements are equivalent:*

(a) *There is a continuous surjection $f\colon X \to Y$.*
(b) *$CO(X)$ contains a subalgebra isomorphic to $CO(Y)$.*

Proof. The proof of (a) ⇒ (b) is simple. If $f\colon X \to Y$ is a continuous surjection, then $F\colon CO(Y) \to CO(X)$ defined by $F(C) = f^{-1}[C]$ is a one-to-one homomorphism.

For (b) ⇒ (a), let $\varphi\colon CO(Y) \to CO(X)$ be a one-to-one homomorphism. If $\mathcal{N}$ is an ultrafilter in $CO(X)$ then $\mathcal{N} \cap \varphi[CO(Y)]$ is an ultrafilter in $\varphi[CO(Y)]$. Define $f\colon \mathrm{st}(CO(X)) \to \mathrm{st}(CO(Y))$ by

$$f(\mathcal{N}) = \varphi^{-1}(\mathcal{N} \cap \varphi[CO(Y)]) \qquad (\mathcal{N} \in \mathrm{st}(CO(X))).$$

- *Exercise* 27. Prove that f is a continuous surjection.

An application of Corollary 6.8 now finishes the proof. □

Closed subspaces can be dealt with similarly.

Theorem 7.2. *Let X and Y be zero-dimensional compact spaces. The following statements are equivalent:*

(a) *Y can be embedded in X as a (closed) subspace.*
(b) *There is a surjective homomorphism $\varphi\colon CO(X) \to CO(Y)$.*

Proof. If Y is a closed subspace of X then define $\varphi\colon CO(X) \to CO(Y)$ by $\varphi(C) = C \cap Y$. Then φ is clearly a homomorphism, and is surjective since by compactness every clopen subset of Y is the trace on Y of some clopen subset of X.

Now assume that there is a surjective homomorphism $\varphi\colon CO(X) \to CO(Y)$. Fix an element $\mathcal{N} \in \mathrm{st}(CO(Y))$ and put

$$\mathcal{M}(\mathcal{N}) = \{C \in CO(X) : \varphi(C) \in \mathcal{N}\}.$$

- *Exercise* 28. Prove that $\mathcal{M}(\mathcal{N})$ is an ultrafilter, and that the function $\mathcal{N} \mapsto \mathcal{M}(\mathcal{N})$ is one-to-one and continuous, hence an embedding (by compactness).

This completes the proof of the theorem. □

Let fin denote the ideal of finite subsets of ω.

- *Exercise* 29. Let $\gamma\omega$ be a zero-dimensional compactification of ω, and let $X = \gamma\omega \setminus \omega$. Prove the following statements:

(a) If $C \subseteq \gamma\omega$ is clopen and $C \subseteq \omega$ then C is finite.
(b) For every $C \in CO(X)$ there exists an element $C' \in CO(\gamma\omega)$ such that $C' \cap X = C$.
(c) $CO(X)$ is isomorphic to $CO(\gamma\omega)/\,\mathrm{fin}$.

8. Complete Ba's and extremally disconnected spaces

Let B be a Ba and let $A \subseteq B$. If A has a least upperbound (with respect to the natural partial order $\leq$ on B defined in §5)) then it is denoted by $\bigvee A$. Similarly, if A has a greatest lower bound then it is denoted by $\bigwedge A$. (Observe that if $A = \{x_1, \ldots, x_n\}$ is finite then this notation agrees with our earlier notation because then $\bigwedge A = x_1 \wedge \cdots \wedge x_n$, etc.)

If every subset $A \subseteq B$ has a least upper bound then B is said to be *complete*.

Let X be a set and consider the Ba $\mathcal{P}(X)$. The natural partial order on $\mathcal{P}(X)$ coincides with the inclusion relation. It therefore follows that if $\mathcal{A} \subseteq \mathcal{P}(X)$ then $\bigcup \mathcal{A}$ is the least upper bound of $\mathcal{A}$. Hence $\mathcal{P}(X)$ is complete.

- *Exercise* 30. Give an example of an incomplete Ba.

Lemma 8.1. *Let B be a Ba. The following statements are equivalent:*

(a) *Every subset of B has a least upper bound.*
(b) *Every subset of B has a greatest lower bound.*

Proof. If $A \subseteq B$ then $x \in B$ is the least upper bound of A if and only if a' is the greatest lower bound of $\{a' : a \in A\}$. ☐

A topological space X is called *extremally disconnected* if the closure of every open set is again open.

Lemma 8.2. *Let X be a space. The following conditions are equivalent:*

(a) *X is extremally disconnected.*
(b) *Disjoint open subsets in X have disjoint closures.*

- *Exercise* 31. Prove Lemma 8.2.

Corollary 8.3. *Let X be a compact extremally disconnected space, and let $Y \subseteq X$ be dense. Then $\beta Y = X$.*

Proof. Let A and B be disjoint zero-sets in Y. These sets can be separated by disjoint open sets in Y, say U and V (because they can be separated by a realvalued continuous function, see Exercise 9). Let U' and V' be open subsets of X such that $U' \cap Y = U$ and $V' \cap Y = V$. Then U' and V' are disjoint because Y is dense. But then U' and V' have disjoint closures in X (Lemma 8.2), which immediately implies that A and B have disjoint closures in X. Now apply Corollary 4.6. ☐

Theorem 8.4. *Let B be a Ba. The following statements are equivalent:*

(a) *B is complete.*
(b) *$\mathrm{st}(B)$ is extremally disconnected.*

Proof. Assume that B is complete, and let $U \subseteq \mathrm{st}(B)$ be open. There is a subset $A \subseteq B$ such that

$$U = \bigcup_{a \in A} a^*.$$

Let b be the least upper bound of A. We claim that the closure of U is equal to b^* which suffices since b^* is open. If $x \in A$ then $x \leq b$ from which it follows that $x^* \subseteq b^*$. As a consequence, $U \subseteq b^*$ and so $\overline{U} \subseteq b^*$. Striving for a contradiction, assume that there exists $N \in b^* \setminus \overline{U}$. Pick $n \in N$ such that $n^* \cap \overline{U} = \emptyset$. Then $n \wedge a = 0$ for every $a \in A$, and so $a \leq n'$ for every $a \in A$. But then $b \leq n'$ from which it follows that $n' \in N$ since b is. But this is a contradiction since N is an ultrafilter and contains both n and n'.

Now assume that $\mathrm{st}(B)$ is extremally disconnected, and let $A \subseteq B$. Then $U = \bigcup_{a \in A} a^*$ is an open subset of $\mathrm{st}(B)$, and hence has clopen closure. By Lemma 6.6 there exists an element $b \in B$ such that $b^* = \overline{U}$. It is now easily seen that b is the least upper bound of A. ☐

Let X be a topological space. A subset $U \subseteq X$ is called *regular-open* provided that it is the interior of its own closure. In symbols:

$$U = \mathrm{int}(\overline{U}).$$

So it follows in particular that a regular-open subset is open, and also that a clopen set is regular-open. For a space X, $RO(X)$ denotes the collection of regular-open subsets of X.

Lemma 8.5. *If X is extremally disconnected then $RO(X) = CO(X)$.*

Proof. This is clear since if $U \in RO(X)$ then $\overline{U}$ is open since X is extremally disconnected. ☐

Lemma 8.6. *A set $U \subseteq X$ is regular-open if and only if there is an open set $V \subseteq X$ such that $U = \mathrm{int}(\overline{V})$.*

Proof. Suppose there is an open set $V \subseteq X$ such that $U = \mathrm{int}(\overline{V})$. We claim that U is dense in $\overline{V}$. This is clear since $V \subseteq \overline{V}$ and hence $V \subseteq \mathrm{int}(\overline{V})$. But then

$$\mathrm{int}(\overline{U}) = \mathrm{int}(\overline{V}) = U,$$

as required. ☐

• *Exercise* 32. Give an example of an open set which is not regular-open. Give an example of regular-open sets U and V such that $U \cup V$ is not regular-open.

Lemma 8.7. *Let X be a space and let $U, V \subseteq X$.*

(a) *If U is closed then $\mathrm{int}(U)$ is regular-open.*
(b) *If $U, V \subseteq X$ are regular-open then so is $U \cap V$.*
(c) *If U is open then $X \setminus \overline{U}$ is regular-open.*

Proof. For (a), let $V = \mathrm{int}(U)$. Then $V \subseteq U$ and so $\overline{V} \subseteq U$ from which it follows that

$$\mathrm{int}(\overline{V}) \subseteq \mathrm{int}(U) = V \subseteq \mathrm{int}(\overline{V}),$$

as required.

For (b), first observe that

$$\mathrm{int}(\overline{U \cap V}) \subseteq \overline{U \cap V} \subseteq \overline{U} \cap \overline{V}$$

and hence that

$$\mathrm{int}(\overline{U \cap V}) \subseteq \mathrm{int}(\overline{U} \cap \overline{V}) \subseteq \mathrm{int}(\overline{U}) \cap \mathrm{int}(\overline{V}) = U \cap V. \tag{8.1}$$

Next, observe that $U \cap V$ is an open subset of $\overline{U \cap V}$ from which it follows that

$$U \cap V \subseteq \mathrm{int}(\overline{U \cap V}). \tag{8.2}$$

So (8.1) and (8.2) show that $U \cap V = \mathrm{int}(\overline{U \cap V})$, as required.

For (c), observe that $X \setminus U$ is closed, so $\mathrm{int}(X \setminus U)$ is regular-open by (a). So now it simply suffices to observe that on the one hand

$$X \setminus \overline{U} \subseteq \mathrm{int}(X \setminus U),$$

while on the other hand $\mathrm{int}(X \setminus U)$ is an open subset of X which misses U, and hence $\overline{U}$, so that

$$\mathrm{int}(X \setminus U) \subseteq X \setminus \overline{U}.$$

This completes the proof of the lemma. □

For $U, V \in RO(X)$ we define

$$\begin{aligned} U \wedge V &= U \cap V, \\ U \vee V &= \mathrm{int}(\overline{U \cup V}), \\ U' &= X \setminus \overline{U}. \end{aligned}$$

From the previous lemma it follows that $U \wedge V$, $U \vee V$ and U' are elements of $RO(X)$.

We also put $0 = \emptyset$ and $1 = X$.

Theorem 8.8. *Let X be a space. Then $RO(X)$ with the above operations is a complete Ba. In fact, if $\mathcal{A} \subseteq RO(X)$ then*

$$\bigvee \mathcal{A} = \mathrm{int}(\overline{\bigcup \mathcal{A}}).$$

$CO(X)$ is a subalgebra of $RO(X)$.

The verification of this theorem is tedious but routine.

- *Exercise* 33. Prove Theorem 8.8.

The Boolean algebra $RO(X)$ is quite interesting. By Stone duality, it is (isomorphic to) a field of sets. But the precise structure of this field is not clear at all, not even for simple spaces X such as the unit interval $\mathbb{I}$.

Let X be a compact space. We are interested in the Stone space of the Ba $RO(X)$. This space is called the *projective cover* of X and is denoted by EX.

Let $\mathcal{N}$ be an arbitrary ultrafilter in $RO(X)$. If $U, V \in \mathcal{N}$ then $U \wedge V = U \cap V \neq \emptyset$ since $\mathcal{N}$ is a proper filter. We conclude that

$$\overline{\mathcal{N}} = \{\overline{U} : U \in \mathcal{N}\}$$

is a collection of closed subsets of X with the finite intersection property. So by compactness,

$$\bigcap \overline{\mathcal{N}} \neq \emptyset. \tag{8.3}$$

We claim that the intersection in (8.3) consists in fact of a single point. Striving for a contradiction, assume that x and y are distinct points in the intersection (8.3). There is an open set $U \subseteq X$ such that $x \in U \subseteq \overline{U} \subseteq X \setminus \{y\}$. By replacing U by $\operatorname{int}(\overline{U})$ if necessary, we may assume that U is regular-open. Put $V = X \setminus \overline{U}$. Then V is regular-open as well (see Lemma 8.7) and $y \in V$. Observe that $U \wedge V = 0$ and

$$U \vee V = \operatorname{int}(\overline{U \cup V}) = X = 1.$$

So $U' = V$ and we may therefore assume without loss of generality that $U \in \mathcal{N}$ (Lemma 6.3). But this is a contradiction since $y \notin \overline{U}$.

So each element $\mathcal{N} \in \operatorname{st}(RO(X))$ "clusters" or "converges" to a unique point in X, say $\pi(\mathcal{N})$.

Lemma 8.9. *If $\pi(\mathcal{N}) \in V$ and V is regular-open then $V \in \mathcal{N}$.*

Proof. If not then $V' \in \mathcal{N}$ since $\mathcal{N}$ is an ultrafilter (Lemma 6.3). But then $\pi(\mathcal{N}) \in \overline{V'}$ which is a contradiction since $V \cap \overline{V'} = \emptyset$. ☐

Let $f\colon X \to Y$ be a continuous surjection. We say that f is *irreducible* provided that for every proper closed subset $A \subseteq X$ we have that $f[A] \neq Y$.

Theorem 8.10. *The function $\pi\colon EX \to X$ is continuous and irreducible.*

Proof. Let $U \subseteq X$ be open and assume that for some $\mathcal{N} \in EX$ we have $x = \pi(\mathcal{N}) \in U$. There is an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. As before, we may assume that V is regular-open. Then $V \in \mathcal{N}$ by Lemma 8.9. So V^* is a neighborhood of $\mathcal{N}$ in EX. If $\mathcal{M} \in V^*$ then $\pi(\mathcal{M}) \in \overline{V} \subseteq U$. It therefore follows that $V^* \subseteq \pi^{-1}[U]$. Since $\mathcal{N}$ was arbitrary, it follows that $\pi^{-1}[U]$ is open. This proves continuity.

Now let $A \subseteq EX$ be a proper closed set, and pick $\mathcal{N} \in EX \setminus A$. Since A is closed, there exists an element $V \in \mathcal{N}$ such that $V^* \cap A = \emptyset$. We claim that $V \cap \pi[A] = \emptyset$. If this were true then we are done since $V \neq \emptyset$. Striving for a contradiction, assume that there exists an element $\mathcal{M} \in A$ such that $\pi(\mathcal{M}) \in V$. By Lemma 8.9 it follows that $V \in \mathcal{M}$. But this is a contradiction since $V^* \cap A = \emptyset$. ☐

Since $RO(X)$ is complete, EX is an extremally disconnected compact space which admits an irreducible surjection onto X. In fact, these properties characterize EX, as we will now aim at proving.

Lemma 8.11. *Let $f\colon X \to Y$ be irreducible and closed. If $U \subseteq X$ is non-empty and open then there exists a non-empty open set $V \subseteq X$ such that $f^{-1}[V] \subseteq U$.*

Proof. Put $V = Y \setminus f[X \setminus U]$. Observe that $V \neq \emptyset$ since f is irreducible. ☐

Lemma 8.12. *Let $f\colon X \to Y$ be irreducible and closed. If $U \in RO(X)$ then*

$$U^{\#} = Y \setminus f[X \setminus U] \in RO(Y).$$

Moreover, $f^{-1}[U^{\#}]$ is dense in U.

Proof. It is clear that $U^\#$ is open. Put $V = \operatorname{int}(\overline{U^\#})$. Observe that $U^\# \subseteq V$.

Suppose that $f^{-1}[V] \subseteq \overline{U}$. Then $f^{-1}[V] \subseteq \operatorname{int}(\overline{U}) = U$. As a consequence,

$$f^{-1}[V] \cap (X \setminus U) = \emptyset,$$

and so $V \cap f[X \setminus U] = \emptyset$. But then $V \subseteq U^\#$ and hence $V = U^\#$, so we are done.

Let $E = f^{-1}[V] \setminus \overline{U} \neq \emptyset$. By Lemma 8.11 there is a non-empty open set $W \subseteq Y$ such that $f^{-1}[W] \subseteq E$. Then on the one hand $f^{-1}[W] \subseteq f^{-1}[V]$ which implies that $W \subseteq V$. On the other hand, $f^{-1}[W] \subseteq X \setminus \overline{U}$. This implies that $W \subseteq f[X \setminus \overline{U}]$, hence misses $U^\#$, hence misses $\overline{U^\#}$. So this is a contradiction since $\emptyset \neq W \subseteq V \subseteq \overline{U}^\#$.

First observe that $f^{-1}[U^\#] \subseteq U$. Now if $f^{-1}[U^\#]$ is not dense in U then there exists a non-empty set $V \subseteq U \setminus f^{-1}[U^\#]$. But this is impossible since $V^\# \subseteq U^\#$ and so $f^{-1}[V^\#] \subseteq f^{-1}[U^\#]$, and $f^{-1}[V^\#] \subseteq V$. □

Let $f\colon X \to Y$ be irreducible and closed. If $U \in RO(X)$ then the previous result tells us that $U^\# \in RO(Y)$. So the assignment $U \mapsto U^\#$ is a function from $RO(X)$ to $RO(Y)$. This function shall be denoted by $f^\#$. So: $f^\#(U) = U^\#$ for all $U \in RO(X)$.

Corollary 8.13. *Let $f\colon X \to Y$ be irreducible and closed. If $U \in RO(X)$ and $V \in RO(Y)$ is such that V is dense in $f[U]$ then $V = U^\#$.*

Proof. This is clear since the previous lemma shows that $f^{-1}[U^\#]$ is dense in U. As a consequence, both V and $U^\#$ are dense in $f[U]$, and so

$$V = \operatorname{int}(\overline{V}) = \operatorname{int}(\overline{f[U]}) = \operatorname{int}(\overline{U^\#}) = U^\#,$$

as required. □

Lemma 8.14. *Let $f\colon X \to Y$ be irreducible and closed. Then the function $f^\#\colon RO(X) \to RO(Y)$ is an isomorphism of Boolean algebra's.*

Proof. By Lemma 8.12 it follows that $f^\#$ is well-defined.

Claim 8.14.1. $f^\#$ is surjective.

Take an arbitrary $V \in RO(Y)$, and let $U = \operatorname{int}(\overline{f^{-1}[V]})$. Then V is dense in $f[U]$ and so $V = U^\#$ by Corollary 8.13.

Claim 8.14.2. $f^\#$ is one-to-one.

Take distinct $U, V \in RO(X)$. If $\overline{U} = \overline{V}$ then $U = V$. We may therefore assume without loss of generality that $O = U \setminus \overline{V} \neq \emptyset$. But this implies that $\emptyset \neq O^\# \subseteq U^\# \setminus V^\#$.

Claim 8.14.3. $f^\#$ is a homomorphism.

Let $U, V \in RO(X)$. Then $U^\# \cap V^\# \in RO(Y)$ (Lemma 8.7(b)). In addition, $f^{-1}[U^\#] \cap f^{-1}[V^\#]$ is dense in $U \cap V$ by Lemma 8.12 and the fact that both U and V are open. But then $U^\# \cap V^\#$ is a regular-open set which is dense in $f[U \cap V]$. This implies that

$$U^\# \cap V^\# = (U \cap V)^\#$$

(Corollary 8.13), and so

$$f^\#(U) \wedge f^\#(V) = f^\#(U \wedge V).$$

- *Exercise* 34. Complete the proof of Claim 8.14.3.

So we are done. ☐

Theorem 8.15. *Let X be a compact space, let Y be an extremally disconnected compact space, and assume that there exists an irreducible surjection $f\colon Y \to X$. Then there exists a homeomorphism $\varphi\colon Y \to EX$ such that $f = \pi \circ \varphi$.*

Proof. By Lemma 8.5 it follows that $RO(EX) = CO(EX)$ and, similarly, that $RO(Y) = CO(Y)$. By Lemma 8.14 it follows that the function $CO(Y) \ni C \mapsto C^{\#} \in CO(X)$ is an isomorphism between $CO(Y)$ and $RO(X)$ (observe that the operation # is performed by using the function f). It follows similarly that the function

$$RO(X) \ni U \mapsto \operatorname{int}(\overline{\pi^{-1}[U]}) = \overline{\pi^{-1}[U]} \in CO(EX) \tag{8.4}$$

(for this we need to know the inverse of the isomorphism $\pi^{\#}$; see the proof of Claim 8.14.2 for that and conclude that (8.4) is correct). So it follows that the function

$$C \mapsto \overline{\pi^{-1}[C\#]}$$

is an isomorphism between the Ba's $CO(Y)$ and $CO(EX)$. So by Stone duality, $Y \approx EX$.

- *Exercise* 35. Prove that the homeomorphism $\varphi\colon Y \to EX$ we get from Stone duality satisfies $f = \pi \circ \varphi$.

So we are done. ☐

We finish this section with a few observations on irreducible maps. If $A \subseteq X$ and $r\colon X \to A$ is such that $r \restriction A$ is the identity on A then we say that A is a *retract* of X, and the function r is called a *retraction.*

- *Exercise* 36. Prove that a retract of X is a closed subspace of X.

Proposition 8.16. *Let X and Y be compact spaces.*

(a) *If $f\colon X \to Y$ is a continuous surjection then there exists a closed subset $A \subseteq X$ such that $f \restriction A \to Y$ is irreducible.*
(b) *If Y is extremally disconnected and $f\colon X \to Y$ is irreducible then f is a homeomorphism.*
(c) *If Y is extremally disconnected and $f\colon X \to Y$ is a continuous surjection then there is a closed subset $A \subseteq X$ such that $f \restriction A\colon A \to Y$ is a homeomorphism. Hence X contains a closed homeomorph of Y which is a retract of X.*

Proof. For (a), let $\mathcal{A}$ be the collection of all closed subsets of X satisfying $f[A] = Y$. Then $\mathcal{A} \neq \emptyset$ since X belongs to it. Partially order $\mathcal{A}$ by reverse inclusion, that is, for $A, A' \in \mathcal{A}$ define

$$A \leq A' \Leftrightarrow A \supseteq A'.$$

By compactness of X, for each chain $\mathcal{L} \subseteq \mathcal{A}$, $\bigcap \mathcal{L}$ is an upperbound. So Zorn's Lemma implies that $\mathcal{A}$ contains a maximal element, say A. Then A is clearly as required.

For (b) it suffices to prove that f is one-to-one. Let x and y be distinct points in X. Fix open neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Then

$U^\# \cap V^\# = \emptyset$, and hence $\overline{U^\#} \cap \overline{V^\#} = \emptyset$ since Y is extremally disconnected (Lemma 8.2). So $f(x) \neq f(y)$ since $f(x) \in f[U] \supseteq U^\#$ and $f(y) \in f[V] \supseteq V^\#$, and by Lemma 8.12 $U^\#$ and $V^\#$ are dense in $f[U]$ and $f[V]$, respectively, and $U^\#$ and $V^\#$ have disjoint closures.

For (c), we first use (a) to find a closed subset $A \subseteq X$ such that $f \restriction A \colon A \to Y$ is irreducible. Then we use (b) to conclude that $f \restriction A \colon A \to Y$ is a homeomorphism. The rest is clear. ☐

• *Exercise* 37. Let X be a compact space. Prove that $E(EX)$ and EX are homeomorphic.

Let X be a space, and put

$$d(X) = \min\{|D| : D \subseteq X \text{ is dense}\}.$$

The cardinal number $d(X)$ is called the *density* of X.

Lemma 8.17. *Let $f \colon X \to Y$ be irreducible and closed. Then X and Y have the same density.*

Proof. That $d(Y) \leq d(X)$ is clear. Now let $D \subseteq Y$ be dense such that $d(Y) = |Y|$. For every $d \in D$ pick $x_d \in X$ such that $f(x_d) = d$, and put $E = \{x_d : d \in D\}$. Then E is of the same cardinality as D since $f \restriction E \colon E \to D$ is a bijection. Also, E is dense. For if E is not dense, then $f[\overline{E}]$ is a proper closed subset of Y. Hence there exists $d \in D$ such that $d \notin f[\overline{E}]$. But then $x_d \notin E$, which is a contradiction. We conclude that $d(X) \leq d(Y)$. ☐

• *Exercise* 38. Let X be a space with compactification γX. Prove that the canonical continuous surjection $f \colon \beta X \to \gamma X$ which is the identity on X is irreducible. Conclude that all compactifications of X have the same density.

A π-basis for a space X is a collection $\mathcal{U}$ of non-empty open subsets of X such that every non-empty open subset of X contains a member from $\mathcal{U}$. The π-*weight* $\pi(X)$ is the minimum of the set $\{|\mathcal{B}| : \mathcal{B}\}$ is a π-basis for $X\}$.

• *Exercise* 39. Let $f \colon X \to Y$ be irreducible and closed. Prove that $\pi(X) = \pi(Y)$.

9. HOW $\beta\omega$ LOOKS LIKE

Consider the Ba $\mathcal{P}(\omega)$ and its Stone space X. So the points of X are ultrafilters in $\mathcal{P}(\omega)$. That is, $\mathcal{N} \in X$ if and only if $\mathcal{N} \subseteq \mathcal{P}(\omega)$ and has the following properties:

(a) $\emptyset \notin \mathcal{N}$,
(a) $P, Q \in \mathcal{N} \Rightarrow P \cap Q \in \mathcal{N}$,
(c) If $P \subseteq \omega$ then either $P \in \mathcal{N}$ or $\omega \setminus P \in \mathcal{N}$

(Lemma 6.3).

There are a few simple but important characterizations of ultrafilters.

Theorem 9.1. *Let $\mathcal{F}$ be a collection of subsets of ω. The following properties are equivalent:*

(a) *$\mathcal{F}$ is an ultrafilter.*
(b) *$\mathcal{F}$ is a subfamily of $\mathcal{P}(\omega)$ which is maximal with respect to the finite intersection property.*

(c) *$\mathcal{F}$ has the finite intersection property, and if $A \subseteq \omega$ is such that for every finite subfamily $\mathcal{G} \subseteq \mathcal{F}$ we have $A \cap \bigcap \mathcal{G} \neq \emptyset$ then $A \in \mathcal{F}$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\mathcal{F}$ is an ultrafilter, $\mathcal{F} \subseteq \mathcal{G}$, and $\mathcal{G}$ has the finite intersection property. If for some $G \in \mathcal{G}$ we have $G \notin \mathcal{F}$ then $G' = \omega \setminus G \in \mathcal{F} \subseteq \mathcal{G}$. But then $G \cap G' = \emptyset$ violates our assumption that $\mathcal{G}$ has the finite intersection property.

(b) $\Rightarrow$ (c). If A is such that $A \cap \bigcap \mathcal{G} \neq \emptyset$ for every finite $\mathcal{G} \subseteq \mathcal{F}$ Then $\mathcal{F} \cup \{A\}$ has the finite intersection property, and so $\mathcal{F} = \mathcal{F} \cup \{A\}$ by (b).

(c) $\Rightarrow$ (a). It is clear that $\emptyset \notin \mathcal{F}$. Now let $P, Q \in \mathcal{F}$. If $P \cap Q \notin \mathcal{F}$ then by (c) there exists a finite subcollection $\mathcal{G} \subseteq \mathcal{F}$ such that $(P \cap Q) \cap \bigcap \mathcal{G} = \emptyset$. But then $\mathcal{F}$ does not have the finite intersection property. Finally, let $A \subseteq \omega$. It is not true that both A and $\omega \setminus A \in \mathcal{F}$ for otherwise $\mathcal{F}$ does not have the finite intersection property. If $A \notin \mathcal{F}$ and $\omega \setminus A \notin \mathcal{F}$ then by (c) there are finite subcollections $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ such that

$$\bigcap \mathcal{G} \cap A = \emptyset = \bigcap \mathcal{H} \cap (\omega \setminus A).$$

But then $\bigcap \mathcal{G} \cap \bigcap \mathcal{H} = \emptyset$, which again violates the finite intersection property for $\mathcal{F}$. ☐

An ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called *fixed* provided that $\bigcap \mathcal{F} \neq \emptyset$ and is called *free* if it is not fixed.

• *Exercise* 40. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a fixed ultrafilter. Prove that there exists a unique $n \in \omega$ such that $\mathcal{F}$ coincides with the collection $\mathcal{F}_n = \{A \subseteq \omega : n \in A\}$.

Lemma 9.2. *An ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is free if and only if every $F \in \mathcal{F}$ is infinite.*

• *Exercise* 41. Prove Lemma 9.2.

Now consider the space X again. From the above discussions it is clear that $\{n\}$ is contained in precisely one ultrafilter, namely $\mathcal{F}_n$. This means that $\{n\}^* = \{\mathcal{F}_n\}$ is clopen, hence $\mathcal{F}_n$ is an isolated point of X. We conclude that the function $n \mapsto \mathcal{F}_n$ is an embedding of ω in X. The collection $\{\mathcal{F}_n : n \in \omega\}$ is also dense in X. For if A^* for $A \subseteq \omega$ is a non-empty basic open set in X then A^* contains $\mathcal{F}_n$ for every $n \in A$.

It will be convenient to identify n and the ultrafilter $\mathcal{F}_n$ for every $n \in \omega$.

We conclude that X is a compactification of ω. But is it $\beta\omega$? The answer to this question is in the affirmative. This can be seen as follows. Since $\mathcal{P}(\omega)$ is complete, X is extremally disconnected (Theorem 8.4). But then $X \equiv \beta Y$ for every dense subset $Y \subseteq X$ (Corollary 8.3). Since X contains ω as a dense subspace, $X = \beta\omega$.

This gives us a clear picture of $\beta\omega$. Its elements can be identified with ultrafilters on ω. There are two types of ultrafilters: the fixed ones, which correspond to the points in ω, and the free ones, which correspond to points in $\beta\omega \setminus \omega$. The topology of $\beta\omega$ is defined as follows: for every $A \subseteq \omega$ the set $A^* = \{q \in \beta\omega : A \in q\}$ is (cl)open, and a neighborhood base for a point $p \in \beta\omega$ is the collection $\{A^* : A \in p\}$.

It will be convenient to change our notation slightly. For $A \subseteq \omega$ now put $\overline{A} = \{p \in \beta\omega : A \in p\}$.

- *Exercise* 42. Prove that for $A \subseteq \omega$, $\overline{A}$ is the closure of A in $\beta\omega$. In addition, prove that the following equality holds:

$$\text{If } A, B \subseteq \omega \text{ then } \overline{A} \cap \overline{B} = \overline{A \cap B}.$$

In addition, put $A^* = \overline{A} \cap (\beta\omega \setminus \omega)$ for $A \subseteq \omega$. So $\beta\omega \setminus \omega = \omega^*$.

- *Exercise* 43. Prove that the operator * satisfies the following equalities:
  1. $A^* \subseteq B^*$ iff $A \setminus B$ is finite.
  2. $A^* = B^*$ iff $A \triangle B$ is finite.
  3. $A^* \cap B^* = \emptyset$ iff $A \cap B$ is finite.

- *Exercise* 44. Let $A \subseteq \omega$ be infinite. Prove that $\overline{A}$ is homeomorphic to $\beta\omega$.

- *Exercise* 45. Prove that ω^* is the Stone space of the Ba $\mathcal{P}(\omega)/\operatorname{fin}$. (See Exercise 29.)

10. More notation

If $A, B \subseteq \omega$ then

$$\begin{array}{lcl}
A \subseteq_* B & \Leftrightarrow & |A \setminus B| < \omega \\
A =_* B & \Leftrightarrow & A \subseteq_* B \wedge B \subseteq_* A \\
 & \Leftrightarrow & |A \setminus B| < \omega \wedge |B \setminus A| < \omega \\
 & \Leftrightarrow & |A \triangle B| < \omega \\
A \subset_* B & \Leftrightarrow & A \subseteq_* B \wedge A \neq_* B \\
 & \Leftrightarrow & A \subseteq_* B \wedge B \not\subseteq_* A \\
 & \Leftrightarrow & |A \setminus B| < \omega \wedge |B \setminus A| = \omega \\
A \cap B =_* \emptyset & \Leftrightarrow & |A \cap B| < \omega.
\end{array}$$

If $A \cap B =_* \emptyset$ then A and B are said to be *almost disjoint.*

Consider the Ba $B = \mathcal{P}(\omega)/\operatorname{fin}$. If $x \in B$ then there is a subset $A \subseteq \omega$ such that $x = [A]$. If $B \subseteq \omega$ also has the property that $x = [B]$ then $A \triangle B \in \operatorname{fin} \Leftrightarrow A =_* B$. So A is in fact determined modulo a finite set.

A space X satisfies the *countable chain condition* (abbreviated: ccc) provided that every family consisting of pairwise disjoint nonempty open sets is at most countable.

If X is a space that a subspace $A \subseteq X$ is called C^**-embedded* in X provided that every continuous function $f\colon A \to \mathbb{I}$ can be extended to a continuous function $\bar{f}\colon X \to \mathbb{I}$.

- *Exercise* 46. Prove that if $A \subseteq X$ is C^*-embedded and if K is compact then every continuous function $f\colon A \to X$ can be extended to a continuous function $\bar{f}\colon X \to K$.

Let X be a space. We call a set $A \subseteq X$ a G_δ*-subset* of X provided there is a countable family $\mathcal{U}$ consisting of open subsets of X such that $A = \bigcap \mathcal{U}$. The complement of a G_δ-subset of X is called an F_σ*-subset* of X. Observe that a subset $A \subseteq X$ is F_σ if and only if there exists a countable family $\mathcal{F}$ consisting of closed subsets of X such that $A = \bigcup \mathcal{F}$.

An *atom* in a Ba B is an element $x \in B \setminus \{0\}$ such that if $y \in B \setminus \{0\}$ is such that $y \leq x$ then $y = x$. We say that B is *atomless* provided that it has no atoms.

- *Exercise* 47. Let B be a Ba. Prove that the following statements are equivalent:

(1) B has no atoms.
(2) $\mathrm{st}(B)$ has no isolated points.

11. The topology of βX

In this section we will prove some useful facts on the topology of βX.

If X is a space then X^* denotes $\beta X \setminus X$ and if $U \subseteq X$ is open, then

$$\mathrm{Ex}(U) = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U).$$

Observe that $\mathrm{Ex}(U)$ is open in βX and that $\mathrm{Ex}(U) \cap X = U$.

• *Exercise* 48. Prove that if $U \subseteq X$ is open then $\mathrm{Ex}(U)$ is the largest open subset $V \subseteq \beta X$ such that $V \cap X = U$.

Lemma 11.1. *The collection*

$$\{\mathrm{Ex}(U) : U \subseteq X \text{ open}\}$$

is a base for the topology of βX.

Proof. Let $p \in \beta X$ and let $V \subseteq \beta X$ be an open neighborhood of p. There is an open neighborhood W of p such that $p \in W \subseteq \overline{W} \subseteq V$. Now put $U = W \cap X$. Since W is an open subset of βX such that $W \cap X = U$ it follows that $W \subseteq \mathrm{Ex}(U)$ (Exercise 48) and hence that $p \in \mathrm{Ex}(U)$. Since U is dense in $\mathrm{Ex}(U)$ as well as in W it also follows that $\overline{\mathrm{Ex}(U)} = \overline{W}$. So we conclude that

$$p \in \mathrm{Ex}(U) \subseteq \overline{\mathrm{Ex}(U)} = \overline{W} \subseteq V,$$

which is as required. □

If $U \subseteq X$ is open then we put

$$U' = \mathrm{Ex}(U) \cap X^*.$$

So by Lemma 11.1 it follows that the collection

$$\{U' : U \subseteq X \text{ open}\}$$

is a base for X^*.

Lemma 11.2. *Let X be a space, let $U \subseteq X$ be open and let $K \subseteq X$ be compact. Then $U' = (U \setminus K)'$.*

Proof. Put $V = U \setminus K$. Then

$$X \setminus V = X \setminus (U \setminus K) = (X \setminus U) \cup K,$$

from which it follows that

$$\mathrm{cl}_{\beta X}(X \setminus V) = \mathrm{cl}_{\beta X}(X \setminus U) \cup K.$$

But then

$$\begin{aligned} \mathrm{Ex}(V) &= \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus V) \\ &= \beta X \setminus (\mathrm{cl}_{\beta X}(X \setminus U) \cup K) \\ &= (\beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U)) \cap (\beta X \setminus K) \\ &= \mathrm{Ex}(U) \cap (\beta X \setminus K) \end{aligned}$$

from which it follows that

$$V' = \mathrm{Ex}(V) \cap X^* = \mathrm{Ex}(U) \cap ((\beta X \setminus K) \cap X^*) = U'.$$

□

Corollary 11.3. *Let X be a space, let $U, V \subseteq X$ be open such that $U \setminus V$ has compact closure in X. Then $U' \subseteq V'$.*

Proof. Let K be the closure in X of $U \setminus V$. Then by Lemma 11.2 we have $U' = (U \setminus K)'$. But $U \setminus K \subseteq V$ from which it follows that $(U \setminus K)' \subseteq V'$. □

Lemma 11.4. *Let X be any space and let $U, V \subseteq X$ be open.*

(a) $\mathrm{Ex}(\emptyset) = \emptyset$ *and* $\mathrm{Ex}(U \cap V) = \mathrm{Ex}(U) \cap \mathrm{Ex}(V)$.
(b) $\mathrm{cl}_{\beta X} \mathrm{Ex}(U) = \mathrm{cl}_{\beta X}(U)$.
(c) *If X is normal then* $\mathrm{Ex}(U) \cup \mathrm{Ex}(V) = \mathrm{Ex}(U \cup V)$.

Proof. (a) follows from a straightforward computation.

• *Exercise* 49. Prove (a).

For (b), observe that X is dense in βX, and $\mathrm{Ex}(U)$ is open in βX, hence $\mathrm{cl}_{\beta X} \mathrm{Ex}(U) = \mathrm{cl}_{\beta X}(X \cap \mathrm{Ex}(U)) = \mathrm{cl}_{\beta X}(U)$.

• *Exercise* 50. If X is normal then $\mathrm{cl}_{\beta X}(F) \cap \mathrm{cl}_{\beta X}(G) = \mathrm{cl}_{\beta X}(F \cap G)$ for any two closed subsets F and G of X. Prove this.

Observe that from this (c) immediately follows. □

Corollary 11.5. *Let X be a space and let $U \subseteq X$ be clopen. Then $\mathrm{Ex}(U)$ is equal to $\mathrm{cl}_{\beta X}(U)$, and hence is clopen in βX.*

Proof. If U is clopen then $\mathrm{cl}_{\beta X}(U)$ and $\mathrm{cl}_{\beta X}(X \setminus U)$ are disjoint by Corollary 4.6. In adition, $\mathrm{cl}_{\beta X}(U) \cup \mathrm{cl}_{\beta X}(X \setminus U) = \beta X$. This implies that both $\mathrm{cl}_{\beta X}(U)$ and $\mathrm{cl}_{\beta X}(X \setminus U)$ are clopen. So $\mathrm{cl}_{\beta X}(U) \subseteq \mathrm{Ex}(U)$ and $\mathrm{cl}_{\beta X}(X \setminus U) \subseteq \mathrm{Ex}(X \setminus U)$. Since $\mathrm{Ex}(U) \cap \mathrm{Ex}(X \setminus U) = \emptyset$ (Lemma 11.4(a)), we now easily get what we want. □

A space X is called *strongly zero-dimensional* provided that βX is zero-dimensional.

Lemma 11.6. *A zero-dimensional Lindelöf space has the property that for all disjoint closed subsets $A, B \subseteq X$ there exists a clopen set $C \subseteq X$ such that $A \subseteq C \subseteq X \setminus B$.*

Proof. Let A and B be disjoint closed subsets of X. Every $x \in X$ has a clopen neighborhood C_x such that $C_x \cap A = \emptyset$ or $C_x \cap B = \emptyset$. The cover $\{C_x : x \in X\}$ has a countable subcover, say $\mathcal{C} = \{C_{x_i} : i < \omega\}$. By replacing every C_{x_i} by

$$C_{x_i} \setminus \bigcup_{j<i} C_{x_j}$$

we may even assume that the elements of the cover $\mathcal{C}$ are pairwise disjoint.

Put $U_A = \bigcup\{C \in \mathcal{C} : C \cap A \neq \emptyset\}$. Then U_A is an open neighborhood of A which is contained in $X \setminus B$. Moreover, U_A is closed, since its complement is equal to $\bigcup\{C \in \mathcal{C} : C \cap A = \emptyset\}$. ☐

Lemma 11.7. *X is strongly zero-dimensional if and only if for all disjoint zero-sets A and B in X there exists a clopen set $C \subseteq X$ such that $A \subseteq C \subseteq X \setminus B$.*

Proof. Let βX be zero-dimensional, and let A and B be disjoint zero-sets in X. Then $\overline{A} \cap \overline{B} = \emptyset$ (Corollary 4.6). So we are done by Lemma 11.6.

Conversely, assume that disjoint zero-sets in X can be separated by clopen sets. Let $p \in \beta X$, and let U be an open neigborhood of p in βX. There is a continuous function $f\colon \beta X \to \mathbb{I}$ such that $f(p) = 0$ and $f \restriction \beta X \setminus U \equiv 1$. Put $E = f^{-1}[[0, 1/4]]$ and $F = f^{-1}[[3/4, 1]]$, respectively. Let C be a clopen subset of X such that $E \cap X \subseteq C \subseteq X \setminus F$. Then $\mathrm{cl}_{\beta X}(C) \subseteq f^{-1}[[0, 3/4]]$ and hence it misses $\beta X \setminus U$. Since clearly $p \in \mathrm{cl}_{\beta X}(C)$, this shows that βX is zero-dimensional at p. ☐

Corollary 11.8. *Every zero-dimensional Lindelöf space is strongly zero-dimensional.*

Lemma 11.9. *Let X be a zero-dimensional space of weight κ. Then there is a base $\mathcal{B}$ for X consisting entirely of clopen sets such that $|\mathcal{B}| = \kappa$.*

Proof. Let $\mathcal{F}$ be a base for X such that $|\mathcal{F}| = \kappa$. Consider the collection of all pairs (F_0, F_1) of elements of $\mathcal{F}$ for which there exists a clopen set E such that $F_0 \subseteq E \subseteq F_1$. For each such pair pick a fixed clopen set C for which $F_0 \subseteq C \subseteq F_1$. In this way we identified a collection $\mathcal{C}$ of at most $\kappa \times \kappa = \kappa$ many clopen sets. We claim that those sets form a base for X. To see this, let $U \subseteq X$ be open, and let $x \in U$. There is an element $F_1 \in \mathcal{F}$ such that $x \in F_1 \subseteq U$. There is also a clopen set E in X such that $x \in C \subseteq F_1 \subseteq U$. There also is an element $F_0 \in \mathcal{F}$ such that $x \in F_0 \subseteq E$. So (F_0, F_1) is one of the pairs that we condidered in our construction. So for some $C \in \mathcal{C}$ we have $F_0 \subseteq C \subseteq F_1$. But then $x \in C \subseteq U$. We conclude that $\mathcal{C}$ is a base. (We proved that $\mathcal{C}$ has cardinality $\leq \kappa$ but why does it hace cardinality $\geq \kappa$?.) ☐

Corollary 11.10. *Let X be a zero-dimensional Lindelöf space of weight κ. Then X has at most κ^ω many clopen subsets. So βX is zero-dimensional and has weight at most κ^ω.*

Proof. Let $\mathcal{B}$ be a base for X consisting of clopen sets such that $|\mathcal{B}| = \kappa$ (Lemma 11.9). Since X is Lindelöf, every clopen subset $C \subseteq X$ is the union of a countable subfamily of $\mathcal{B}$. So there are at most κ^ω many clopen subsets.

That βX is zero-dimensional follows from Corollary 11.8.

The function $C \mapsto C \cap X$ from $CO(\beta X)$ to $CO(X)$ is one to one. Hence $|CO(\beta X)| \leq \kappa^\omega$. ☐

Corollary 11.11. *Let X be a zero-dimensional Lindelöf space with weight at most $\mathfrak{c}$. Then βX is zero-dimensional and has weight at most $\mathfrak{c}$.*

12. The spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under CH

In this section we will see how $\beta\omega$ and ω^* behave under CH.

12.1. **A characterization of $\mathcal{P}(\omega)/\operatorname{fin}$.** Let B be a Ba and let $F, G \subseteq B$. We say that $F < G$ provided that for all $F' \in [F]^{<\omega}, G' \in [G]^{<\omega}$ we have $\bigvee F' < \bigwedge G'$.

Definition 12.1. Let B be a Ba. We say that B satisfies *condition H_ω* provided that for all $F, G \in [B]^{\leq\omega}$ such that $F < G$ there is an element $x \in B$ such that $F < \{x\} < G$.

Observe that condition H_ω for B implies that B is atomless.

Lemma 12.2. *Let B satisfy condition H_ω.*

1. *If $F \in [B]^{\leq\omega}$ has the finite intersection property then there exists an element $x \in B$ such that $0 < \{x\} < F$.*
2. *If F and G are disjoint countable subsets of B such that for all $\tilde{F} \in [F]^{<\omega}$ and $\tilde{G} \in [G]^{<\omega}$,*
$$\bigvee \tilde{F} \wedge \bigvee \tilde{G} = 0,$$
then there exists an element $x \in B$ such that $f \leq x$ for every $f \in F$ and $x \leq g'$ for every $g \in G'$.

Proof. For (1), simply observe that $0 < F$.

For (2), put $G' = \{g' : g \in G\}$. Then if $F < G'$ we get what we want from condition H_ω. If $F \not< G'$ then for some $\tilde{F} \in [F]^{<\omega}$ and $\tilde{G} \in [G]^{<\omega}$ we have $\bigvee \tilde{F} = (\bigvee \tilde{G})'$. Then $x = \bigvee \tilde{F}$ is clearly as required. ☐

Corollary 12.3. *Let B be a Ba satisfying condition H_ω. Then $|B| \geq \mathfrak{c}$. In fact, if $a \in B \setminus \{0\}$ then there exists an pairwise disjoint subset $E \subseteq B$ such that $|E| = \mathfrak{c}$ while moreover for every $e \in E$ we have $e \leq a$.*

Proof. Pick an arbitrary element $a \in B \setminus \{0\}$. Since B is atomless, it is easy to construct by induction on n for every $f \in 2^n$ an element $a(f) \in B \setminus \{0\}$ such that

1. If $f, g \in 2^n$ are distinct then $a(f) \wedge a(g) = 0$.
2. If $f \in 2^n$ then $a(f) \leq a$.
3. If $f \in 2^n$ and $g \in 2^{n+1}$ is such that $g \restriction n = f$ then $a(g) \leq a(f)$.

Observe that if $f \in 2^\omega$ then the collection
$$\{a(f \restriction n) : n < \omega\}$$
has the finite intersection property. There consequently exists by Lemma 12.2(1) an element $a(f) \in B \setminus \{0\}$ such that $a(f) \leq a(f \restriction n)$ for every $n < \omega$. The collection $\{a(f) : f \in 2^\omega\}$ is clearly as required. ☐

• *Exercise* 51. Let X be a compact zero-dimensional space.

(1) Prove that for every open F_σ-subset $F \subseteq X$ there is a countable subfamily $\mathcal{C} \subseteq CO(X)$ such that $F = \bigcup \mathcal{C}$.
(2) Prove that $CO(X)$ satisfies condition H_ω if and only if the following hold:
   (a) Every nonempty G_δ in X has infinite interior.
   (b) Every two disjoint open F_σ's in X have disjoint closures.

Can we replace (a) by: Every nonempty G_δ in X has nonempty interior?

Let A be a subset of a Ba B. We say that A is *pairwise disjoint* provided that for all distinct $a_0, a_1 \in A$ we have $a_0 \wedge a_1 = 0$.

Let $A = \{a_i : i \in I\}$ be an indexed subset of a Ba B such that $a_i \neq 0$ for every $i \in I$. We say that A admits a *disjoint refinement* provided that for every $i \in I$ there exists an element $b_i \in B \setminus \{0\}$ such that $b_i \leq a_i$ while moreover for all distinct $i, j \in I$ we have $b_i \wedge b_j = 0$.

Observe that in this definition we do not require the a_i's to be pairwise distinct.

Lemma 12.4. *Let B satisfy condition H_ω. Then every countable subset $A = \{a_i : i \in I\} \subseteq B \setminus \{0\}$ with $|I| \leq \omega$ admits a disjoint refinement.*

Proof. By induction on $n = |I|$ we will first show that if $|I|$ is finite then A admits a disjoint refinement.

If $|I| = 1$ then there is nothing toprove. So assume the statement to be true if $|I| = n \geq 1$, and assume that we are delaing with the situation that $|I| = n+1$, say $I = \{1, 2, \ldots, n+1\}$. Let $A = \{a_1, \ldots, a_{n+1}\}$. First assume that $n = 2$. If $a_1 \neq a_2$ then we may assume without loss of generality that $a_1 \wedge a_2' \neq 0$. So then $\{a_1 \wedge a_2', a_2\}$ is a disjoint refinement of $\{a_1, a_2\}$. If $a_1 = a_2$ then we use the fact that B is atomless to find an element $a \in B \setminus \{0\}$ such that $a < a_1 = a_2$. Then $b_1 = a$ and $b_2 = a_2 \wedge a'$ form a disjoint refinement of a_1 and a_2. Now assume that $n > 2$. By our inductive assumption we may assume that $\{a_1, \ldots, a_n\}$ is pairwise disjoint. Since we already dealt with the case that $n = 2$, we can find a disjoint refinement of $\{a_1, a_{n+1}\}$. So we may assume without loss of generality that $a_1 \wedge a_{n+1} = 0$. Then we do the same thing with $\{a_2, a_{n+1}\}$, etc.

We now consider the general case of our assertion. Without loss of generality, assume that $I = \omega$. For each $m < \omega$ we will construct elements $a_n^m \in B \setminus \{0\}$ having the following properties:

(a) $a_n^0 = a_n$ for all n.
(b) If $k \leq \ell$ then $a_n^\ell \leq a_n^k$ for all n.
(c) $a_i^n \wedge a_j^n = 0$ for all n and $i, j \leq n$.

Suppose that for some $m < \omega$ the elements a_n^m are defined for all n. Then by the above there is a disjoint refinement $\{b_0, \ldots, b_{m+1}\}$ of the elements $\{a_0^m, \ldots, a_{m+1}^m\}$. So by putting

$$a_n^{m+1} = \begin{cases} b_n & (n \leq m+1), \\ a_n^m & (n > m+1), \end{cases}$$

we satisfy our inductive demands.

Now since $B_n = \{a_n^m : m < \omega\}$ has the finite intersection property for every n, by Lemma 12.2(1) there exists for each n an element $b_n \in B \setminus \{0\}$ such that $0 < \{b_n\} < B_n$. Then $\{b_n : n < \omega\}$ is a disjoint refinement of A by (c) above. □

• *Exercise* 52. Observe that in the first part of this proof we only used that B is atomless. So a finite subset of an atomless Ba admits a disjoint refinement. Give an example of a Ba containing an infinite subset without disjoint refinement.

Definition 12.5. Let B be a Ba. We say that B satisfies *condition* R_ω provided that for all $F, G, H \in [B]^{\leq\omega}$ such that

(1) $F < G$,
(2) $\forall \tilde{F} \in [F]^{<\omega}\, \forall \tilde{G} \in [G]^{<\omega}\, \forall h \in H : h \not\leq \bigvee \tilde{F}$ and $\bigwedge \tilde{G} \not\leq h$,

then there is an element $x \in B$ such that

3. $F < \{x\} < G$, and
4. $\forall h \in H : h \not\leq x$ and $x \not\leq h$.

The reason why ω^* is relatively easy to deal with under CH is, as we will see later, because of the following lemma.

Lemma 12.6. *If a Boolean Algebra satisfies condition H_ω then it satisfies condition R_ω.*

Proof. Let B be a Ba satisfying condition H_ω.

Let $F, G, H \subseteq B$ be as in 12.5. By applying condition H_ω, there exists an element $y_0 \in B$ such that $F < \{y_0\} < G$. By applying it once more, there exists an element $y_1 \in B$ such that $\{y_0\} < \{y_1\} < G$. Put $z = y_1 \wedge y_0'$. Observe that $z \wedge f = 0$ for every $f \in F$, and $z < g$ for every $g \in G$.

Enumerate F as $\{f_n : n < \omega\}$, G as $\{g_n : n < \omega\}$ and H as $\{h_n : n < \omega\}$.

For all $h \in H$ and $\tilde{F} \in [F]^{<\omega}$ we have $(\bigvee \tilde{F})' \wedge h \neq 0$ and there consequently exists by applying condition H_ω for all $n < \omega$ an element $d_n \in B \setminus \{0\}$ such that

(1) $d_n < h_n$ and $\forall f \in F, f \wedge d_n = 0$.

Similarly we can find $e_n \in B \setminus \{0\}$ such that

(2) $\{e_n\} < G$ and $e_n \wedge h_n = 0$

for every $n < \omega$. By applying Lemma 12.4, we can find a disjoint refinement of the set $\{z\} \cup \{d_n : n < \omega\} \cup \{e_n : n < \omega\}$. This shows that we may additionally assume that the d_n's and e_n's are pairwise disjoint, and that $d_n \wedge z = 0 = z \wedge e_n$ for every n.

Now define for all $n < \omega$,

$$\tilde{f}_n = f_n \vee e_n \qquad \text{and} \qquad \tilde{g}_n = g_n \wedge d_n'.$$

Fix $n, m < \omega$. Then $\bigvee_{i\leq n} f_i < \bigwedge_{i\leq m} g_i$, $\bigvee_{i\leq n} f_i \leq \bigwedge_{i\leq m} d_i'$ (by (1)), $\bigvee_{i\leq n} e_i \leq \bigwedge_{i\leq m} g_i$ (by (2)), and $\bigvee_{i\leq n} e_i \leq \bigwedge_{i\leq m} d_i'$ (for this last inequality, observe that the e_i's and the d_j's are pairwise disjoint). As a consequence,

$$\bigvee_{i\leq n} \tilde{f}_i = \bigvee_{i\leq n} f_i \vee \bigvee_{i\leq n} e_i \leq \bigwedge_{i\leq m} g_i \wedge \bigwedge_{i\leq m} d_i' = \bigwedge_{i\leq m} \tilde{g}_i.$$

So by Lemma 12.2(2) there exists an element $p \in B$ such that for all $m, m < \omega$,

$$\bigvee_{i\le n} \tilde{f}_i \le \{p\} \le \bigwedge_{i\le m} \tilde{g}_i. \tag{12.1}$$

By replacing p by $p \wedge z'$ we may assume without loss of generality that $p \wedge z = 0$. That we may indeed assume this, observe that for every n,

$$\big(\bigvee_{i\le n} \tilde{f}_i\big) \wedge z = \big(\bigvee_{i\le n} f_i \wedge z\big) \wedge \big(\bigvee_{i\le n} e_i \wedge z\big) = 0,$$

so that $\bigvee_{i\le n} \tilde{f}_i \le z'$ and so $\bigvee_{i\le n} \tilde{f}_i \le p \wedge z'$. Since $p \wedge z' \le p$, the second inequality in (12.1) gives no problems.

Since B is atomless, there are elements $a, b \in B \setminus \{0\}$ such that $a, b \le z$ and $a \wedge b = 0$. Now put $x = p \vee a$. We claim that x is as required.

Let $n < \omega$. Then $\bigvee_{i\le n} f_i \le \bigvee_{i\le n} \tilde{f}_i \le p < p \vee a = x$. In addition, if $m < \omega$ then $x \le \bigwedge_{i\le m} \tilde{g}_i \le \bigwedge_{i\le m} g_i$. Moreover, $0 < b \le \bigwedge_{i\le m} g_i$ and $b \wedge x = 0$. This proves that $x < \bigwedge_{i\le m} g_i$.

Now if $h_n \in H$ then by (2) and the fact that $e_n \le x$ it follows that $x \not\le h_n$. It follows similarly by (1) and the fact that $d_n \wedge x = 0$ that $h_n \not\le x$. So we conclude that x is as required. ☐

By Corollary 12.3 it follows that every Ba satisfying condition H_ω has cardinality at least $\mathfrak{c}$. The question naturally arises whether there exist Ba's satisfying condition H_ω.

Proposition 12.7. $\mathcal{P}(\omega)/\operatorname{fin}$ *satisfies condition* H_ω.

Proof. Let $F = \{x_n : n < \omega\}$ and $G = \{y_n : n < \omega\}$ be countable subsets of $B = \mathcal{P}(\omega)/\operatorname{fin}$ such that $F < G$. For every $n < \omega$ pick subsets $A_n, B_n \subseteq \omega$ such that $x_n = [A_n]$ and $y_n = [B_n]$. Observe that by assumption for every $n < \omega$ we have

$$\bigcup_{i\le n} A_i \subset_* \bigcap_{i\le n} B_i.$$

This implies that for every $n < \omega$ we have

$$|\bigcap_{i\le n} B_i \setminus \bigcup_{i\le n} A_i| = \omega.$$

So by induction on $n < \omega$ it is possible to pick elements $e_n < \omega$ such that

$$e_n \in \bigcap_{i\le n} B_i \setminus \Big(\bigcup_{i\le n} A_i \cup \{e_0, \dots, e_{n-1}\}\Big).$$

Put $E = \{e_n : n < \omega\}$.

Claim 12.7.1. $E \subseteq_* \bigcap_{i\le n} B_i$ for every $n < \omega$.

This is clear since $e_m \in \bigcap_{i\le n} B_i$ for every $m \ge n$.

Claim 12.7.2. $E \cap_* \bigcup_{i\le n} A_i = \emptyset$ for all $n < \omega$.

This is also clear since $e_m \notin \bigcup_{i\le n} A_i$ for all $m \ge n$.

Since E is an infinite set, we can spit it into two disjoint infinite sets E_0 and E_1.

Put

$$X = \bigcup_{n<\omega} \Big(\bigcup_{i\le n} A_i \cap \bigcap_{i\le n} B_i \Big).$$

Claim 12.7.3. $\bigcup_{i\le n} A_i \subseteq_* X$ for all $n < \omega$.

This is clear since

$$\bigcup_{i\le n} A_i \setminus X \subseteq \bigcup_{i\le n} A_i \setminus \bigcap_{i\le n} B_i$$

which by assumption is a finite set.

Claim 12.7.4. $X \subseteq_* \bigcap_{i\le n} B_i$ for every $n < \omega$.

This is also clear since

$$X \setminus \bigcap_{i\le n} B_i \subseteq \bigcup_{i<n-1} \Big(\bigcup_{j\le i} A_j \setminus \bigcap_{i\le n} B_i \Big).$$

which is also a finite set.

Now put $X' = (X \setminus E_0) \cup E_1$. Then $[X']$ is clearly as required. $\square$

- *Exercise* 53. Prove that the set X' is indeed as required.

We finish here by proving that under CH there exists up to isomorphism only one Ba satisfying property H_ω of the smallest possible cardinality ($= \mathfrak{c}$). This is the famous Parovičenko Theorem from 1963.

If B is a Ba and $A \subseteq B$ then $\langle A \rangle$ denotes the subalgebra of B generated by A.

Theorem 12.8 (CH). *All Ba's of size $\mathfrak{c}$ which satisfy condition H_ω are isomorphic.*

Proof. Let B and E be Ba's of size $\mathfrak{c}$ which satisfy condition H_ω. By CH, list B as $\{b_\alpha : \alpha < \omega_1\}$ and E as $\{e_\alpha : \alpha < \omega_1\}$, respectively. Without loss of generality we may assume that $b_0 = 0$ and $e_0 = 0$.

By transfinite induction we will construct for $\alpha < \omega_1$ countable subalgebras $B_\alpha \subseteq B$ and $E_\alpha \subseteq E$ and an isomorphism $\sigma_\alpha \colon B_\alpha \to E_\alpha$ such that

(1) $b_\alpha \in B_\alpha$ and $e_\alpha \in E_\alpha$,
(2) if $\beta < \alpha$ then $B_\beta \subseteq B_\alpha$, $E_\beta \subseteq E_\alpha$ and $\sigma_\alpha \restriction B_\beta = \sigma_\beta$.

Let $B_0 = \{0,1\}$ and $E_0 = \{0,1\}$ and let $\sigma_0 \colon B_0 \to E_0$ be defined in the obvious way. Suppose that B_β, E_β and σ_β are defined for all $\beta < \alpha < \omega_1$ satisfying (1) and (2). If $b_\alpha \in \bigcup_{\beta<\alpha} B_\beta$ and $e_\alpha \in \bigcup_{\beta<\alpha} E_\beta$ then define $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$, $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ and $\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta$. It is easily seen that these choices are as required. Suppose next that e.g. $b_\alpha \notin F = \bigcup_{\beta<\alpha} B_\beta$. Let $\sigma = \bigcup_{\beta<\alpha} \sigma_\beta$ and put

$$F_0 = \{f \in F : f < b_\alpha\}, \quad F_1 = \{f \in F : b_\alpha < f\}, \quad \text{and} \quad F_2 = F \setminus (F_0 \cup F_1).$$

By Lemma 12.6 there exists an element $e \in E$ such that $\sigma[F_0] < \{e\} < \sigma[F_1]$, and for all $\tilde{e} \in \sigma[F_2]$, $\tilde{e} \not\le e$ and $e \not\le \tilde{e}$. If we put $\sigma(b_\alpha) = e$ then σ can be extended to an isomorphism $\bar{\sigma} \colon \langle F \cup \{b_\alpha\} \rangle \to \langle \sigma[F] \cup \{e\} \rangle$.

- *Exercise* 54. Fill in all the missing details.

If $e_\alpha \notin \langle \sigma[F] \cup \{e\} \rangle$ then we do the same thing as above with $\bar{\sigma}$ replaced by $\bar{\sigma}^{-1}$. It is now clear how to define B_α, E_α and σ_α, respectively. This completes the transfinite induction.

So $\bigcup_{\alpha<\omega_1} \sigma_\alpha$ is an isomorphism $B \to E$. ☐

Corollary 12.9 (CH)**.** $\mathcal{P}(\omega)/\operatorname{fin}$ *is up to isomorphism the unique Ba of size* $\mathfrak{c}$ *which in addition satisfies condition* H_ω.

Proof. Apply Proposition 12.7 and Theorem 12.8. ☐

12.2. **Topological applications.** In this section we present interesting topological applications of the algebraic results obtained in the previous section.

Definition 12.10. A space X is called an *F-space* if every cozero-set in X is C^*-embedded in X.

Lemma 12.11. *Let X be a space.*

(a) *If X is an F-space then βX is an F-space*[1].
(b) *A normal space is an F-space iff any two disjoint open F_σ-subsets of X have disjoint closures in X.*
(c) *Each extremally disconnected space is an F-space.*
(d) *Any closed subspace of a normal F-space is again an F-space.*
(e) *If an F-space X satisfies the countable chain condition then it is extremally disconnected.*

Proof. For (a), let $U \subseteq \beta X$ be a cozero-set, and let $f\colon U \to \mathbb{I}$ be continuous. Then $V = U \cap X$ is a cozero-set in X and $g = f \restriction f\colon V \to \mathbb{I}$ is continuous. Since by assumption V is C^*-embedded in X, the function g can be extended to a continuous function $\bar{g}\colon X \to \mathbb{I}$. This function can in turn be extended to a continuous function $\beta\bar{g}\colon \beta X \to \mathbb{I}$. It now suffices to observe that V is dense in U since U is open. Since the functions $f\colon U \to \mathbb{I}$ and $\beta\bar{g} \restriction U$ agree on V, they consequnetly have to agree on all of U. In other words, $\beta\bar{g}$ extends f.

For (b), let X be a normal F-space, and let U and V be disjoint open F_σ-subsets of X. Write $U = \bigcup_{n<\omega} U_n$ with each U_n closed. Similarly write $V = \bigcup_{n<\omega} V_n$ with each V_n closed. By induction on $n < \omega$, we will construct increasing sequences cozero-sets $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ and $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n \subseteq \cdots$ in X such that for all n,

(1) $U_n \subseteq F_n$, $V_n \subseteq G_n$,
(2) $\overline{F}_n \cap (\overline{V} \cup \bigcup_{i\leq n} \overline{G}_i) = \emptyset$, and $\overline{G}_n \cap (\overline{U} \cup \bigcup_{i\leq n} \overline{F}_i) = \emptyset$.

Since $U_0 \cap \overline{V} = \emptyset$, the normality of X gives us a cozero-set F_0 such that $U_0 \subseteq F_0$ while moreover $\overline{F}_0 \cap \overline{V} = \emptyset$. Observe that $V_0 \cap (\overline{U} \cup \overline{F}_0) = \emptyset$. So by applying normality again, we get a cozero-set G_0 such that $V_0 \subseteq G_0 \subseteq \overline{G}_0 \subseteq (X \setminus (\overline{U} \cup \overline{F}_0))$. Observe that

$$(\overline{F}_0 \cup U_1) \cap (\overline{G}_0 \cup \overline{V}) = \emptyset.$$

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[1] The converse is also true, but the proof is much more complicated.

So by normality we can proceed as above, creating F_1 and then G_1, etc.

Now put $F = \bigcup_{n<\omega} F_n$ and $G = \bigcup_{n<\omega} G_n$, respectively. Then F and G are clearly disjoint cozero-sets (Exercise 9) such that $U \subseteq F$ and $V \subseteq G$. Now define $f\colon F \cup G \to \mathbb{I}$ by $f \restriction F \equiv 0$ and $f \restriction G \equiv 1$. Then f is continuous since $F \cap G = \emptyset$ and F and G are open. Since X is an F-space and $F \cup G$ is a cozero-set this function can be extended to a continuous function $\bar{f}\colon X \to \mathbb{I}$. But then clearly F and G have disjoint closures in X, and in turn U and V have disjoint closures in X.

Now assume that X is a normal space, having the property that disjoint open F_σ-subsets of X have disjoint closures. Let U be a cozero-set in X and let $f\colon U \to \mathbb{I}$ be continuous. If A and B are disjoint closed subsets of $\mathbb{I}$ then there exist disjoint open subsets $E, F \subseteq \mathbb{I}$ such that $A \subseteq E$ and $B \subseteq F$. Observe that both E and F are F_σ-subsets of $\mathbb{I}$. So $f^{-1}[E]$ and $f^{-1}[F]$ are disjoint open F_σ-subsets of U. But since U is an open F_σ-subset of X, it follows that both $f^{-1}[E]$ and $f^{-1}[F]$ are disjoint open F_σ-subsets of X. So they have disjoint closures by assumption. As a consequence, $f^{-1}[A]$ and $f^{-1}[B]$ have disjoint closures in X. Lemma 4.5 therefore implies that f can be extended to a continuous function $\bar{f}\colon \overline{U} \to \mathbb{I}$. Since X is normal, this function can in turn be extended to a continuous function $\tilde{f}\colon X \to \mathbb{I}$.

For (c), let X be extremally disconnected. Let U be a cozero-zet of X and let $f\colon U \to \mathbb{I}$ be continuous. If A and B are disjoint closed subsets of $\mathbb{I}$ then, as in the proof of (b), $f^{-1}[A]$ and $f^{-1}[B]$ can be separated by disjoint open neighborhoods. But disjoint open subsets of X have disjoint closures. So by again appealing to Lemma 4.5 it follows that f can be extended to a continuous function $\bar{f}\colon \overline{U} \to \mathbb{I}$. But $\overline{U}$ is clopen, and the union of $\bar{f}$ and the function on $X \setminus \overline{U}$ with constant value 0, is continuous and extends f.

For (d), first complete the following exercise.

- *Exercise* 55. Let A be a closed subspace of a normal space X. If U and V are two disjoint open (in A) F_σ-subsets of A then there exist disjoint open F_σ-subsets U' and V' of X such that $U' \cap A = U$ and $V' \cap A = V$.

So (d) immediately follows from this and (b).

For (e), let X be an F-space satisfying the countable chain condition. From Lemma 8.2 it follows that it suffices to prove that disjoint open subsets in X have disjoint closures. So let U and V be disjoint nonempty open subsets of X. Let $\mathcal{M}$ be a family of nonempty cozero-sets in X all members of which are contained in U and which is maximal with respect to the property of being pairwise disjoint.

- *Exercise* 56. Prove that such a family exists. (Hint: apply Zorn's Lemma.)

Since X is ccc, $\mathcal{M}$ is countable. Since the cozero-subsets of X form a base, it follows that $M = \bigcup \mathcal{M}$ is a dense subset of U. Since a countable union of cozero-sets is again a cozero-set (Exercise 9), it follows that M is a cozero-set.

Similarly, let $N \subseteq V$ be a dense cozero-set. Then $M \cup N$ is a cozero-set (Exercise 9), and the function $f\colon M \cup N \to \mathbb{I}$ defined by

$$f(x) = \begin{cases} 0 & (x \in M), \\ 1 & (x \in N), \end{cases}$$

is continuous since M and N are disjoint open sets. By assumption we may extend f to a continuous function $\bar{f}\colon X \to \mathbb{I}$. Then $\bar{f}^{-1}(0)$ and $\bar{f}^{-1}(1)$ are disjoint closed subsets of X containing M and N, respectively. Since M is dense in U and N is dense in V this shows that U and V have disjoint closures, as desired. ☐

The following result gives a topological translation of condition H_ω.

Lemma 12.12. *Let X be a compact zero-dimensional space. The following statements are equivalent:*

1. *$CO(X)$ satisfies condition H_ω.*
2. *X is an F-space and each nonempty G_δ in X has infinite interior.*

Proof. This follows from Exercise 51 and Lemma 12.11(b) (observe that X is normal, being compact and Hausdorff). ☐

Corollary 12.13 (CH). *Let X be a space. The following statements are equivalent:*

1. *$X \approx \omega^*$.*
2. *X is a compact zero-dimensional F-space of weight $\mathfrak{c}$ in which each nonempty G_δ has infinite interior.*

Proof. This follows immediately from Theorem 12.8, Corollary 11.10 and Lemma 12.12. ☐

A compact zero-dimensional F-space of weight $\mathfrak{c}$ in which every nonempty G_δ has infinite interior, will be called a *Parovičenko space* from now on. Corollary 12.13 says that under CH ω^* is, up to homeomorphism, the only Parovičenko space.

Corollary 12.13 is a very useful result since it turns out that the class of Parovičenko spaces is quite large. The following result, which is of independent interest, is tht key to finding more Parovičenko spaces.

Theorem 12.14. *Let X be a locally compact, σ-compact and noncompact space. Then X^* is an F-space of which each nonempty G_δ has infinite interior.*

Proof. Let $F \subseteq X^*$ be any F_σ and let $f\colon F \to \mathbb{I}$ be continuous. Since $Y = X \cup F$ is σ-compact, it is Lindelöf and hence normal and therefore, since F is closed in Y, the Tietze Extension Theorem implies that f can be extended to a continuous function $\bar{f}\colon Y \to \mathbb{I}$. Put $g = \bar{f} \restriction X$. Then g can be extended to a continuous function $\bar{g}\colon \beta X \to \mathbb{I}$. Since X is dense in βX, and since $\bar{g} \restriction Y$ and $\bar{f}$ agree on X, it follows that they must agree everywhere. In onthe words, $\bar{g} \restriction F = f$. So we see that $\bar{g} \restriction X^*$ is the desired extension of f.

Let $S \subseteq X^*$ be a nonempty G_δ. Since the collection $\{U' : U \subseteq X \text{ open}\}$ is a base for X^* (Lemma 11.1), it is clear that we can find open sets $U_n \subseteq X$ for all $n < \omega$, such that

$$\overline{U}_{n+1} \subseteq U_n \quad \text{and} \quad \emptyset \neq \bigcap_{n<\omega} U'_n \subseteq S$$

(the bar means closure in X). Onserve that no U_n has compact closure in X.

• *Exercise* 57. Let X be a locally compact and σ-compact space. Prove that X can be written as $X = \bigcup_{n<\omega} K_n$, where each K_n is compact while moreover each compact $K \subseteq X$ is contained in some K_n.

Let the K_n's be as in Exercise 57. For each $n < \omega$ choose a nonempty open set $V_n \subseteq U_n$ such that

$$\overline{V}_n \text{ is compact and misses } K_n.$$

Put $V = \bigcup_{n<\omega} V_n$. If $n < \omega$ then $V \backslash U_n$ has compact closure in X, whence by Corollary 11.3 it follows that

$$V' \subseteq \bigcap_{n<\omega} U_n' \subseteq S.$$

In addition, $V' \neq \emptyset$ since V does not have compact closure in X.

• *Exercise* 58. Prove that V' is infinite.

This completes the proof of the theorem. ☐

• *Exercise* 59. Let X be a zero-dimensional Lindelöf space. Prove that X is strongly zero-dimensional and has at most $\mathfrak{c}$ clopen sets. Conclude that βX is a compact zero-dimensional space of weight at most $\mathfrak{c}$.

We now present an interesting topological consequence of Theorem 12.8.

Theorem 12.15 (CH)**.** *Let X be a zero-dimensional, locally compact, σ-compact, non-compact space of weight at most $\mathfrak{c}$. Then X^* and ω^* are homeomorphic.*

Proof. Since X is a zero-dimensional Lindelöf space, βX is zero-dimensional and has weight at most $\mathfrak{c}$ (Corollary 11.11). So X^* is a zero-dimensional compact space of weight at most $\mathfrak{c}$. By Theorem 12.14 and Lemma 12.12 it follows that $CO(X^*)$ satisfies condition H_ω. This implies that $CO(X^*)$ and $\mathcal{P}(\omega)/\,\text{fin}$ are isomorphic by Theorem 12.8. So by Stone duality (Corollary 6.8), X^* and ω^* are homeomorphic. ☐

Corollary 12.16 (CH)**.** *Each of the following spaces X has the propery that X^* is homeomorphic to ω^*.*

(1) $\omega \times (\omega + 1)$.
(Here $\omega + 1$ is endowed with the order topology.)
(2) $\omega \times (\omega_1 + 1)$.
(Here $\omega_1 + 1$ is endowed with the order topology.)
(3) $\omega \times 2^\omega$.
(4) $\omega \times 2^\mathfrak{c}$.
(5) *The Cantor set minus a single point.*

12.3. **The converse to Parovičenko's Theorem.** The main result in §12.1, which is Theorem 12.8, is false under $\neg$CH. In fact, Theorem 12.8 is equivalent to CH.

Theorem 12.17. CH is equivalent to the statement that all Ba's of cardinality $\mathfrak{c}$ which satisfy condition H_ω are isomorphic.

The proof of this result in in several steps.

Lemma 12.18. *There is an ω_1-sequence $\{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of ω^* with $C_\alpha \subset C_\beta$ if $\beta < \alpha < \omega_1$.*[2]

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[2] Observe that "$\subset$" means *proper* inclusion.

Proof. Put $C_0 = \omega^*$ and assume that the C_β are constructed for every $\beta < \alpha$, for some $\alpha < \omega_1$. Since α is a countable ordinal number and the sequence $\{C_\beta : \beta < \alpha\}$ is decreasing, it follows that

$$C = \bigcap_{\beta<\alpha} C_\beta$$

is a nonempty G_δ-subset of ω^* (for the nonempty part of this, use that ω^* is compact). As a consequence, C has nonempty interior by Exercise 51. So there is a nonempty clopen set $D \subseteq C$. Since D has no isolated points, we can make D a little smaller if necessary. So we may assume that D is a proper subset of C, and hence is a proper subset of every C_β. So we conclude that by putting $C_\alpha = D$ we satisfy our inductive demands. □

Let $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ be as in the previous lemma, and put

$$P = \bigcap_{\alpha<\omega_1} C_\alpha.$$

Now let $S = \omega^*/P$, the quotient space obtained from ω^* by vollapsing P to a single point. Let $\pi\colon \omega^* \to S$ be the quotient map, and let p be the unique point in $\pi[P]$.

Lemma 12.19. *The collection* $\{\pi[C_\alpha] : \alpha < \omega_1\}$ *is a neighborhood base for* p *consisting of clopen sets.*

Proof. If U is an open neighborhood of p then $\pi^{-1}[U]$ is an open neighborhood of P. Since $\mathcal{C}$ is decreasing, by compactness for some $\alpha < \omega_1$ we have that $C_\alpha \subseteq U$. Since $\pi[C_\alpha] \subseteq U$ it suffices to prove that $\pi[C_\alpha]$ is clopen. But this is clear since

$$\pi^{-1}\Big[\pi[C_\alpha]\Big] = C_\alpha.$$

Hence $\pi[C_\alpha]$ is open by definition of the quotient topology on S. In addition, $\pi[C_\alpha]$ is closed in S by continuity of π. So it follows that $\pi[C_\alpha]$ is clopen. □

If X is a space and $p \in X$ then the minimum cardinality of a neighborhood base for p is called the *character* of p in X and is denoted by $\chi(p, X)$. In addition,

$$\chi(X) = \sup\{\chi(p, X) : p \in X\}$$

is called the *character* of X. So a space is first countable if and only if $\chi(X) \leq \omega$.

So by Lemma 12.19, $\chi(p, S) \leq \omega_1$.

Lemma 12.20. *S is a zero-dimensional compact F-space.*

Proof. It is clear that S is compact, being a continuous image of a compact space.

We already know by Lemma 12.19 that S is zero-dimensional at p. But S is also zero-dimensional at all other points since the function $\pi\colon \omega^* \setminus P \to S \setminus \{p\}$ is a homeomorphism. So we conclude that S is zero-dimensional.

To prove that S is an F-space, all we need to show by Lemma 12.11(b) is that disjoint open F_σ-subsets of S have disjoint closures. To this end, let U and V be disjoint open F_σ-subsets of S. Then $U' = \pi^{-1}[U]$ and $V' = \pi^{-1}[V]$ are disjoint open F_σ-subsets of ω^* and consequently have disjoint closures.

Suppose that $p \notin U$. Then $U' \cap P = \emptyset$. We claim that then $\overline{U'} \cap P = \emptyset$. To see this, write $U' = \bigcup_{n<\omega} F_n$ with each F_n closed in ω^*. For every n, $\omega^* \setminus F_n$ is an open neighborhood of P. Since $\mathcal{C}$ is decreasing, by compactness there exists an index $\alpha_n < \omega_1$ such that $C_{\alpha_n} \subseteq (\omega^* \setminus F_n) = \emptyset$, or, equivalently, $C_{\alpha_n} \cap F_n = \emptyset$. Nut let $\alpha = \sup\{\alpha_n : n < \omega\}$. Then $C_\alpha \cap F_n = \emptyset$ for every n. But since C_α is clopen, this implies that $\overline{U'} \cap P = \emptyset$.

It follows similarly that $p \notin V$ implies that $\overline{V'} \cap P = \emptyset$.

Now suppose first that $p \notin U \cup V$. Then $P \cap (\overline{U'} \cup \overline{V'}) = \emptyset$ and $\overline{U'} \cap \overline{V'} = \emptyset$ since ω^* is an F-space. But then

$$\pi[\overline{U'}] \cap \pi[\overline{V'}] = \emptyset, \tag{12.2}$$

which obviously implies that $\overline{U} \cap \overline{V} = \emptyset$.

Suppose next that e.g. $p \in U$. Then $P \subseteq U'$ and so $P \subseteq \overline{U'}$. Since as above $\overline{U'} \cap \overline{V'} = \emptyset$ and P is the only non-degenerate fibre of π, it again clearly follows that (12.2) holds. □

Lemma 12.21. *Every nonempty G_δ-subset of S has infinite interior.*

Proof. Let G be a nonempty G_δ-subset of S, and write $G = \bigcap_{n<\omega} U_n$, where each U_n is open in S. We consider two cases. Suppose first that $p \in G$. Then for every $n < \omega$, $\pi^{-1}[U_n]$ is an open neighborhood of P in ω^*. It therefore follows by similar arguments as in the proof of the previous lemma that there exists an index $\alpha < \omega_1$ such that $P \subseteq C_\alpha \subseteq \bigcap_{n<\omega} \pi^{-1}[U_n]$. But then $\pi[C_\alpha]$ is contained in G and is open by Lemma 12.19. So in this case G has nonempty interior. Suppose next that $p \notin G$. Then $\pi^{-1}[G]$ is a nonempty G_δ-subset of ω^* and hence has nonempty interior. Let $C \subseteq \pi^{-1}[G]$ be a nonempty clopen set. But then $\pi[C]$ is contained in G and is clopen in S because $C \cap P = \emptyset$.

So it suffices to prove that S has no isolated points. It is clear that p is the only point in S which could be isolated. So assume that p is isolated. Then $\pi^{-1}[\{p\}](= P)$ is an open neighborhood of P in ω^* and, as before, there exists an $\alpha < \omega_1$ such that $P \subseteq C_\alpha \subseteq \pi^{-1}[\{p\}] = P$. So $C_\alpha = P$, and so $C_{\alpha+1} = P$ as well. But this contradicts the fact that $C_{\alpha+1}$ is a proper subset of C_α. □

Corollary 12.22. *There exists a Parovičenko-space S containing a point p with $\chi(p, S) = \omega_1$.*

Proof. By Lemmas 12.20 and 12.21 it follows that S is a Parovičenko-space. In addition, $\chi(p, S) \leq \omega_1$, as observed above. Assume that $\chi(p, S) \neq \omega_1$; then $\chi(p, S) \leq \omega$. But then $\{p\}$ is a nonempty G_δ-subset of S, which therefore has nonempty interior, and so $\{p\}$ is an isolated point of S, which is a contradiction. □

Lemma 12.23. *There exists a Parovičenko-space T such that $\chi(x, T) = \mathfrak{c}$ for every $x \in T$.*

Proof. We put $T = X^*$, where $X = \omega \times 2^{\mathfrak{c}}$. First observe that X is a zero-dimensional Lindelöf space with weight $\mathfrak{c}$. So by Corollary 11.11 it follows that βX is a zero-dimensional compact space with weight at most $\mathfrak{c}$. We conclude that T is a Parovičenko-space by Theorem 12.14 and Exercise 51.

- *Exercise* 60. Complete this by proving that X^* has weight $\mathfrak{c}$.

It therefore suffices to show that for every $x \in T$ we have $\chi(x, T) \geq \mathfrak{c}$.

For $\alpha < \mathfrak{c}$ denote the α-th projection $2^{\mathfrak{c}} \to 2$ by π_α. For $\alpha < \mathfrak{c}$ and $i = 0, 1$ define

$$K(\alpha, i) = T \cap \mathrm{cl}_{\beta X}(\omega \times \pi_\alpha^{-1}(\{i\}).$$

Note that $K(\alpha, i)$ is a nonempty clopen subset of T (Corollary 4.7).

Observe that $K(\alpha, i) = (\omega \times \pi_\alpha^{-1}(\{i\})' = \mathrm{Ex}(\omega \times \pi_\alpha^{-1}(\{i\}) \cap T$ (Corollary 11.5).

- *Exercise* 61. Prove that $K(\alpha, i) = K(\alpha', i')$ iff $\alpha = \alpha'$ and $i = i'$.

Define

$$\mathcal{K} = \{K(\alpha, i) : \alpha < \mathfrak{c}, i = 0, 1\}.$$

Claim 12.23.1. Any intersection of ω_1 distinct members of $\mathcal{K}$ has empty interior in T.

For symmetry reasons it suffices to prove that $I = \bigcap_{\alpha<\omega_1} K(\alpha, 0)$ has empty interior in T. Suppose that this is not true. Then there is an open $U \subseteq X$ such that $\emptyset \neq U' = \mathrm{Ex}(U) \cap T \subseteq I$. Since $\mathrm{Ex}(U) \neq \emptyset$, it follows that U does not have compact clusure in X for otherwise $U' \subseteq \emptyset' = \emptyset$ by Corollary 11.3. So U must intersect infinitely may sets of the form $\{n\} \times 2^{\mathfrak{c}}$. For every $n < \omega$ for which $U \cap (\{n\} \times 2^{\mathfrak{c}}) \neq \emptyset$, let $V_n \subseteq \{n\} \times 2^{\mathfrak{c}}$ be clopen such that $V_n \subseteq U$. The union of all the V_n's is a noncompact clopen set V in X such that $V \subseteq U$. As a consequence, $V' \neq \emptyset$ (Corollary 11.5) and since $V' \subseteq U'$ we may therefore assume without loss of generality that U is clopen.

We claim that for every $\alpha < \omega_1$ the set

$$U \setminus (\omega \times \pi_\alpha^{-1}(\{0\}))$$

has compact closure in X. For if $F = U \setminus (\omega \times \pi_\alpha^{-1}(\{0\}))$ does not have compact closure in X then $F' \neq \emptyset$ by Corollary 11.5 and since $F \cap (\omega \times \pi_\alpha^{-1}(\{0\})) = \emptyset$ it follows that $F' \cap (\omega \times \pi_\alpha^{-1}(\{0\}))' = \emptyset$ (Lemma 11.4). But since $F' \subseteq U'$ this contradicts the fact that $U' \subseteq (\omega \times \pi_\alpha^{-1}(\{0\}))'$.

Since $U \cap X$ is not compact, for every $\alpha < \omega_1$ there is an integer n_α such that

$$\emptyset \neq U \cap (\{n_\alpha\} \times 2^{\mathfrak{c}}) \subseteq \{n_\alpha\} \times \pi_\alpha^{-1}(\{0\}).$$

Since ω_1 is uncountable, there is an integer n such that $A = \{\alpha < \omega_1 : n_\alpha = n\}$ is infinite. But then

$$\emptyset \neq U \cap (\{n\} \times 2^{\mathfrak{c}}) \subseteq \{n\} \times \bigcap_{\alpha \in A} \pi_\alpha^{-1}(\{0\}) \subseteq \{n\} \times 2^{\mathfrak{c}}$$

has nonempty interior, which is absurd.

- *Exercise* 62. Complete this by showing that if $A \subseteq \omega_1$ is infinite then the set

$$\bigcap_{\alpha \in A} \pi_\alpha^{-1}(\{0\})$$

has empty interior in $2^{\mathfrak{c}}$.

Now let $x \in T$ be arbitrary, and let $\mathcal{U}$ be a neighborhood base for x in T. Then the family

$$\mathcal{F} = \{K \in \mathcal{K} : x \in K\}$$

has cardinality $\mathfrak{c}$.

- *Exercise* 63. Why is this true?

For each $K \in \mathcal{F}$ there exists $U(K) \in \mathcal{U}$ with $U(K) \subseteq K$. Hence $|\mathcal{U}| \geq |\mathcal{F}| = \mathfrak{c}$ since the Claim implies that

$$|\{K \in \mathcal{K} : U(K) = U\}| \leq \omega$$

for all $U \in \mathcal{U}$. It therefore follows that $\chi(x, T) = \mathfrak{c}$ since we already know that T has weight at most $\mathfrak{c}$. □

So now the proof of Theorem 12.17 is obvious. For assume that all Parovičenko-spaces are homeomorphic. Then the spaces S and T defined in this section are homeomorphic. But S contains a point p with $\chi(p, S) = \omega_1$ (Corollary 12.22), while $\chi(x, T) = \mathfrak{c}$ for every $x \in T$ (Lemma 12.23). We conclude that $\omega_1 = \mathfrak{c}$, i.e. CH holds.

13. Homogeneity problems

A topological space X is called *homogeneous* provided that for all $x, y \in X$ there is a homemorphism $f \colon X \to X$ such that $f(x) = y$. So from a topological standpoint, in a homogeneous space all points have the same topological behaviour.

Since the points in ω^* are free ultrafilters, and all ultrafilters look alike, the question naturally arises whether ω^* is homogeneous. This question will turn out to be not an easy one.

One can ask a more general question, namely whether there exists a space X for which X^* is not homogeneous. This question will be answered first to get some feel for homogeneity problems. We will prove then that ω^* is not homeomorphic under CH, then that there are "many" different points in ω^* under CH, and then that ω^* is not homogeneous without any addition assumptions.

13.1. A space X for which X^* is not homogeneous.

In §8 on page 25 we introduced the π-weight of a topological space. Observe that every space with a countable base, so every separable metrizable space, has countable π-weight. The spaces with countable π-weight are natural generalizations of spaces with a countable base, and are of particular interest.

Lemma 13.1. *Let Y be a dense subspace of a space X. Then $\pi(X) = \pi(Y)$.*

Proof. If $\mathcal{U}$ is a π-base for Y then $\mathcal{U} \restriction Y = \{U \cap Y : U \in \mathcal{U}\}$ is a π-base for Y. For this, first observe that $\mathcal{U} \restriction Y$ consists of nonempty (relatively) open subsets of Y (here we use that Y is dense in X). Second, let $O \subseteq Y$ be open, and let $O' \subseteq X$ be an open subset such that $O' \cap Y = O$. Since $\mathcal{U}$ is a π-base, there is an element $U \in \mathcal{U}$ such that $U \subseteq O'$. But then $\mathcal{U} \restriction Y \ni U \cap Y \subseteq O$, as desired. This proves that $\pi(Y) \leq \pi(X)$.

Now let $\mathcal{V}$ be a π-base for Y. For every $V \in \mathcal{V}$ pick an open set $U(V) \subseteq X$ such that $U(V) \cap Y = V$. If $U(V) = U(V')$ for certain $V, V' \in \mathcal{V}$ then $V = U(V) \cap Y = Y \cap U(V') = V'$, so the function $V \mapsto U(V)$ is one to one. We claim that the collection

$$\mathcal{U} = \{U(V) : V \in \mathcal{V}\}$$

is a π-base for X, which will prove that $\pi(X) \leq \pi(Y)$.

To this end, let O be a nonempty open subset of X, and let W be a nonempty open subset of X such that $\overline{W} \subseteq O$. There is an element $V \in \mathcal{V}$ such that $V \subseteq W \cap Y$. Since Y is dense in X and V is open in Y it follows that V is dense in $U(V)$. As a consequence, V and $U(V)$ have the same closures in X. But then

$$\overline{U(V)} = \overline{V} \subseteq \overline{W} \subseteq O$$

so that $U(V) \subseteq O$, which is as desired. $\square$

As usual, $\mathbb{Q}$ denotes the subspace of $\mathbb{R}$ consisting of all rational numbers, and $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ denotes the irrational numbers. Our aim is to prove that $\mathbb{Q}^*$ is not homogeneous.

First observe that $\mathbb{Q}$ is a zero-dimensional Lindelöf space. For if $E \subseteq \mathbb{P}$ is a countable dense set, then the collection

$$\{[e_1, e_2] \cap \mathbb{Q} : e_1, e_2 \in E, e_1 < e_2\}$$

is a countable clopen base for $\mathbb{Q}$. As a consequence, $\mathbb{Q}$ is strongly-zero-dimensional by Corollary 11.8.

Lemma 13.2. $\pi(\mathbb{Q}^*) = \omega$.

Proof. Since $\mathbb{Q}$ is dense in $\beta\mathbb{Q}$ and has countable weight, it follows from Lemma 13.1 that $\pi(\beta\mathbb{Q}) = \omega$. We claim that $\mathbb{Q}^*$ is dense in $\beta\mathbb{Q}$. Then by another application of Lemma 13.1 it will follow that $\pi(\mathbb{Q}^*) = \omega$.

To this end, let U be a nonempty clopen subset of $\beta\mathbb{Q}$. Then U is compact since $\beta\mathbb{Q}$ is compact. Since U is nonempty, it therefore cannot be contained in $\mathbb{Q}$.

• *Exercise* 64. Let $U \subseteq \mathbb{Q}$ be nonempty and open. Prove that U is not compact. (Spaces with this property are called *nowhere locally compact.*)

As a consequence, U must intersect $\mathbb{Q}^*$. $\square$

So we conclude that $\mathbb{Q}^*$ has countable π-weight. We will next prove that homogeneous spaces with countable π-weight have cardinality at most $\mathfrak{c}$.

Theorem 13.3. *Let X be a homogeneous space. Then $|X| \leq 2^{\pi(X)}$.*

Proof. We first deal with a trivial case.

• *Exercise* 65. Suppose that $\pi(X) < \omega$. Prove that X is finite and that $|X| = \pi(X)$. So then the inequality $|X| \leq 2^{\pi(X)}$ trivially holds.

So from now on assume that $\pi(X) \geq \omega$. Let $\mathcal{U}$ be a π-base for X. For every homeomorphism $f\colon X \to X$ define a function $\hat{f}\colon \mathcal{U} \to \mathcal{P}(\mathcal{U})$ by

$$\hat{f}(U) = \{V \in \mathcal{U} : V \subseteq f[U]\}.$$

Then $\hat{f}$ is clearly well-defined, and we will prove that the assignment $f \mapsto \hat{f}$ is one to one. To this end, let f_1 and f_2 be two homeomorphisms of X such that $f_1 \neq f_2$. There exists $x \in X$ such that $f_1(x) \neq f_2(x)$. Since X is Hausdorff, there are disjoint open sets V_1 and V_2 of X such that $f_i(x) \in V_i$ for $i = 0, 1$. Put $W = f_1^{-1}[V_1] \cap f_2^{-1}[V_2]$. Then W is an open neighborhood of x. Pick an element $U \in \mathcal{U}$ such that $U \subseteq W$. Now observe that $f_1[U] \cap f_2[U] = \emptyset$. In addition, both $f_1[U]$ and $f_2[U]$ are nonempty open subsets of X. Pick an arbitrary element $V \in \mathcal{U}$ such that $V \subseteq f_1[U]$. Then $V \cap f_2[U] = \emptyset$ from which it follows that $V \notin \hat{f}_2(U)$. We conclude that $\hat{f}_1(U) \neq \hat{f}_2(U)$, as desired.

Now observe that if A is a set of cardinality $\kappa \geq \omega$ then the set of functions from A to $\mathcal{P}(A)$ has cardinality

$$\left|\mathcal{P}(A)|^{|A|}\right| = (2^\kappa)^\kappa = 2^{\kappa\times\kappa} = 2^\kappa.$$

This implies that there are at most $2^{\pi(X)}$ functions from $\mathcal{U}$ to $\mathcal{P}(\mathcal{U})$. So by the above X has at most $2^{\pi(X)}$ homeomorphisms.

We claim that the cardinality of X is less than or equal the number of homeomorphisms of X, which will conclude the proof. Pick an arbitrary element $x \in X$. By homogeneity for every $y \in X$ there is a homeomorphism $f_y \colon X \to X$ such that $f_y(x) = y$. Then for distinct $a \neq b$ we have $f_a \neq f_b$ sich $f_a(x) = a \neq b = f_b(x)$. So the assignment $y \mapsto f_y$ is one to one, which is as desired. ☐

Since $\mathbb{Q}^*$ has countable π-weight by Lemma 13.2, if it is homogeneous it must have cardinality at most $\mathfrak{c}$ by Theorem 13.3. So if we prove that $|\mathbb{Q}^*| > \mathfrak{c}$ then this will disprove its homogeneity.

Thinking about the cardinality of $\mathbb{Q}^*$, or of $\beta\mathbb{Q}$, one naturally wonders about the cardinality of ω^*, or of $\beta\omega$. So we will first try to decide the cardinality of $\beta\omega$ and then come back to $\beta\mathbb{Q}$.

Lemma 13.4. (a) *There is a family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of infinite subsets of ω such that if $\alpha < \beta$ then $A_\alpha \cap A_\beta$ is finite (in other words, there is an almost disjoint family of subsets of ω of size $\mathfrak{c}$).*

(b) *There is a family $\{(A^0_\alpha, A^1_\alpha) : \alpha < \mathfrak{c}\}$ of pairs of disjoint subsets of ω such that for all finite $F \subseteq \mathfrak{c}$ and each $f \colon F \to 2 = \{0,1\}$ we have that*

$$\bigcap_{\alpha\in F} A^{f(\alpha)}_\alpha$$

is infinite (such a family is called independent*).*

(c) $|\beta\omega| = 2^\mathfrak{c}$.

Proof. For each irrational number $r \in \mathbb{R}$ choose a sequence $S(r)$ of rational numbers converging to r. Observe that if r and r' are distinct irrational numbers then $S(r) \cap S(r')$ is finite. The collection $\mathcal{S} = \{S(r) : r \text{ irrational}\}$ consequently is an almost disjoint collection of subsets of $\mathbb{Q}$. Since $|\mathbb{R}| = \mathfrak{c}$ and $|\mathbb{Q}| = \omega$ it follows that $|\mathbb{P}| = \mathfrak{c}$. We conclude that $\mathcal{S}$ has cardinality $\mathfrak{c}$ and is obviously as required in (a), except that it does not consist of subsets of ω, but of the countable set $\mathbb{Q}$. But this causes no problems of course.

For (b), first let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be a family of subsets of ω as in (a). For each $\alpha < \mathfrak{c}$ define

$$B^0_\alpha = \{F \in [\omega]^{<\omega} : F \cap A_\alpha \neq \emptyset\} \quad \text{and} \quad B^1_\alpha = [\omega]^{<\omega} \setminus B^0_\alpha.$$

• *Exercise* 66. Prove that the family $\{(B^0_\alpha, B^1_\alpha) : \alpha < \mathfrak{c}\}$ has the properties of the required family in (b), except that it is not defined on ω, but on the countable set $[\omega]^{<\omega}$.

This again causes no problems of course.

For (c), let $\{(A_\alpha^0, A_\alpha^1) : \alpha < \mathfrak{c}\}$ be a family of subsets of ω such as in (b). For every $f \in 2^{\mathfrak{c}}$, the family

$$\{A_\alpha^{f(\alpha)} : \alpha < \mathfrak{c}\}$$

has the finite intersection property, and consequently is contained in at least one ultrafilter $p_f \in \beta\omega$ (Lemmas 6.1 and 6.2). Now if $f, g \in 2^{\mathfrak{c}}$ are distinct then for some $\alpha < \mathfrak{c}$ we have $f(\alpha) \neq g(\alpha)$. Without loss of generality we may assume that $f(\alpha) = 0$ and $g(\alpha) = 1$. But then $A_\alpha^0 \in p_f$ and $A_\alpha^1 \in p_g$ so that p_f and p_g must be distinct ultrafilters because they contain compelmentary sets. We conclude that $|\beta\omega| \geq 2^{\mathfrak{c}}$.

- *Exercise* 67. Complete the proof by showing that if X is separable then $|X| \leq 2^{\mathfrak{c}}$. Conclude that $|\beta\omega| = 2^{\mathfrak{c}}$.

This finishes the proof. ☐

So we see that $\beta\omega$ has size $2^{\mathfrak{c}}$. Since ω is countable, this implies that $|\omega^*| = 2^{\mathfrak{c}}$ as well.

Now observe that ω is a closed subspace of $\mathbb{Q}$. As a consequence, $\beta\omega$ is a closed subspace of $\beta\mathbb{Q}$ by Corollary 4.8. From this we conclude that $|\beta\mathbb{Q}| \geq 2^{\mathfrak{c}}$. But since $\beta\mathbb{Q}$ is separable it also follows that $|\beta\mathbb{Q}| \leq 2^{\mathfrak{c}}$ by Exercise 67. So we conclude that $\beta\mathbb{Q}$ and $\mathbb{Q}^*$ both have size $2^{\mathfrak{c}}$.

Corollary 13.5. $\mathbb{Q}^*$ *is not homogeneous.*

- *Exercise* 68. Prove that $\mathbb{P}^*$ is not homogeneous.
- *Exercise* 69. Prove that the projective cover EC of the Cantor set C is not homogeneous.

The proof presented here of the inhomogeneity of $\mathbb{Q}^*$ is based on cardinality considerations. It is therefore highly unsatisfactory because it does not show *why* $\mathbb{Q}^*$ is not homogeneous since it does not yield two points with obvious distict topological behaviour. It is possible to present a different proof that does give such points, but it would take us to far to present it.

One naturally wonders whether the arguments presented in this subsection can be used to prove that ω^* is not homogeneous. But this is not the case, as the next exercise shows.

- *Exercise* 70. Prove that $\pi(\omega^*) = \mathfrak{c}$. (Hint: use Lemma 13.4(a)).

So for ω^* we have to use a different method.

13.2. Inhomogeneity of ω^*.

A subset A of a space X is called a *P-set* provided that the intersection of a countable family of its neighborhoods is again a neighborhood. So if U_n is a neighborhood of A for every $n < \omega$ then there is a neighborhood U of A such that $U \subseteq \bigcap_{n<\omega} U_n$. A singleton P-set is called a *P-point.*

Example 13.6. Let $X = \omega_1 + 1$ endowed with the order topology. Then $\{\omega_1\}$ is a P-point in X.

Lemma 13.7. *Let X be an infinite compact space. Then there is a point $x \in X$ which is not a P-point.*

Proof. Let $D \subseteq X$ be a countably infinite subset of X, and consider $\overline{D}$. If every point of $\overline{D}$ is isolated (in $\overline{D}$) then the cover $\{\{x\} : x \in \overline{D}\}$ of $\overline{D}$ has a finite subcover by compactness, which contradicts the fact that D is infinite. So let $x \in \overline{D}$ be non-isolated. We claim that x is not a P-point of X. Striving for a contradiction, assume that x is a P-point. Put $Y = D \setminus \{x\}$, and for every $y \in Y$ let $U_y = X \setminus \{y\}$. Then each U_y is a neighborhood of x. Since x is a P-point and Y is countable, there consequently is a neighborhood V of x such that $V \subseteq \bigcap_{y \in Y} U_y$. But then $U \cap (D \setminus \{x\}) = \emptyset$, and so $x \notin \overline{D \setminus \{x\}}$. Since

$$\overline{D} = \overline{(D \setminus \{x\}) \cup \{x\}} = \overline{D \setminus \{x\}} \cup \overline{\{x\}} = \overline{D \setminus \{x\}} \cup \{x\}$$

this implies that x is an isolated point of $\overline{D}$, which is a contradiction. ☐

So a good method for proving nonhomogeneity is to try to prove that a given compact space contains a P-point. If one succeeds in doing so then this space is not homogeneous because it als contains non-P-points by Lemma 13.7. In addition, proving nonhomogeneity in this way is elegant because this method exhibits two points with different topological behaviour. Unfortunately, P-points are rather rare in compact spaces. Fortunately, for ω^* this method works, but it has, as we will see later, a nasty drawback.

Lemma 13.8. *ω^* cannot be covered by ω_1 nowhere dense sets.*

Proof. Let $\{D_\alpha : \alpha < \omega_1\}$ be a family of ω_1 nowhere dense subsets of ω^*. By transfinite induction, we will construct a family $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of ω^* such that for every $\alpha < \omega_1$:

(1) $C_\alpha \cap D_\alpha = \emptyset$,
(2) if $\beta < \alpha < \omega_1$ then $C_\alpha \subseteq C_\beta$.

The construction of $\mathcal{C}$ is a triviality. For let C_0 be any nonempty clopen subset of ω^* which misses D_0. Suppose that we constructed the sets C_β for all $\beta < \alpha$, where $\alpha < \omega_1$. Then $\bigcap_{\beta<\alpha} C_\beta$ is a non-empty G_δ-subset of ω^*, which consequently has nonempty interior. So let $U \subseteq \bigcap_{\beta<\alpha} C_\beta$ be a nonempty clopen set, and let $C_\alpha \subseteq U$ be nonempty and clopen such that $C_\alpha \cap D_\alpha = \emptyset$. This completes the transfinite construction.

So any point in $\bigcap_{\alpha<\omega_1} C_\alpha$ misses $\bigcup_{\alpha<\omega_1} D_\alpha$. ☐

Observe that the classical Baire Category Theorem says that no compact space can be covered by countably many nowhere dense sets. So Lemma 13.8 says that ω^* satisfies a strong form of the Baire Category Theorem.

Corollary 13.9 (CH)**.** *ω^* contains a P-point.*

Proof. Let $\mathcal{A} = \{\overline{U} \setminus U : U \subseteq \omega^* \text{ is an open } F_\sigma\}$. Since ω^* has only $\mathfrak{c}$ clopen sets, and each open F_σ-subset of it is the union of a countable family clopen sets (Exercise 51(1)), $\mathcal{A}$ has cardinality at most $\mathfrak{c}$, hence at most ω_1 by CH. Since the members of $\mathcal{A}$ are clearly nowhere dense, by Lemma 13.7 it therefore follows that there exists an $x \in \omega^* \setminus \bigcup \mathcal{A}$. We claim that x is a P-point.

To this end, let $\mathcal{C}$ be a countable collection of neighborhoods of x. We may assume without loss of generality that every $C \in \mathcal{C}$ is clopen. Then $U = \bigcup_{C \in \mathcal{C}} \omega^* \setminus C$ is an open

F_σ-subset of ω^* which does noet contain x. By construction, $x \notin \overline{U} \setminus U$. But then $x \notin \overline{U}$, which implies that $\omega^* \setminus \overline{U}$ is a neighborhood of x contained in $\bigcap \mathcal{C}$. $\square$

So Lemmas 13.7 and 13.9 yield the following interesting fact.

Corollary 13.10 (CH). *ω^* is not homogeneous because it contains both P-points and non-P-points.*

Although the method in this subsection is rather elegant, it has a drawback since it is performed under the Continuum Hypothesis. The question naturally arises whether ω^* has P-points in ZFC. This question was one of the most difficult open problems in set theory and set theoretic topology for decades. It was finally solved by Shelah in the negative by producing a model of set theory in which ω^* has no P-points.

So our method of proving the nonhomogeneity of ω^* does not work in ZFC alone. Fortunately, it can be shown in ZFC that ω^* is not homogeneous by a method such as the one in the previous subsection, so by a clever counting argument. This is due to Frolík. See Theorem 14.13. His proof is ingeneous, but it does not yield two points with obvious different topological behaviour. Later, it was shown by Kunen that ω^* contains so- called *weak P-points*, where a point x in a space X is said to be a weak P-point provided that for every countable set $D \subseteq X \setminus \{x\}$ we have that $x \notin \overline{D}$. Observe that the proof of Lemma 13.7 shows that every infinite compact space contains a point which is not a weak P-point. So Kunen's result finally settled the problem of the nonhomogeneity of ω^* in a satisfactory manner.

- *Exercise* 71 (CH). (1) Prove that the P-points form a dense subset of ω^*.
  (2) Prove that there are at least $2^{\mathfrak{c}}$ different P-points in ω^* (Hint: construct a binary tree of height ω_1 consisting of clopen subsets of ω^*, such that its members at level α all miss the boundary of the α-th open F_σ-subset of ω^*).
  (3) (Optional) Prove that if x and y are P-points in ω^* then there exists an autohomeomorphism $f\colon \omega^* \to \omega^*$ such that $f(x) = y$ (Hint: go through the proof of Parovičenko's Theorem in such a way that the final isomorphism sends x onto y).

13.3. **Continuous images of ω^*.** In this subsection we characterize the continuous images of ω^*.

- *Exercise* 72. Let $f\colon X \to Y$ be a continuous surjection between compact spaces. Prove that $\mathrm{w}(Y) \le \mathrm{w}(X)$.

Since $\mathrm{w}(\omega^*) = \mathfrak{c}$, a necessary condition for a continuous image of ω^* is that it is of weight at most $\mathfrak{c}$. Interestingly, this necessary condition is under CH also sufficient.

Theorem 13.11. *Let B be a Ba of cardinality at most ω_1. Then B can be embedded in $\mathcal{P}(\omega)/\operatorname{fin}$.*

Proof. Let us return to the proof of Parovičenko's Theorem 12.8. We were dealing there with two Ba's B and E of cardinality ω_1 and both satisfying condition $H_\omega \Leftrightarrow R_\omega$. An isomorphism $f\colon B \to E$ was created by transfinite induction. In the inductive step we were in the following situation. There were countable subalgebras $\hat{B}$ and $\hat{E}$ of B and E, respectively, and an isomorphism $g\colon \hat{B} \to \hat{E}$. There were also elements $b \in B$ and $e \in E$. Our task was to extend the isomorphism $g\colon \hat{B} \to \hat{E}$ over countable subalgebras containing b and e, respectively. Property R_ω was used in E to find an element that served there as the image of b, and property R_ω was used in B to find an element that served there as the preimage of e.

Now we are in a similar situation. The Ba $\mathcal{P}(\omega)/\operatorname{fin}$ satisfies R_ω, but from B we only know that it has cardinality ω_1. We aim to use the same technique as in the proof of Parovičenko's Theorem to construct an embedding $f\colon B \to \mathcal{P}(\omega)/\operatorname{fin}$. Since B has size ω_1, in our inductive process we are now dealing with a countable subalgebra $\hat{B}$ of B, an embedding $g\colon \hat{B} \to \mathcal{P}(\omega)/\operatorname{fin}$, and an element $b \in B$. As before, we use R_ω for $\mathcal{P}(\omega)/\operatorname{fin}$ to find an element in $\mathcal{P}(\omega)/\operatorname{fin}$ that serves as the image of b. We cannot do anything with preimages since B need not have property R_ω. But this is irrelevant since in this theorem we do not aim at an isomorphism but at an embedding. So by performing half of the proof of Parovičenko's Theorem, we get what we want. ☐

By Stone duality, this theorem is equivalent to the statement that each compact, zero-dimensional space of weight at most ω_1 is a continuous image of ω^*. This result suggests the obvious question whether the same result holds without the assumption on zero-dimensionality. This is indeed the case. Before we prove this, we first need a simple lemma.

Lemma 13.12. *Let X be a compact space with weight κ. Then there is a compact zero-dimensional space Y of weight κ which can be mapped onto X.*

Proof. If κ is finite then X is finite and there is nothing to prove. So assume that $\kappa \geq \omega$. Let $\mathcal{B}$ be a base for X of cardinality κ. Replace each $B \in \mathcal{B}$ by $\operatorname{int}\overline{B}$. The new collection of open sets is easily seen to be a basis for X as well. So we may assume that $\mathcal{B} \subseteq RO(X)$. Now let $\mathcal{E}$ be the subalgebra of $RO(X)$ generated by $\mathcal{B}$. Then $\mathcal{E}$ has cardinality at most $\kappa \cdot \omega = \kappa$, hence has cardinality κ. Now let $Y = \operatorname{st}(\mathcal{E})$, the Stone space of Y. That Y admits a continuous map onto Y follows by the same argumentation as in the proof that the projective cover EX maps onto X (Theorem 8.10). ☐

So we arrive at the following

Theorem 13.13. *Every compact space of weight at most ω_1 is a continuous image of ω^*.*

Corollary 13.14 (CH)**.** *For a compact space X the following statements are equivalent:*

(a) *X is a continuous image of ω^*.*
(b) $\mathrm{w}(X) \leq \mathfrak{c}$.

Proof. The implication (a) $\Rightarrow$ (b) follows form Exercise 72, and (b) $\Rightarrow$ (a) follows from Theorem 13.13. ☐

Corollary 13.14 is not true in ZFC. Kunen produced a model of set theory in which $\mathfrak{c} = \omega_2$ and the space $\omega_2 + 1$ endowed with the order topology is not a continuous image of ω^*.

13.4. **Closed subspaces of ω^*.** We now aim at characterizing all closed subspaces of $\beta\omega$ topologically. If X is a closed subspace of $\beta\omega$, then it must clearly be of weight at most $\mathfrak{c}$ and it must be a zero-dimensional compact F-space by Lemma 12.11(c),(d). It turns out that, under CH, these conditions are nor only necessary but also sufficient.

Let X and Y be compact spaces. Let $A \subseteq X$ be closed, and let $f\colon A \to Y$ be a continuous surjection. It is easily seen that the collection

$$\mathcal{B} = \{f^{-1}(y) : y \in Y\} \cup \{\{x\} : x \in X \setminus A\}$$

is an upper-semicontinuous decomposition of X. The decomposition space $X/\mathcal{B}$ will be denoted by $X \cup_f Y$. It $\pi\colon X \to X \cup_f Y$ is the decomposition map, then $\pi[A]$ is homeomorphic to Y. It will be convenient to identify it with Y. It will also be convenient to identify $X \setminus A$ and $(X \cup_f Y) \setminus Y$. One should think of $X \cup_f Y$ as X where the set A has been replaced by Y by using the map f.

Lemma 13.15. *Let X and Y be compact zero-dimensional F-spaces, let $A \subseteq X$ be a closed P-set, and let $f\colon A \to Y$ be a continuous surjection. Then $X \cup_f Y$ is a compact zero-dimensional F-space.*

Proof. It is clear that $Z = X \cup_f Y$ is compact.

- *Exercise* 73. Prove that Z is zero-dimensional and that Y is a P-set in Z.

Since Z is compact, it consequently suffices to prove by Lemma 12.11(b) that disjoint open F_σ-subsets of it have disjoint closures. So let U and V be disjoint open F_σ-subsets of Z. Since $Y \subseteq Z$ is an F-space, $\overline{U \cap Y} \cap \overline{V \cap Y} = \emptyset$. Let E and F be disjoint clopen neighborhoods of $\overline{U \cap Y}$ and $\overline{V \cap Y}$, respectively. Consider E for a moment. Observe that $E \cap V$ is an open F_σ-subset of Z which misses Y. Since Y is a P-set in Z by Exercise 73, it follows that $\overline{E \cap V} \cap Y = \emptyset$. Now let C and D be disjoint clopen subsets of Z such that $Y \subseteq C$ and $\overline{E \cap V} \subseteq D$. Replace E by $E' = E \cap C$. Then E' is a clopen neighborhood of $\overline{U \cap Y}$ which misses $\overline{V}$. It follows similarly that we can replace F by a smaller clopen neighborhood F' of $\overline{F \cap Y}$ such that $F' \cap \overline{U} = \emptyset$. Now observe that $U \setminus E'$ and $U \setminus F'$ are disjoint open F_σ-subsets of X which do not meet A. Since X is an F-space and A is a P-set, we get

(1) $\overline{U \setminus E'} \cap \overline{U \setminus F'} = \emptyset$, and
(2) $A \cap \big(\overline{U \setminus E'} \cup \overline{U \setminus F'}\big) = \emptyset$.

This easily implies that $\overline{U} \cap \overline{V} = \emptyset$. ☐

Lemma 13.16. *Let X be a compact space with the property that each nonempty G_δ subset has infinite interior. If $A \subseteq X$ is closed and nowhere dense and if $f\colon A \to Y$ is a continuous surjection, then the space $X \cup_f Y$ also has the property that each nonempty G_δ has infinite interior.*

- *Exercise* 74. Prove Lemma 13.16.

Lemma 13.17 (CH)**.** *ω^* contains a nowhere dense closed P-set A which is homeomorphic to ω^*.*

Proof. By Corollary 12.16(2), we can represent ω^* by

$$Z = (\omega \times (\omega_1 + 1))^*.$$

Let $A = (\omega \times \{\omega_1\})^*$. Trivially, $A \approx \omega^*$ (Corollary 4.8) and that A is a nowhere P-set follows easily from the fact that $\{\omega_1\}$ is a nonisolated P-point of $\omega_1 + 1$.

- *Exercise* 75. Prove this.

This completes the proof. □

We now come to the main result in this subsection.

Theorem 13.18 (CH)**.** *Let X be a space. The following statements are equivalent:*

(a) *X is a compact zero-dimensional F-space of weight $\mathfrak{c}$.*
(b) *X can be embedded in $\beta\omega$ as a closed subspace.*
(c) *X can be embedded as a nowhere dense closed P-set in ω^*.*

Proof. The implications (b) ⇒ (a) and (c) ⇒ (b) are trivial, so it suffices to prove that (a) ⇒ (c). To this end, let X be a compact zero-dimensional F-space of weight at most $\mathfrak{c}$. By Lemma 13.17 we can find a closed nowhere dense P-set $A \subseteq \omega^*$ such that $A \approx \omega^*$. In addition, by Corollary 13.14, there is a continuous surjection $f\colon A \to X$. Put $Z = \omega^* \cup_f X$. It is routine to verify that X is zero-dimensional, has weight at most $\mathfrak{c}$, and that X is a nowhere dense closed P-set of Z (see Exercise 73). Lemma 13.15 follows by Lemma 13.16 imply that Z is a compact zero- dimensional F-space of weight at most $\mathfrak{c}$ in which nonempty G_δ's have infinite interior. Consequently, by Corollary 12.13 it follows that $Z \approx \omega^*$. □

Theorem 13.18 is false in ZFC. Under MA and $\mathfrak{c} = \omega_2$ there is a compact zero-dimensional F-space of weight $\mathfrak{c}$ which cannot be embedded in any extremally disconnected space. This is due to van Douwen and van Mill.

13.5. **Some other points of interest.** So far we saw that there are at least two special types of points in ω^*, the so-called P-points and the non-P-points. But there is actually a variety of 'special' points in ω^*. In this subsection we will construct a few of them.

A space X is called *basically disconnected* if the closure of every cozero-set is again open.

- *Exercise* 76. Prove that every basically disconnected space is an F-space.

A Ba B is said to satisfy the *countable chain condition* provided that every paiwise disjoint subset of it is at most countable.

- *Exercise* 77. Prove that a Ba B satisfies the countable chain condition if and only if its Stone space satisfies the countable chain condition.

A Boolean σ-algebra is a Boolean algebra in which every countable set has a supremum. A Boolean σ-algebra is also called a *countably complete* Ba.

Just as the extremally disconnected compacta are the Stone spaces of the complete Ba's, the basically disconnected compacta are the Stone spaces of the Boolean σ-algebras. To prove this, one can follow the proof of Theorem 8.4 verbatim.

Theorem 13.19. *Let B be a Ba. The following statements are equivalent:*
(a) *B is a σ-algebra.*
(b) *$\operatorname{st}(B)$ is basically disconnected.*

Corollary 13.20. *Every countably complete Ba satisfying the countable chain condition is complete.*

Proof. We present a topological proof. Let B be a countably complete Ba, and consider its Stone space $X = \operatorname{st}(B)$. Then X is basically disconnected by Exercise 76 and satisfies the countable chain codition by Exercise 77. So X is extremally disconnected by Lemma 12.11(d). So B is complete by Theorem 8.4. ☐

We now present a particularly interesting example of a countably complete Ba.

Let $\mathcal{M}$ be the Ba of Lebesgue measurable subsets of $\mathbb{I}$, and let $\mathcal{N}$ be its ideal of nullsets. Consider the quotient algebra $\mathcal{B} = \mathcal{M}/\mathcal{N}$. Then clearly $\mathcal{M}$ is countably complete, and so is $\mathcal{B}$. If $M \in \mathcal{M}$ then the $\mathcal{N}$-equivalence class of M is denoted by $[M]$. λ denotes Lebeague measure.

Lemma 13.21. *$\mathcal{B}$ satisfies the countable chain condition.*

Proof. Let $\mathcal{A} \subseteq \mathcal{M}$ be uncountable such that $\lambda(A) > 0$ for every $A \in \mathcal{A}$ while moreover the family

$$\{[A] : A \in \mathcal{A}\} \tag{13.1}$$

is paiwise disjoint. Let $\mathcal{U}$ be a countable open basis of $\mathbb{I}$ which is closed under finite unions. For all $U \in \mathcal{U}$ put

$$\mathcal{A}(U) = \{A \in \mathcal{A} : \lambda(A \cap U) > 1/2\lambda(U)\}.$$

If $A \in \mathcal{A}$ then there is a compact $K \subseteq A$ with $\lambda(K) > 0$. For this K there is an element $U \in \mathcal{U}$ with $K \subseteq U$ and $\lambda(K) > 1/2\lambda(U)$. We conclude that $A \in \mathcal{A}(U)$ and since A is arbitrarily chosen, this implies that

$$\bigcup_{U \in \mathcal{U}} \mathcal{A}(U) = \mathcal{A}.$$

Hence there must be an element $U \in \mathcal{U}$ such that $\mathcal{A}(U)$ is uncountable. But this easily contradicts (13.1). ☐

Lemma 13.22. *$\mathcal{B}$ has cardinality $\mathfrak{c}$.*

Proof. Let $M \in \mathcal{M}$ be such that $\lambda(M) > 0$. For every $n < \omega$ there is a compact subset $K_n \subseteq M$ such that $\lambda(M) - \lambda(K_n) < 2^{-n}$. Put $F = \bigcup_{n<\omega} K_n$. Then F is an F_σ-subset of $\mathbb{I}$ such that $F \subseteq M$ and $\lambda(M \setminus F) = 0$. As a consequence, $[M] = [F]$. Since there are at most $\mathfrak{c}$ closed subsets of $\mathbb{I}$, there are at most $\mathfrak{c}^\omega = \mathfrak{c}$ F_σ-subsets of $\mathbb{I}$. From this we conclude that $|\mathcal{B}| \leq \mathfrak{c}$. That $|\mathcal{B}| \geq \mathfrak{c}$ is clear since the collection $\{[(t, 1)] : 0 < t < 1\}$ has cardinality $\mathfrak{c}$. ☐

• *Exercise* 78. Prove that $\mathbb{I}$ has $\mathfrak{c}$ closed subsets. (Hint: let $\mathcal{U}$ be a countable open base for $\mathbb{I}$ and observe that every open subset of $\mathbb{I}$ is the union of a countable subfamily of $\mathcal{U}$.)

Now let $X = \mathrm{st}(\mathcal{B})$.

Lemma 13.23. *Every countable $D \subseteq X$ is nowhere dense.*

Proof. Take $M \in \mathcal{M}$ and list D as $\{d_n : n < \omega\}$. Since d_n is an ultrafilter in the Ba $\mathcal{B}$, there exists an element $M_n \in \mathcal{B}$ such that

(1) $[M_n] \in d_n$, and
(2) $\lambda(M_n) < 2^{-2-n} \cdot \lambda(M)$.

Then $\{x \in X : [M \setminus \bigcup_{n<\omega} M_n] \in x\}$ is a nonempty open subset of $[M]$ which misses D. ☐

Lemma 13.24. *There is a family $\mathcal{D}$ of $\mathfrak{c}$ nowhere dense subsets of X such that each nowhere dense subset of X is contained in an element of $\mathcal{D}$.*

Proof. For every nowhere dense set $E \subseteq X$ we let $\mathcal{C}_E$ be maximal family pairwise disjoint clopen subsets in $X \setminus E$. Then $\bigcup \mathcal{C}_E$ is dense by maximality. In addition, $\mathcal{C}_E$ is countable by Lemma 13.21. Since X has $\mathfrak{c}$ clopen sets only (Lemma 13.22), there are at most $\mathfrak{c}^\omega = \mathfrak{c}$ families of the form $\mathcal{C}_E$. We conclude that

$$\mathcal{D} = \{X \setminus \bigcup \mathcal{C}_E : E \subseteq X \text{ nowhere dense}\}$$

is as required. ☐

Put $Y = \omega \times X$.

Lemma 13.25 (CH). *There is a point $x \in Y^*$ such that*

(1) *x is a P-point of Y^*.*
(2) *if $E \subseteq \omega \times X$ is nowhere dense, then $x \notin \mathrm{cl}_{\beta Y} E$.*

Proof. By CH and Lemma 13.24 there is a family $\mathcal{E}$ consisting of ω_1 nowhere dense subsets of Y such that each nowhere dense subset of Y is contained in an element of $\mathcal{E}$.

• *Exercise* 79. Let Xbe a locally compact space, and let $D \subseteq X$ be nowhere dense in X. Prove that $Y^* \cap \mathrm{cl}_{\beta X} D$ is nowhere dense in X^*.

So the collection

$$\mathcal{E}^* = \{Y^* \cap \mathrm{cl}_{\beta X} E : E \in \mathcal{E}\} \tag{13.2}$$

consists of ω_1 nowhere dense subsets of Y^*.

Since X has weight $\mathfrak{c}$, so does Y, and so βY as well as Y^* also have weight $\mathfrak{c}$ (Corollary 11.11).

• *Exercise* 80. This is slightly incorrect because Corollary 11.11 gives us that $\mathrm{w}(Y^*) \leq \mathfrak{c}$. Why is it true that $\mathrm{w}(Y^*) \geq \mathfrak{c}$.

So the collection

$$\{\overline{U} \setminus U : U \subseteq Y^* \text{ is an open } F_\sigma\} \tag{13.3}$$

does also have cardinality ω_1 (under CH). By Theorem 12.15 it follows that $Y^* \approx \omega^*$ and so by Lemma 13.8 it follows that there is a point $x \in Y^*$ which is not in the union of the collection (13.2) and not in the union of the collection (13.3). Then x is clearly as required. ☐

We now come to the main results in this subsection.

Theorem 13.26 (CH)**.** *There is a weak P-point in ω^* which is not a P-point.*

Proof. Observe that X is extremally disconnected. So Y is extremally disconnected as well, so βY is extremally disconnected by the following exercise.

• *Exercise* 81. Let X be extremally disconnected. Prove that βX is extremally disconnected.

We already observed that βY has weight at most $\mathfrak{c}$. So by Theorem 13.18, we may think of βY as a closed nowhere dense P-set in ω^*. Now let x be the point we get from Lemma 13.25. We claim that x is as required.

First observe that x is not a P-point in ω^* because x is in the closure of the F_σ-set $Y \subseteq \omega^* \setminus \{x\}$.

To prove that x is a weak P-point, let $D \subseteq \omega^* \setminus \{x\}$ be countable. Then $D_1 = D \setminus \beta Y$ is countable, hence $\overline{D_1} \cap \beta Y = \emptyset$ since βY is a P-set. Next, $D_2 = D \cap Y$ is countable, and hence nowhere dense (Lemma 13.23). This implies that $x \notin \overline{D_2}$ by construction. Finally, $D_3 = D \cap Y^*$ is countable and contained in $Y^* \setminus \{x\}$. But then $x \notin \overline{D_3}$ since x is a P-point of Y^*. So we finally arrive at the conclusion that $x \notin \overline{D}$, which is as desired. □

Using the technique in this subsection it is possible to construct various other points in ω^*. For example, under CH there exists a point $x \in \omega^*$ such that x is in the closure of some countable subset of $\omega^* \setminus \{x\}$ but not of any countable *discrete* subset of $\omega^* \setminus \{x\}$. This can be done by replacing the space X in this subsection by EC, the projective cover of the Cantor set. By precisely the same argumentation as the one above we get the desired point.

Theorem 13.26 is true in ZFC. This was proved by Kunen.

14. ORDERINGS ON $\beta\omega$ AND APPLICATIONS

14.1. **The Rudin-Keisler (pre)ordering on** $\beta\omega$**.**

In this subsection we will define an interesting "order" on $\beta\omega$ and derive some of its properties.

Lemma 14.1. *Let $f \in \omega^\omega$ and $p, q \in \beta\omega$. Then the following statements are equivalent:*

(a) $\beta f(p) = q$.
(b) $\forall P \in p : f[P] \in q$.
(c) $\forall Q \in q : f^{-1}[Q] \in p$.

Proof. For (a) ⇔ (b), let $P \in p$. Then $p \in \overline{P}$ (Exercise 42) and hence, by continuity of βf, $q \in \beta f[\overline{P}] \subseteq \overline{\beta f[P]} = \overline{f[P]}$ and so $f[P] \in q$.

For (b) ⇒ (a), take $Q \in q$, and assume that $f^{-1}[Q] \notin p$. Then $P = \omega \setminus f^{-1}[Q] \in p$, from which it follows by (b) that $f[P] \in q$. But since $f[P] \cap Q = \emptyset$, this is a contradiction.

For (c) ⇒ (a), assume that $\beta f(p) \neq q$. There is a clopen neighborhood U of q such that $\beta f(p) \notin U$. We may assume without loss of generality that U has the form $\overline{Q}$ for certain $Q \in q$. But then by (c), $f^{-1}[Q] \in p$, which implies that

$$\beta f(p) \in \beta f\left[\overline{f^{-1}[Q]}\right] \subseteq \overline{\beta f f^{-1}[Q]} = \overline{f f^{-1}[Q]} \subseteq \overline{Q}.$$

This contradicts the fact that $\beta(f)(p) \notin \overline{Q}$. $\square$

Define an equivalence relation $\sim$ on $\beta\omega$ by

$$p \sim q \quad \text{iff} \quad \exists\, \text{permutation } \pi \in \omega^\omega \text{ with } \beta\pi(p) = q.$$

It is clear that $\sim$ is indeed an equivalence relation. Equivalent ultrafilters under this equivalence relation are combinatorially equivalent, and hence are intuitively the same ultrafilters. Not all ultrafilters are equivalent. Simply observe that $|\beta\omega| = 2^{\mathfrak{c}}$ (Lemma 13.4(c)) and each $\sim$-equivalence class has size at most $\mathfrak{c}$ since $|\omega^\omega| = \mathfrak{c}$.

Let $p, q \in \beta\omega$ and write

$$p \prec q \quad \text{iff} \quad \exists\, f \in \omega^\omega \text{ with } \beta f(p) = q.$$

Let X be a set and let $f\colon X \to X$. A subset B of X is said to be *f-invariant* if $f[B]$ is contained in B and *f-coinvariant* if $f^{-1}[B]$ is contained in B. Call a subset B of X *3-decomposable* if it is f-invariant and is the pairwise disjoint union of three sets B_1, B_2 and B_3 which satisfy $B_i \cap f[B_i] = \emptyset$ for each i.

In order to formulate and prove some nice properties of the relation $\prec$, we first need to formulate and prove the following interesting result.

Theorem 14.2. *Let X be a set and let $f\colon X \to X$ be such that $f(x) \neq x$ for every $x \in X$. Then X is 3-decomposable. In other words, there is a partition $\{E, F, G\}$ of X such that $f[E] \cap E = \emptyset$, $f[F] \cap F = \emptyset$ and $f[G] \cap G = \emptyset$.*

Proof. Let $\mathcal{D}$ be a maximal collection of pairwise disjoint 3-decomposable sets. Then $D = \bigcup \mathcal{D}$ is easily seen to be 3-decomposable since every $D \in \mathcal{D}$ is f-invariant. It is clear that if $S \in \mathcal{D}$ and $S \cap D = \emptyset$ then $S = \emptyset$.

Let $\{B_1, B_2, B_3\}$ be a 3-decomposition of D. Consider the following collection $\mathcal{F}$ of all quadruples $\langle A, E, F, G\rangle$ such that

(1) $A \subseteq X$ and $\{E, F, G\}$ is a 3-decomposition of A.
(2) $D \subseteq A$, $E \cap D = B_0$, $F \cap D = B_1$ and $G \cap D = B_2$.

We partially order $\mathcal{F}$ as follows:

$$\langle A, E, F, G\rangle \leq \langle A', E', F', G'\rangle \quad \text{iff} \quad A \subseteq A', E' \cap A = E, F' \cap A = F, G' \cap A = G.$$

It is clear that every chain in $\mathcal{F}$ has an upperbound. By Zorn's Lemma 2.5 there is a maximal element $\langle A, E, F, G\rangle$ in $\mathcal{F}$. We claim that $A = X$.

We first claim that A is f-coinvariant. To see this, define $\hat{A} = A \cup f^{-1}[A]$, and

$$\begin{aligned} \hat{E} &= E \cup (f^{-1}[F \cup G] \setminus A), \\ \hat{F} &= F \cup (f^{-1}[E] \setminus A), \text{and} \\ \hat{G} &= G. \end{aligned}$$

Let $x \in \hat{A}$. If $x \in A$ then $x \in E \cup F \cup H \subseteq \hat{E} \cup \hat{F} \cup \hat{G}$. So assume that $x \in \hat{A} \setminus A$. Then $x \in f^{-1}[A]$ and so then either $x \in f^{-1}[F \cup G] \setminus A \subseteq \hat{E}$ or $x \in f^{-1}[E] \setminus A \subseteq \hat{F}$. We conclude that $\hat{A} = \hat{E} \cup \hat{F} \cup \hat{G}$.

It is clear that $\hat{A}$ is f-invariant because A is.

Now consider e.g. $\hat{E}$. Then $f[E] \subseteq f[E] \cup F \cup G$. In addition, $\hat{E} = E \cup (f^{-1}[E] \setminus A) \cup (f^{-1}[G] \setminus A)$. Since $E \cap f[E] = \emptyset$, $E \cap (F \cup G) = \emptyset$, $(f^{-1}[E] \setminus A) \cap f[E] \subseteq (X \setminus A) \cap A = \emptyset$, etc., it follows that $\hat{E} \cap f[\hat{E}] = \emptyset$. Similarly for $\hat{F}$ and $f[\hat{F}]$, and trivially for $\hat{G}$ and $f[\hat{G}]$. So we conclude that $\hat{A}$ is 3-decomposable.

Since $\hat{E} \cap A = E$, etc., it follows that $\langle \hat{A}, \hat{E}, \hat{F}, \hat{G} \rangle$ is in $\mathcal{F}$. By maximality of $\langle A, E, F, G \rangle$ we therefore conclude that $\hat{A} = f^{-1}[A] \cup A = A$, or, equivalently, $f^{-1}[A] \subseteq A$.

Put $B = X \setminus A$. Since $f^{-k}[A] \subseteq A$ for every $k \geq 1$ it follows that B is f-invariant. Pick an arbitrary element $x \in B$. If x is a periodic point, i.e. if $f^n(x) = x$ for certain $n \geq 1$, then $n \geq 2$ since $f(x) \neq x$. The orbit $O(x)$ of x under f then clearly can be spit into two disjoint sets missing their own images, hence is 3-decomposable (take the two sets just found and the empty set). But this shows that $O(x) \in \mathcal{D}$, which implies that $O(x) = \emptyset$ by the above. So this is a contradiction. So x is not a periodic point. In other words, $f^n(x) \neq f^m(x)$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Now put $S = \{f^n(x) : n \in \mathbb{N}\}$, $E = \{f^n(x) : n \text{ even}\}$ and $F = \{f^n(x) : n \text{ odd}\}$. Then E and F show that $S \in \mathcal{D}$, and since $S \cap A = \emptyset$, this again yields a contradiction. □

Lemma 14.3. *Let $f \in \omega^\omega$, and let $p \in \beta\omega$. Then*

(a) *$\beta f(p) = p$ if and only if $\{n < \omega : f(n) = n\} \in p$.*
(b) *$\beta f(p) \sim p$ if and only if there exists $A \in p$ such that $f \restriction A$ is one-to-one.*

Proof. (a) We set $A = \{n < \omega : f(n) = n\}$ and let id denote the identity function on $\beta\omega$. Both βf and id are continuous function son $\beta\omega$ which are equal on A and hence on $\overline{A}$. If $A \in p$ then $p \in \overline{A}$ so that $\beta f(p) = \text{id}(p) = p$. Suppose conversely that $A \notin p$. Then $\omega \setminus A \in p$ and $f(n) \neq n$ for every $n \in \omega \setminus A$. We assume without loss of generality that $A = \emptyset$ and hence $f(n) \neq n$ for all $n < \omega$. By Theorem 14.2 there are subsets A_0, A_1 and A_2 of ω such that

(1) $\omega = A_0 \cup A_1 \cup A_2$,
(2) $A_i \cap A_j = \emptyset$ for $i, j \leq 2$ and $i \neq j$, and
(3) $A_i \cap f[A_i] = \emptyset$ for $i \leq 2$.

It follows by (1) that exactly one of the sets A_0, A_1 and A_2 is an element of p. We may assume without loss of generality that $A_0 \in p$. Then $p \in \overline{A_0}$ and

$$\beta f(p) \in \beta f[\overline{A_0}] \subseteq \overline{\beta f[A_0]} = \overline{f[A_0]} \subseteq \overline{A_1 \cup A_2}.$$

Since $A_0 \cap (A_1 \cup A_2) = \emptyset$ and disjoint subsets of ω have disjoint closures in $\beta\omega$, it follows that $\beta f(p) \neq p$, which is a contradiction.

For (b), assume that $\beta f(p) \sim p$ and let $\pi \in \omega^\omega$ be a permutation such that $\beta\pi(\beta f(p)) = p$. Then $\beta(\pi \circ f)(p) = p$ and hence by (a)

$$A = \{n < \omega : \pi \circ f(n) = n\} \in p.$$

It is clear that $f \restriction A$ is one-to-one.

Suppose conversely that there is $A \in p$ such that $f \restriction A$ is one-to- one.

Suppose first that $p \in \omega$. Then $\beta f(p) = f(p) \in \omega$. So then there is clearly a permutation $\pi \in \omega^\omega$ such that $\pi(\beta f(p)) = p$.

We may therefore assume without loss of generality that $p \in \omega^*$. Then A is infinite (Lemma 9.2) and so we can spit it in two infinite subsets, say A_0 and A_1. We may assume without loss of generality that $A_0 \in p$. So $\omega \setminus A_0$ is infinite, and so is $\omega \setminus f[A_0]$ because $f \restriction A$ is one-to-one. So there is a permutation $\pi \in \omega^\omega$ such that $\pi[\omega \setminus A_0] = \omega \setminus f[A_0]$ and $\pi \restriction A_0 = f \restriction A_0$. Then $\beta\pi(p) = \beta f(p)$ and thus $\beta f(p) \sim p$. □

The reason why we did all this work is to prove part (3) of the theorem below.

Theorem 14.4. *Let $p, q, r \in \beta\omega$. Then*

(1) $p \prec p$,
(2) *if* $p \prec q$ *and* $q \prec r$ *then* $p \prec r$,
(3) *if* $p \prec q$ *and* $q \prec p$ *then* $p \sim q$.

Proof. First observe that (1) and (2) are trivial.

For (3), let $f, g \in \omega^\omega$ be such that $\beta f(p) = q$ and $\beta g(q) = p$. Thus $\beta(g \circ f)(p) = \beta g(\beta f(p)) = \beta g(q) = p$, and thus according to Lemma 14.3(a) we have

$$A = \{n < \omega : g \circ f(n) = n\} \in p.$$

Then $f \restriction A$ is one-to-one, and $p \sim q$ by Lemma 14.3(b). □

Observe that Theorem 14.4(c) shows that the quotient relation defined by $\prec$ on $\beta\omega/\sim$ is a partial ordering.

The relation $\prec$ is called the *Rudin-Keisler ordering* on $\beta\omega$.

If $p \in \beta\omega$ then the set $\mathcal{P}(p) = \{q \in \beta\omega : q \prec p\}$ is equal to $\{\beta f(p) : p \in \omega^\omega\}$ and therefore has cardinality at most $\mathfrak{c}$, since $|\omega^\omega| = \mathfrak{c}$. Several natural questions arise. For example, is there for every $p \in \beta\omega$ a point $q \in \beta\omega$ such that $q \not\prec p$? This question can easily be answered in the affirmative since $|\beta\omega| = 2^{\mathfrak{c}}$ by Lemma 13.4(3). So there are many points $q \in \beta\omega \setminus \mathcal{P}(p)$. Let us specify the question a little bit: given $p \in \omega^*$ does there exist a point $q \in \beta\omega$ such that $p \not\prec q$ and $q \not\prec p$? It may come as a shock, but the answer to this question it now known. Under CH it is easy to see that the answer is in the affirmative, but in ZFC we do not know the answer. It is known, however, that there are points $p, q \in \beta\omega$ such that $p \not\prec q$ and $q \not\prec p$. This is a celabrated theorem of Kunen and in the remaining part of this section we will prove it and present interesting applications.

By a *filter* in $\mathcal{P}(\omega)$ we mean a proper filter in the Ba $\mathcal{P}(\omega)$.

Definition 14.5. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter no element of which is finite. An indexed family $\{(A_1^0, A_i^1) : i \in I\}$ of pairs of disjoint subsets of ω is called an *independent family with repect to* $\mathcal{F}$ provided that for all $\sigma \in [I]^{<\omega}$, $f \in 2^\sigma$ and $F \in \mathcal{F}$ the set

$$F \cap \bigcap_{i \in \sigma} A_i^{f(i)}$$

is infinite

Let $\mathcal{CF}$ denote the filter of cofinite subsets of ω.

Lemma 14.6. *There is an independent family $\{(A_\alpha^0, A_\alpha^1) : \alpha < \mathfrak{c}\}$ with repect to $\mathcal{CF}$.*

Proof. This is Lemma 13.4(b). $\square$

If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote by $\langle\mathcal{A}\rangle$ the (possibly improper) filter on ω generated by $\mathcal{A}$.

Lemma 14.7. *Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(\omega)$ be filters and assume that $\{(A_i^0, A_i^1) : i \in I\}$ is independent with repect to $\mathcal{F}$ as well as to $\mathcal{G}$. For each $f \in \omega^\omega$ there are a finite subset $J \subseteq I$ and a subset $A \subseteq \omega$ such that $\{(A_i^0, A_i^1) : i \in I \setminus J\}$ is independent with repect to $\langle\mathcal{F} \cup \{A\}\rangle$ as well as to $\langle\mathcal{G} \cup \{\omega \setminus f^{-1}[A]\}\rangle$.*

Proof. Fix an arbitrary $a \in I$.

Case 1: $\{(A_i^0, A_i^1) : i \in I \setminus \{a\}\}$ is independent with repect to $\langle\mathcal{G} \cup \{\omega \setminus f^{-1}[A_a^0]\}\rangle$.

We then put $A = A_\alpha^0$ and $J = \{a\}$. An easy check shows that A and J are as required.

Case 2: Not Case 1. Then there are a finite $K \subseteq I \setminus \{a\}$ and a function $\varphi \in 2^K$ and an element $G \in \mathcal{G}$ such that

$$\Big|\bigcap_{i\in K} A_i^{\varphi(i)} \cap G \cap (\omega \setminus f^{-1}[A_a^0])\Big| < \omega. \tag{14.1}$$

Now put $A = \omega \setminus A_a^0$ and $J = K \cup \{a\}$. It is clear that $\{(A_i^0, A_i^1) : i \in I \setminus J\}$ is independent with respect to $\langle\mathcal{F} \cup \{A\}\rangle$, so it remains to verify that it is also independent with repsect to $\langle\mathcal{G} \cup \{\omega \setminus f^{-1}[A]\}\rangle$. To this end, let $L \subseteq I \setminus J$ be finite, and let $g \in 2^L$. Choose an arbitrary $G_0 \in \mathcal{G}$. Then

$$\begin{aligned}\bigcap_{i\in L} A_i^{g(i)} \cap G_0 \cap (\omega \setminus f^{-1}[A]) &= \bigcap_{i\in L} A_i^{g(i)} \cap G_0 \cap f^{-1}[A_a^0] \\ &\supseteq \bigcap_{i\in L} A_i^{g(i)} \cap \bigcap_{i\in K} A_i^{\varphi(i)} \cap (G_0 \cap G) \cap f^{-1}[A_a^0],\end{aligned}$$

which is infinite by (14.1) and our assumption that $\{(A_i^0, A_i^1) : i \in I\}$ is independent with respect to $\mathcal{G}$. $\square$

We now come to the main result in this section.

Theorem 14.8. *There are points $p, q \in \beta\omega$ such that $p \not\leq q$ and $q \not\leq p$.*

Proof. By Lemma 14.6 there is an independent family $\{(A_\alpha^0, A_\alpha^1) : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ with respect to $\mathcal{CF}$. Let $\{f_\alpha : 1 \leq \alpha < \mathfrak{c}\}$ enumerate ω^ω. By transfinite induction on α we will construct $\mathcal{F}_\alpha$, $\mathcal{G}_\alpha$ and $K_\alpha \subseteq \mathfrak{c}$ so that

(1) $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are filters on ω and $\{(A_\xi^0, A_\xi^1) : \xi \in K_\alpha\}$ is independent with respect to $\mathcal{F}_\alpha$ as well as $\mathcal{G}_\alpha$,
(2) $K_0 = \mathfrak{c}$ and $\mathcal{F}_0 = \mathcal{G}_0 = \mathcal{CF}$.
(3) $\kappa < \alpha$ implies that $\mathcal{F}_\kappa \subseteq \mathcal{F}_\alpha$, $\mathcal{G}_\kappa \subseteq \mathcal{G}_\alpha$ and $K_\alpha \subseteq K_\kappa$,
(4) for each α, $|\mathfrak{c} \setminus K_\alpha| \leq |\alpha| \cdot \omega$,
(5) for each $\alpha \geq 1$ there are sets $A, B \subseteq \omega$ with $\{A, \omega \setminus f_\alpha^{-1}[B]\} \subseteq \mathcal{F}_\alpha$ and $\{B, \omega \setminus f_\alpha^{-1}[A]\} \subseteq \mathcal{G}_\alpha$.

Suppose that we have completed the construction for all $\kappa < \alpha$, $\alpha < \mathfrak{c}$. Put $K = \bigcap_{\kappa<\alpha} K_\kappa$, $\mathcal{G} = \bigcup_{\kappa<\alpha} \mathcal{G}_\kappa$ and $\mathcal{F} = \bigcup_{\kappa<\alpha} \mathcal{F}_\alpha$. Observe that $\mathcal{F}$ and $\mathcal{G}$ are filters, that $|K| = \mathfrak{c}$ by (4), and that by (1), $\{(A_\xi^0, A_\xi^1) : \xi \in K\}$ is independent with respect to $\mathcal{F}$ as well as to $\mathcal{G}$. Using Lemma 14.7 twice, it is now easy to construct $\mathcal{F}_\alpha$, $\mathcal{G}_\alpha$ and K_α satisfying (1) through (5).

Now let $p \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ and $q \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathcal{G}_\alpha$ (Lemma 6.2). By (5) it easily follows that $p \not\prec q$ and $q \not\prec p$, as desired. ☐

Points p and q as in the above theorem are called *$\prec$-incomparable*, or *RK-incomparable.* Observe that if $p, q \in \beta\omega$ are $\prec$-incomparable then they both belong to ω^*.

Kunen actually proved that there is a set in ω^* consisting of $\mathfrak{c}$ pairwise incomparable points. This was later generalized by Shelah and Rudin, who proved that there is such a set of size $2^{\mathfrak{c}}$. A simpler proof of this fact was later discovered by Dow.

14.2. **The Rudin-Frolík (pre)ordering on $\beta\omega$.** The *Rudin-Frolík (pre)order* $\sqsubseteq$ on ω^* is defined as follows:

$$p \sqsubseteq q \quad \text{iff} \quad \text{there is an embedding} h\colon \beta\omega \to \omega^* \text{ with } h(p) = q.$$

This order and the Rudin-Keisler order are related, as the following lemma shows.

Lemma 14.9. *If $p, q \in \omega^*$ and $p \sqsubseteq q$ then $p \prec q$.*

Proof. Let $h\colon \beta\omega \to \omega^*$ be an embedding with $h(p) = q$. Since $h[\omega]$ is relatively discrete, there is a sequence C_n of subsets of ω such that for all $n < \omega$,

(1) $h(n) \in \overline{C_n}$, and
(2) if $n \neq m$ then $C_n \cap C_m = \emptyset$.

By adding $\omega \setminus \bigcup_{n<\omega} C_n$ to C_0 we may even assume that the sequence $\{C_n : n < \omega\}$ is a partition of ω. Define $g\colon \omega \to \omega$ by

$$g(k) = n \quad \text{if} \quad k \in C_n,$$

and consider βg. We claim that $\beta g(q) = p$. Indeed, take an arbitrary $Q \in q$. Put $A = \{n < \omega : h(n) \in \overline{Q}\}$. Then clearly $q \in \overline{h[A]}$ since $\overline{Q}$ is a clopen neighborhood of q. Since h is embedding, and $h(p) = q$ it therefore follows that $p \in \overline{A}$, or, equivalently, $A \in p$. But if $n \in A$ then $h(n) \in \overline{C_n} \cap \overline{Q}$ from which it follows that $C_n \cap Q \neq \emptyset$, and so $n \in g[Q]$. We conclude that $A \subseteq g[Q]$ from which it follows that $g[Q] \in p$ since $A \in p$. But now $\beta g(q) = p$ by Lemma 14.1(b). ☐

Corollary 14.10. *If $p, q \in \omega^*$ are such that $p \sqsubseteq q$ and $q \sqsubseteq p$ then $p \sim q$.*

Proof. This follows from Lemmas 14.9 and 14.4(3). ☐

Lemma 14.11. *If $p, q, r \in \omega^*$ and $p \sqsubseteq q$ and $q \sqsubseteq r$ then $p \sqsubseteq r$.*

- *Exercise* 82. Present a proof of Lemma 14.11.

So, similarly to the RK-order, we find that the quotient relation defined by $\sqsubseteq$ on $\omega^*/\sim$ is a partial ordering.

We now show that the orderd $\prec$ and $\sqsubseteq$ are powerful tools if one wishes to study $\beta\omega$. First a preliminary lemma.

Lemma 14.12. *Let* $f\colon \omega^* \to \omega^*$ *be a homeomorphism, and let* $q \in \omega^*$. *Then*

$$\{p \in \omega^* : p \sqsubseteq q\} = \{p \in \omega^* : p \sqsubseteq f(q)\}.$$

Proof. This is obvious. If $p \sqsubseteq q$ then there is an embedding $e\colon \beta\omega \to \omega^*$ such yhat $e(p) = q$. But then $f \circ e$ is such an embedding with $f \circ e(p) = f(q)$, etc. ☐

This enables us to give our first 'real' proof that ω^* is not homogeneous.

Theorem 14.13. ω^* *is not homogeneous.*

Proof. Let $D = \{d_n : n < \omega\}$ be a relatively discrete faithfully indexed (= no repetitions) subset of ω^* and take a point $x \in \overline{D} \setminus D$. In Theorem 12.14 we proved among other things that ω^* is an F- space. But an inspection of the proof shows that we in fact proved that every F_σ-subset of ω^* is C^*-embedded in ω^*. So D is C^*-embedded in ω^* and hence its closure can be identified with βD. Now put

$$A = \{y \in \overline{D} \setminus D : \exists \text{ homeomorphism } f\colon \omega^* \to \omega^* \text{ with } f(x) = y\}.$$

Let $h\colon \omega \to D$ be defined by $h(n) = d_n$. Its Stone extension $e = \beta h\colon \beta\omega \to \overline{D} = \beta D$ is an embedding. If $y \in A$ then clearly

$$e^{-1}(y) \sqsubseteq y$$

and so by Lemma 14.12 we get

$$e^{-1}(y) \sqsubseteq x.$$

Since e is one-to-one and since by Lemma 14.9 it follows that $|\{q \in \beta\omega : q \sqsubseteq x\}| \le \mathfrak{c}$, we conclude that $|A| \le \mathfrak{c}$. But $|\beta\omega| = 2^{\mathfrak{c}}$ (Lemma 13.4(c)), and since $\beta D \approx \beta\omega$ we can therefore find $2^{\mathfrak{c}}$ points in $\overline{D} \setminus A$. ☐

14.3. **More nonhomogeneity results.** We now turn to an interesting application of the fact that RK-incomparable points exist. We will derive a surprisinly general nonhomogeneity result.

• *Exercise* 83. Let X be a normal F-space. Prove that every countable subset of X is C^*-embedded in X.

So if X is a compact F-space and $D \subseteq X$ is a countable discrete set then $\overline{D}$ can be identified with $\beta\omega$.

Theorem 14.14. *Let* X *be an infinite compact space in which all countable discrete subspaces are* C^**-embedded. Then* X *is not homogeneous.*

Proof. Since X is infinite, it contains an infinite (relatively) discrete subset. So for convenience assume that ω is a subspace of X. The conditions on X imply that $\overline{\omega}$ can be identified with $\beta\omega$. By Theorem 14.8 there are RK-incomparable points $p, q \in \beta\omega$. We claim that there is no homeomorphism $h\colon X \to X$ with $h(p) = q$. Striving for a contradiction, assume that such an h exists.

Let $\{U_n : n < \omega\}$ be a family of open subsets of X such that

(1) $n \in U_n \subseteq \overline{U}_n \subseteq X \setminus \omega^*$,
(2) if $n \neq m$ then $\overline{U}_n \cap \overline{U}_m = \emptyset$.

Put $E = \{n < \omega : h(n) \notin \bigcup_{m<\omega} U_m \cup \overline{\omega}\}$.

Case 1: $q \in \overline{h[E]}$.

Since $h[E] \cup \omega$ is clearly a discrete subset of X, and since $h[E] \cap \omega = \emptyset$, the assumptions on X imply that $\overline{h[E]} \cap \overline{\omega} = \emptyset$. But this is impossible since $q \in \overline{\omega}$.

Put $F = \{n < \omega : h(n) \in \omega^*\}$.

Case 2: $q \in \overline{h[F]}$.

Then $p \in \overline{F}$ so that $p \sqsubseteq q$ which implies $p \prec q$ by Lemma 14.9. But this is a contradiction.

Put $G = \{n < \omega : h(n) \in \bigcup_{m<\omega} U_m$.

Case 3: $q \in \overline{h[G]}$.

Define a function $f\colon \omega \to \omega$ by

$$\begin{cases} f(k) = 0 & \text{if } k \notin G \\ f(k) = n & \text{if } k \in G \text{ and } h(k) \in U_n. \end{cases}$$

An easy check shows that $\beta f(p) = q$, i.e. $q \prec p$, which is also a contradiction.

Since $E \cup F \cup G = \omega$, $p \in \overline{\omega}$ and $q = h(p) \in \overline{h[E]} \cup \overline{h[F]} \cup \overline{h[G]}$, we have obtained a contradiction. ☐

Corollary 14.15. *No infinite compact F-space is homogeneous. In particular, no infinite compact extremally disconnected space is homogeneous.*

Corollary 14.16. *If X is noncompact, locally compact and σ-compact, then X^* is not homogeneous.*

Proof. This is clear since by Theorem 12.14, X^* is an F-space. ☐

It can be shown that X^* is not homogeneous for any nonpseudocompact space X. This very general and beautiful nonhomogeneity result is due to Frolík.

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